Communications

Roots in real and complex numbers: a case of unacceptable discrepancy

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Please calculate √9 in the field of complex numbers. Having no access to your actual responses, I would dare to predict that they are diverse: Some probably responded with 3 because of the principal branch of the multivalent root function; others, possibly, determined –3 and 3 with De Moivre's theorem; and some, rebelled perhaps against using the √ symbol in the context of complex numbers. The root symbol instantiates *polysemy*—a phenomenon in which the same concept name or symbol can be interpreted in discrepant manners depending on the context in which they are used and on the curricular norms associated with the context. An additional source of complexity is that the meanings of polysemous entities are deeply related, which forces one to pay attention to small contextual details that can make a big difference. Clearly, such mindfulness requires one to be aware that such polysemy exists in mathematics.

How do students cope with and make sense of polysemy in mathematics? Zazkis (1998) tackled these questions in the case of 'divisor' and 'quotient'. When requested to determine the quotient in the division of 12 by 5, some of her pre-service teachers operated in the domain of integers and argued for 2, while others adhered to rational numbers and suggested 2.4. Zazkis also reported that the teachers did not reach a consensus around *the* quotient in the assigned division and turned to her for 'the right answer' that would resolve the discrepancy. In this way, polysemy does not seem to shape their individual concept images (Tall & Vinner, 1981), which may indicate insufficient awareness of it.

My central claim in this communication is that while mathematics is replete with discrepancies in general and polysemy in particular, school and university may not provide students with enough opportunities to become aware of it. I support the claim with an episode from my conversation with Anna—a high-achieving mathematics major who distorted her perfectly valid mathematical conceptions for the sake of resolving discrepancies between roots of real and complex numbers.

Selected perspectives on the root concept

Anna has been well familiar with the root concept from her studies in an Israeli high-school and university. In the field of real numbers, the √ symbol is used in Israel for denoting a non-negative square root. This convention is particularly useful for supporting students' expansion of the concept image from roots as algebraic operations to roots as univalent functions. The situation becomes more complicated when students encounter complex numbers, where concept definitions and symbol meanings become highly dependent on teachers, lecturers and textbooks (Kontorovich, 2016). For instance, the textbook of Yaquel (2004) distinguishes between roots of real radicands, which are single-valued, and non-real radicands, which are multi-valued. The distinction is never articulated and textbook readers seem to be expected to pick it up from multiple examples of roots that Yaquel assigns for calculation. In linear algebra courses, a link between roots of numbers and roots of polynomials is often emphasized, which draws attention to multivalence of the concept. In complex analysis courses, a root is a traditional example of multivalent functions; the example becomes univalent in the topic on inverse functions.

Radical changes in definitions and properties of *the same* polysemous entity can be easily viewed as discrepancies. But who is to blame for their emergence: a student who engaged with the entity or mathematics that accommodates such anomalies? The next section shows that one student's answer to this question can prescribe the way she copes with discrepancies.

Anna copes with discrepancies

When extracting square roots of positive numbers, Anna replied with single positive results. For example, she used the graph of a function for explaining that √9 = 3 (see Figure 1a). Then, we turned to square roots in complex numbers.

1 *Igor'* Can you please compute square root [1] of nine in the field of complex numbers?

2 *Anna* OK, so according to De Moivre [*wrote*]

$$9 = 9e^{2\pi ki}$$
$$\sqrt{9} = \sqrt{9e^{2\pi ki}} = \sqrt{9}\left(e^{2\pi ki}\right)^{1/2} = 3e^{\pi ki} = \pm 3$$

nine *e* to two *pi k i*. So a root now. Three *e* to *pi k i*. *k* can be 1 or 2, so we get 3 and –3.

3 *Igor'* So you got two results.

4 *Anna* Right.

5 *Igor'* Does it bother you that you got just 3 before?

6 *Anna* Oh right [*paused*] Obviously it shouldn't be like this, any real number is also complex [*paused*] Mmm [*paused*] Let me check [*Went over her calculations for half a minute*] I know, I should have also gotten two results then, it was a mistake.

7 *Igor'* Can you explain?

8 *Anna* By definition, *b* is a square root of a if *b* squared equals *a*. So 3 and –3 work.

9 *Igor'* And what about the graph?

10 *Anna* [*After half a minute*] All your numbers are squared. Root of *a* squared is an absolute value

of *a*, then we got absolute value of 3 which is plus minus three.

11 *Igor'* An absolute value of 3 is two numbers?

12 *Anna* Yes.

13 *Igor'* Can you show it on a graph?

14 *Anna* [*Sketched the graph of the absolute value shown in Figure 1b and pointed at y = 3*] Here, we get 3 and −3.

15 *Igor'* How about? [*wrote*]

$$\sqrt{(-5)^2}$$

16 *Anna* [*Paused for 15 seconds*] It's a tricky function, it should be 5 and −5.

17 *Igor'* Can you show it on the graph?

18 *Anna* [*Paused for 10 seconds*] I told you it's a bit tricky: we know that the absolute value of −5 is the same as the absolute value of 5 [*pointed at y = 5*] so you get two values.

Anna seemed to be sensitive to the field in which roots were extracted: in the domain of real numbers she used the graph of the square root function; in the domain of complex numbers in line 2 she employed De Moivre's theorem. However, this difference cannot be an acknowledgment of polysemy because in line 6 she interpreted the obtaining of discrepant results, in the fields of complex and real numbers, as a mistake that needed to be fixed.

In her first attempt to resolve the discrepancy in line 8, Anna turned to the definition of square root that "should have also gotten two results" in both fields. This created a new discrepancy with the single value that she obtained from the graph in Figure 1a. In her second attempt in lines 10-14, she exchanged her use of the input and output of the graph. When she determined √9 using the graph in Figure 1a, Anna located the input 9 on the x-axis and extracted 3 on the y-axis. However, when she simplified √9 into |3| in her second attempt (Figure 1b) she located the input 3 on the y-axis and extracted 3 and −3 on the x-axis. This exchange is remarkable because just a few minutes earlier Anna operated correctly with the graph of the square root. Anna's interpretation of the absolute value in Figure 1b entailed that the notion does not exist for negative inputs. This discrepancy was creatively resolved in lines 16 and 18 by substituting negative inputs with their positive counterparts. Overall, Anna systematically revised and renounced her perfectly valid ways of thinking in order to achieve at least a temporary coherence in her work. It is also notable that at no point did Anna question why the mathematics that she perceived as valid a short time ago should be renounced. This tendency was also evident at the end of our conversation where I shared my approach to the root concept with Anna. There, she mostly concentrated on "what is right" rather than on how her initial approaches made perfect sense together.

Anna had good reasons for anticipating that complex and real roots of 9 would be the same. Indeed, complex numbers are often introduced as an extension of real numbers, which may result in an image that the wider numerical set contains 'new' mathematics that does not contradict the results obtained from the 'old' one. This image is graphically captured in Figure 2 and it resonates with line 6 where Anna stated that any real number is also complex. With such a coherent image in mind, there seems to be no room for discrepancies.

On opportunities to experience the discrepancy of mathematics

The episode with Anna illustrates that even an experienced and successful student can interpret a discrepancy as an unacceptable mistake in *her* mathematics rather than polysemy rooted in mathematics itself. Coherence, consistency and connectedness are fundamental principles of mathematics curricula in different countries and I share the view of many mathematicians who emphasize the importance of these ideas. Indeed, Anna's recognition of discrepancies and attempts to resolve them can be justly viewed as coherence-driven acts of monitoring and control. My point is that students' learning experiences cannot be complete without awareness of discrepancies in mathematics—the ones acknowledged by the mathematics community not just the ones perceived as students' mistakes. The episode above shows that confusion between the two may lead to a systematic renunciation of perfectly valid ways of thinking.

The first curricular (*i.e.* unavoidable) opportunities to experience discrepancies in mathematics are embedded in university courses, such as set theory. Consequently, an enormous number of students can finish their mathematics education journey without experiencing discrepancy even once. The 'lucky ones', on the other hand, can encounter the discrepancy only after more than a decade of engagement with the discipline. I wonder why it is so and whether

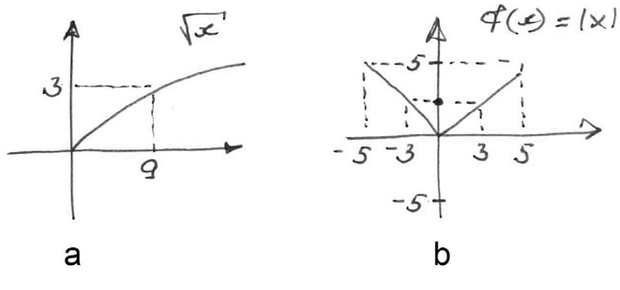

Figure 1. Anna's sketches of functions.

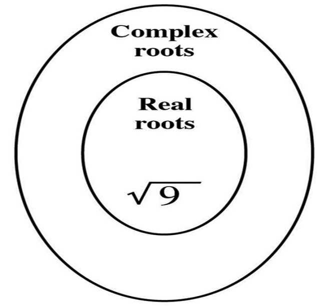

Figure 2: Schematic representation of Anna's concept image

such experiences can come sooner. The archive of *FLM* is particularly insightful in this regard with articles on incompatible definitions (*e.g.,* Jayakody & Zazkis, 2015), historical debates and mistakes (*e.g.,* Arcavi, Bruckheimer, & Ben-Zvi, 1983; Kleiner, 1986) and ambiguities of different sorts (*e.g.,* Foster, 2011; Rathouz, 2010). My hope with this communication is to encourage students and educators to challenge sterilized mathematics in the textbooks and to learn to enjoy its discrepancies.

Note

[1] Anna and I were speaking Hebrew, and I was intentionally using an article-less form when referring to the root symbol.

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If Trump were an applicant to your mathematics education program, would you accept him?

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After the stunning result of the 2016 presidential election, many are wondering how Donald Trump could have been elected president of the United States. While the realization that Trump is the U.S. president is mind numbing, what would you do if he was an applicant to your secondary mathematics program? Knowing what we know about Trump, many of us would recommend rejection without even needing to review his application. After all, he established himself quite clearly during the election as a racist, xenophobe, misogynist, and Islamophobe. He raises obvious red flags, so this is an easy decision, right? But, hold on…

The dilemma

In the United States, prospective teachers (PTs) must apply for admission into a teacher education program and are not automatically accepted or denied admission based on test results. Faculty generally have the final say about who to accept into the program they oversee. Assuming Trump, or more realistically, an applicant with a similar profile applied to your program, how much would you actually know about the applicant's past, beliefs, or how well the applicant works with diverse student populations? Perhaps, the applicant may not come across as particularly culturally sensitive in his or her essay and may even come across as somewhat insensitive during the interview. Is the applicant's lack of cultural sensitivity and experience working with diverse students enough to deny acceptance into your teacher education program? Many mathematics teacher educators would argue yes, since many PTs will be working with low-income, culturally and linguistically diverse 'underserved' students. Furthermore, many mathematics teacher educators probably agree with Haberman (1995) that we should take great care to admit only the very best to teach underserved students in schools located in the most impoverished communities. Nevertheless, the question does beg some level of reflection concerning our work to prepare PTs to work with underserved student populations. If we are in the business of preparing PTs to do exactly this, then is accepting a Trump-like applicant something we should even consider doing?

In this article, we engage in a conversation about accepting students, with biases against some of the groups they might be teaching, into our teacher education programs. Our focus on preparing PTs to work well with underserved students is central in our discussion because of the on-going historic legacy of Blacks, Native Americans, and Latinos/as and working class people in the U.S being denied access to a challenging mathematics education (See, for example, Kitchen, DePree, Celedón-Pattichis & Brinkerhoff, 2007). Our arguments here are based on our experiences as mathematics teacher educators rather than on the specifics of our programs. The three of us work at very different institutions. Anthony Rodriguez works at a Catholic Dominican liberal arts college. Richard Kitchen works at a public university with a strong focus on sciences, agriculture and engineering. Jenni Harding works at a public teaching oriented university. Anthony gets us started.

Our discussion

AR: I believe we have to admit such an applicant because we need to be around more people like Donald Trump to understand them, find out why they are so angry and have such deep hate, why immigrants from Mexico scare them so much, why it is seen as acceptable to only define diversity as difference of thought and not color. The applicant clearly does not know what s/he does not know and that is scary. As a faculty member, I need to open the door to such applicants so I can go to work to try to awaken them, push them, expose them to truths their simultaneously fragile and massive egos cannot handle. I need to both understand their deep beliefs, as well as find intersections with my own, to optimize my work with them (Chu & Garcia, 2014).

RK: I think an important question to ask here is just how open such applicants are to being awakened? In my career, I have worked with a number of PTs who were not particularly concerned about their students' backgrounds in general, much less their students' cultural and linguistic backgrounds. I found this to be particularly true of second career individuals, like engineers who decide later in life to make a career change and become teachers. For many, particularly at the secondary level, their focus is simply on imparting mathematical knowledge to their students. I'm not trying to over-generalize about engineers who come into teaching. I think that applicants to a secondary mathematics program fall somewhere on a spectrum of sensitivity when we consider issues such as race, class, gender, and language. My experiences have been that not just second career folks, but the majority of secondary mathematics applicants could be classified as biased, some more extremely than others (Kitchen, 2005). And while it has been my experience that the majority of the more extreme applicants are white and male, others have been white women and people of color as well.

JH: I have also seen this being true for applicants where teaching is their first career because of their lack of sensitivity to students' cultural backgrounds. Some of these applicants view the teaching of mathematics as a content area where it is important to understand and know the academic discipline, but nothing beyond that (Shulman, 1987; Fan 2014). They believe they will teach the mathematics curriculum from the textbook provided, exactly as it is written. Thinking about students' cultural and linguistic backgrounds in order to effectively teach them mathematics is not something they arrive at on their own, because they assume all of their students have a background similar to theirs. PTs need to come to recognize and understand how power and privilege operate so that they can become advocates for culturally and linguistically diverse students in their future classrooms (Swanson, 2005).

AR: I am concerned that if we don't accept a Trump-like applicant, a program with less qualified people will; a program that aligns with the applicant's views—a safe space for racism to go unchallenged. Such applicants may never be ready for teaching in highly diverse, marginalized communities, regardless of training, but I have to believe they will or I won't engage fully in the work that needs to be done. My own teacher efficacy, that is, my belief both in my skill in teaching them as well as in their ability to reach cultural competence (Chu & Garcia, 2014), must be continually honed to engage successfully with such applicants.

JH: It is difficult to deny any 'qualified' applicants into our programs because mathematics is considered a high need area in most states. Moreover, most programs are not at capacity and are not in a position to turn an applicant away if s/he meets all of the admission requirements (Abbott, Ferriso, & Smith, 2010). It comes down to the question: Is a trained math teacher better than not having one at all? In our need to provide schools with mathematics teachers, are we losing quality? There exist exit measures to demonstrate knowledge in the education field for a student to graduate (*e.g.,* content exams, successful student teaching experiences, capstone projects, GPA requirements). Might a culturally responsive measure be required for graduation? This could challenge the status quo within universities where teaching for diversity is an implicit outcome, but not one that is measured.

RK: Many PTs don't really believe classes about equity, diversity and access apply to them (Kitchen, 2005). How do we help PTs understand these issues are important no matter where they teach? After all, even in the most homogenous and isolated White communities, one finds diversity (*e.g.,* low-income students, refugees, special education students).

AR: To do this work, biased applicants, and all PTs for that matter, need to be placed in a rigorous immersion program in the local city schools, ones that are strengths-based, rigorous and reflective, that treat all students as capable and responsible (Chu & Garcia, 2014). Biased applicants need to come to view that every student has funds of knowledge about, for example, knowing one language and learning another, and have experiences that are richly grounded in their culture, and PTs need to understand that every student engages in extensive mathematical experiences in their daily lives (Moll & Ruiz, 2002). In every semester starting in their first year of the program, PTs should spend many hours teaching in diverse schools with a culturally competent liaison to both provide specific feedback on their instruction but also to be there for support. The liaison can show biased applicants that hate and divisiveness, sexism and racism, ignorance and fixed beliefs all do not belong in the classroom. I want them to see that cultural and linguistic diversity strengthens us, sustains life and makes us more connected to the world around us— lowering the anxiety, fear, and lack of self-confidence that clearly is behind all the hate.

RK: One of my primary goals is to foster in PTs a respect for their students' knowledge, particularly the mathematical knowledge of their low-income students and students of color (Kitchen, 2005). In my mathematics methods courses, PTs engage a student in problem solving. As part of the assessment, the PT meets with a student and conducts a one-on-one interview with the student, asking the child to solve 4-6 problems. A requirement of the activity is that the student selected must have a very different profile from the PT (in terms of *e.g.* gender, race, class, and [dis]ability). Every year, PTs are generally genuinely surprised by what their students are capable of doing mathematically. I attempt to build on these initial experiences to support PTs in beginning to develop an assets-based frame that leads them to work on actively seeking opportunities where they can build on their students' mathematical knowledge during instruction.

JH: It is the experiences we create in our programs that allow our students to appreciate, recognize and teach towards diversity. In my classes, PTs participate in an activity to focus on English language learners in the classroom where a video is shown in Farsi allowing my students to become Farsi language learners themselves. This is then followed with focused discussion questions on how to specifically make mathematics instruction comprehensible to students whose first language is not English (Harding-DeKam, 2007). Second, PTs create, teach, transcribe and reflect upon a mathematics lesson that incorporates mathematics concepts and multiculturalism in their practicum classroom (Harding, Hbaci, Loyd, & Hamilton, 2017).

What it means for our programs

To conclude, we found our aforementioned 'easy decision' to reject a Trump-like applicant if s/he applied to our secondary mathematics program was anything but. At a time when racial profiling and race-based violence are on open display again in the United States, it is important that we actually commit to work with biased applicants. We believe that our work must go further than it has gone before. It must involve interrogating bias and discrimination by "identity thieves" (Jett, 2013) to examine the roots of exclusion that ask people of color to check their cultures at the door (Valenzuela, 1999). And it must explore what this means for our academic departments and courses. If we believe our teacher education programs are for everyone, then we must be willing to be changed by those who enter as much as we wish to make a dent in their lives. It will be more difficult now than ever, but if we are going to model inclusion and culturally relevant teaching (Ladson-Billings, 1995; Gay, 2002), we must accept biased applicants' backgrounds as starting points of strength, assume competence and do our best to prepare them to work in highly diverse schools. The goal is for all PTs to not only tolerate difference, but actively work to promote and embrace differences as well as competence among their students.

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Responding to "a new look at the ZPD"

DEBBIE STOTT

This is a response to Yasmine Abtahi's recent article, in **37**(1), about the more knowledgeable other in the ZPD. I recall what Palincsar said about the ZPD nineteen years ago: the ZPD is "probably one of the most used and least understood constructs to appear in contemporary educational literature" (1998, p. 370). My own experiences echo his thoughts. Hence, I believe that it is important to highlight problematics for open discussion, as Abtahi has.

I'd like to open the discussion by briefly reviewing my own experiences of working with the ZPD before discussing some questions that Abtahi's article raised for me. My doctoral research took place in two afterschool mathematics clubs with 9 to 10-year-old children. The notion of the ZPD I used supported a broader understanding of the ZPD in context, which included the participants, their interactions, collaboration and activity, the types of mediation and tools used and the cultural-historical perspective (following Levykh, 2008). My interpretation favoured the ZPD as a symbolic space, which does not exist prior to the learning activity *i.e.,* the ZPD did not 'belong' to the individual child. Rather it was created through the social interactions with others during learning activities and depended on the active contributions of all the participants. Whilst I worked within a broad Vygotskian socio-cultural context of learning and development, my research focused on mathematical progression of the learners in the two afterschool mathematics clubs and the mediation that enabled this progression to take place.

I have since reflected that this focus may or may not lead to the development of higher-level psychological functions (such as the development of deduction, inference, categorisation, classification, generalisation and so on in mathematics). With such a focus on learning and progression within a given mathematical learning trajectory, my theoretical space was ultimately a neo-Vygotskian one: observable progression, perhaps learning (but not development) took place through interaction with others, through mediation as well as through verbal and non-verbal communication and was embedded in the learning activity and the established ethos of the club context.

I, like many others, initially took up Vygotsky's ZPD as if it were a fully formed notion, though it was not. Meira and Lerman (2001) for example, tell us that Vygotsky first spoke about the ZPD at a conference in 1933, a year before he died. The much-quoted definition of the ZPD (see below), was not the only description he used. Moreover, it differs from my own and many subsequent neo-Vygotskian interpretations of the notion currently found in the educational literature. The usual definition points to the ZPD being created at the space between the actual development and the potential development. The possibility for learning occurs once the ZPD comes into existence. Vygotsky stressed that learning and teaching do not coincide with development, but that the developmental process lags behind the learning process. This relationship between learning and development is at the core of Vygotsky's theory.

You will note that I have raised two key points so far. The first is the issue of conflation of development and learning that is so often seen in using the ZPD, my own working definition included. The second highlights the idea that if one is *not* working with development, then one is working with a neo-Vygotskian interpretation. I'd like to use these ideas to structure a response to Abtahi's proposal for a new look at the ZPD.

The version of the ZPD quoted by Abtahi and then referenced in the discussion conflates development and learning from a Vygotskian perspective. As I myself and many others have done, Abtahi draws on the common definition of the ZPD: namely the distance between the actual development level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers (Vygotsky, 1978, p. 86).

Yet, the article highlights that "the Vygotskian perspective assumes that we *learn* in the presence of other people: others who have a better knowledge of certain historical and cultural practices" (p. 35). Throughout the article Abtahi refers to learning. In the final paragraph, there is a call for a new look at the ZPD which is a "zone within which we learn (we come to know) with others" (p.38). I infer from this that the ZPD is seen as a space of learning and perceive that Vygotsky's notions of development and learning are considered as the same process, where they are fused together. The definition of the ZPD used in the article does not reference learning, only development. As already mentioned, the Vygotskian version of development (and the key word in the phrase ZPD) only takes place if the child's learning *develops* maturing psychological functions.

What I did in my research and what Abtahi's article does seems to be common in educational research. But why is Vygotsky's distinction between learning and development not maintained in many of the definitions of the ZPD used in educational literature? Perhaps we as educational researchers and practitioners find that Vygotsky's ZPD is not easy to translate into practice, as it can be difficult to identify development of higher-level psychological functions in his terms. It is often easier to observe and possibly measure learning, thus, the ZPD is often theorised in terms of learning. Consequently, the key aspect of Vygotskian development is notably absent, as we see in the description of my concept and in Abtahi's discussion. With these concerns in mind, Chaiklin (2003) suggests that the ZPD should be considered as a *zone of learning* rather than a zone of development, particularly in educational contexts. The version I now use, described above, could perhaps be more accurately called a *zone of proximal learning* (ZPL).

I turn now to the second challenge. In calling for a new look at the ZPD, I argue that Abtahi is herself entering the realm of neo-Vygotskian conceptualisations of ZPD. Obviously, when choosing a research focus and theoretical frame, one must take a stand about where one works. Reflection on this stance can change how one views notions such as the ZPD. With the complexity and popularity of such a concept as the ZPD and the differing conceptualisations available, contention over the key elements of the ZPD and how it is conceptualised are bound to occur. Therefore, I believe that caution should be taken when calling for changes to the original Vygotskian notion. I propose that, perhaps, it is more useful to acknowledge the problematic areas and move forward with a conceptualisation that differs from the original, calling it something else yet acknowledging that it derives from the original Vygotskian idea.

Indeed, Chaiklin (2003) is vociferous in suggesting that the term zone of proximal development should only be used to refer to the "phenomenon that Vygotsky was writing about" (p.58) and argues that with the availability of more of Vygotsky's texts there is "no excuse to continue to use limited or distorted interpretations of the concept" (p.58). Rowlands (2003), too, acknowledges the disparities between interpretations of the ZPD and the contexts in which they are used. He suggests using the terms 'historical Vygotsky' (read directly from his translated texts [1]) and 'our Vygotsky' (the ways in which the ZPD has been assimilated into other theoretical perspectives).

I think it is important to make these distinctions and to make them explicitly. Mathematics education research has sometimes been criticised for a lack of coherence (see Bikner-Ahsbahs & Prediger, 2010 as an example) as it draws theoretical inspiration from psychology, anthropology and sociology as well as drawing on different views of how people learn and develop. Thus, the theoretical assumptions we make about the ZPD for our own research/work may be rooted in a 'historical Vygotsky' perspective or not; they may or may not align with Vygotsky's intended idea of the ZPD and indeed may well be considered as neo-Vygotskian interpretations.

In moving forward, if we can agree that there is a differentiation between historical and neo-Vygotskian interpretations, we find that there are numerous conceptualisations that deviate and place differing emphasis on the more knowledgeable other which, as Abtahi's article emphasises, can be problematic, as it is difficult to identify who or what is the more knowledgeable other from an analytic perspective. To illustrate this, specifically regarding the more knowledgeable other, I'll share a few examples that I came across in the literature. I draw these from a sample of *non-Vygotskian* research work (primarily in mathematics education) where the notion of the ZPD is used.

Most of us are familiar with Wood, Bruner and Ross's (1976) metaphor of scaffolding where someone more knowledgeable (usually an adult) scaffolds the learning process for the student from assisted through to independent performance, which is seen as operationalising the ZPD. This assistance enables the student to build on his or her own existing knowledge and internalise new information. This view, which Roth and Radford (2010) argue can be attributed to a simplified reading of Vygotsky's original definition, foregrounds "opposition of individuals" (p. 299) and tends to result in unidirectional or unequal status learning as one person is seen as more capable than the other. Van Oers (2011) concurs, describing it as a "discrepancy-formula" (p. 86), and contending that this way of looking at the ZPD can misconstrue it as a place of transmissive teaching and learning, in that it defines the discrepancy between what a child can do alone and what s/he can do under adult guidance.

This brief examination reveals how problematic it is to pin down just this one aspect of the ZPD, concerning with whom does learning / development in the ZPD take place [2].

Back in 1995, Smagorinsky argued that the process of gaining an understanding of Vygotsky is itself mediated by different points of view in "conversation, print, and cyberspace" (p. 193) which can influence one's own approach to theory building and conceptualisation of various notions. It is precisely this idea of mediated viewpoints that presents challenges for researchers and educators, particularly regarding the ZPD. The ZPD and Vygotsky's work have been mediated in many different ways and by many different people over the decades. However, I note that there is a trend in more recent educational literature that stresses the importance of viewing the ZPD in the broader context of Vygotsky's developmental theory (see Van Oers, 2011; Meira and Lerman, 2001 and Levykh, 2008 as examples), as a way to re-framing the way the ZPD is conceptualised.

Despite Chaiklin's (2003) proposal that we call the non-Vygotskian interpretations of the ZPD something else, this does not seem to have been taken up in any significant way. The ZPD continues to be a complex theoretical space. I see it as important to continue engagement with the types of questions Abtahi has raised and I have discussed here in order to constantly gain a finer-grained understanding of ideas that are often misunderstood and re-conceptualised in educational research, as well as acknowledging that our work may well rest on some of the Vygotskian ideas, but with certain points of departure.

Notes

[1] Although translation of his texts are also problematic. See Stott (2016) for an overview.

[2] In Stott (2016), I have created a framework to help understand these differing viewpoints with regards to key aspects of the ZPD.

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Vygotsky: A learning theory or a theory of development?

YASMINE ABTAHI

Vygotsky assumed that some elementary processes in a child, such as attention or basic memory, are biological in origin. Then, over time, by the mediation of signs, tools and other forms of meditational means, the child develops some other functions such as will, interpretation and understanding of different concepts, *e.g.,* mathematics or art, as s/he grows into the settings around her/him. But, how did Vygotsky envision children's development and growth into social worlds around them—in the world of mathematics, for example?

Vygotsky in *Mind in Society* (1978) referred to Koffka's view of learning and development. In that reference he explained that development could be looked at as a formation that makes it possible to transfer the general principles learnt in solving one task to a variety of other tasks. That is, development means the child's creation of *structures* independent of the materials with which s/he is working and regardless of the particular elements involved. This means, for example, a child 'has developed' an understanding for the concept of addition of fractions if she has created structures that allow her to add two fractions independently of the materials she used when she learned the concept. Now, how does a child develop such structures?

Vygotsky (1978) says: "good learning" (p. 89) advances development. To him, learning and development are "interrelated from the child's first day in life" (p. 84). Learning is not development, he says, but "properly organized learning results in development" (p. 68). It was in elaborating the dimension of learning in a child's development that he introduced the ZPD—a concept he said was new and exceptionally important, without which the complex interrelationship between learning and development could not be understood. He described ZPD as "the distance between the actual developmental level (independent problem solving) and the level of potential development (problem solving under adult guidance or in collaboration with more capable peers)" (p. 86).

Let's turn to an example. Tom Kieren has done multiple studies over two decades showing how using mathematical tools such as fraction kits [1] can help students to learn fractions. In one longitudinal study, Pirie and Kieren (1994) described a case of a third grader (Teresa) who learnt to add two fractions using the fraction kit. Pirie and Kieren noted that Teresa began the task of adding two fractions not knowing what to do. She said "I don't know" and "I think you just add the tops and the bottom" (p. 167). This suggests that she had not yet developed a general structure related to the concept of adding two fractions. She was then given the fraction kit and asked to work on and think about a series of tasks. Pirie and Kieren note that "Using her kit she noticed that one fourth, three eights, and two sixteenths together exactly cover three fourths" (p. 167). Later, Teresa could 'add' 1/3 + 1/6 + 6/12 using the kit. Teresa learnt how to add three fractions with the guidance provided by the fraction kit (physical objects, created by a more knowledgable other with a specific aim in mind). After a while, she learnt more; she was able to say, "You can do 2/3 + 5/6 because twelfths fit on both" (p. 168). She could explain making reference to a common denominator, but in the specific context of the fraction kit. Over time, she was able to transfer the general principles that she learnt in adding fractions with her fraction kit to other tasks of adding fractions without the use of the kit. When asked "What is 1/2 + 3/4 + 2/5 + 7/10?" Teresa, without using the kit (which does not include fifths and tenths), said:

> Twentieths will fit on all of them. Two times ten makes twenty, so one times ten or ten twentieths. Four times five makes twenty so three times five is fifteen twentieths (p. 168).

Finally, Pirie & Kieren note that Teresa went beyond the concrete concept of addition, making statements like, "Addition is easy. You can make up the right kind of fractions just by multiplying the denominators and then just get the right numerators by multiplying by the right amounts" (p. 168).

My thought

I used Teresa's example to show her development and growing up in the mathematical world of the addition of fractions—from her starting point when she did not know how to add them, through her use of the fraction kit to add fractions, to her being able to perform the task without the kit. Teresa's development was a result of an organised series of learnings. Teresa learnt how the sizes of the pieces in the fractions kit were related to one another. She learnt that different pieces could be put together to create a new piece (a new fraction). She learnt that 1/12 fits on both 2/3 and 5/6. From a Vygotskian perspective, Teresa's case can be viewed as one example of the interrelationship between learning and development.

I find the ZPD a useful tool to examine children's learning, and especially to examine the assumption that learning happens in the presence of others, more knowledgeable or not (Abtahi, 2017). In closing, I would like to suggest that to go from the level of potential development (problem solving with others) to that of actual development (independent problem solving), a series of properly organised learnings are needed, occurring over time and space. And the Zone of Proximal Development is only a way of highlighting such a zone, within which learning happens with others.

Note

[1] A fraction kit contains rectangles representing halves, thirds, fourths, sixths, eighths, twelfths, and twenty-fourths based on a standard sheet of paper as a unit.

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Continuing the discussion on the ZPD

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The recent article by Yasmine Abtahi in **37**(1) and Debbie Stott's reply to it in this issue are welcome contributions to the discussion of how the notion of the ZPD can be understood and worked with, more than 85 years on from its first appearance near the end of Vygotsky's life. Yasmine introduces an analysis of the role of tools in the ZPD, as carriers of the knowledge of the tool's creator, and therefore of culture too; Debbie highlights the challenge of the notion of development as well as learning within the ZPD. The former is much harder to observe of course, not least of all because it takes place over a longer time period than most studies of learning that are usually brief. The work I did with Luciano Meira, published nearly a decade later as Meira and Lerman (2009), was based on very short instances of communications between a very young child and her nursery teacher, though we endeavoured to discuss there both learning and development in the ZPD.

There is much more that can be said about these two pieces by Yasmine and Debbie and about so much else that has been written in studies of learning and development in the ZPD, both in the field of mathematics education and beyond.

I want to pick up here on what is meant by 'neo-Vygotskian' theorising and research and our struggles with it, that struggle being in part because we want to remain faithful to Vygotsky's own work and ideas as well as make them relevant for today's context. In itself, that is important to recognise. Vygotsky's whole conception is social-cultural-historical and it would therefore make no sense to try to ignore the passage of time when working with ideas from the early 1930s. As Luciano and I wrote:

> It is also the case that Vygotsky's *methodology* (in the sense suggested by Valsiner, 2000) emphasizes the need to relate knowledge and concepts, and indeed consciousness, to time and place, to their historical, cultural and social settings. For both of these reasons we regard it as inevitable and proper that researchers working with the concept of ZPD appropriate and transform it, a process which of necessity engages with one's existing theoretical perspectives, and hence demands some work on the notion of the ZPD. (Meira & Lerman, 2009, p. 200)

It is equally important to recognise that Vygotsky was also neo-Vygotskian! Roth's recent book (Roth, 2017; see also Radford & Roth, 2017) has opened our eyes to how Vygotsky was reviewing and revising his whole theoretical approach as he lay dying. Had tuberculosis not taken him at such an early age, we might have been reading whole new ways of thinking about child development and teaching-learning, set deeply within ethical and monist (versus dualist) conceptions of human relations and human development. Of course, that assumes Stalin would have allowed Vygotsky to live and work, had he lived past 1934.

Radford and Roth (2017) discuss Marx's concept of alienation, in response to William's (2016) accusation that they have not understood the concept. I do not want to get into their debate, but to see how they (Radford and Roth) work with development as well as learning in the ZPD. First, they point out:

> Indeed, Vygotsky (1989) articulates development in terms of the individualization of the social. Ontogenetically, everything that the child does not know and that is not already contained as a possibility in current knowing is alien. (p. 377)

Framing development as "individualization of the social" takes it beyond learning, as Debbie indicates, and gives it methodological substance in terms of visibility for the researcher. Luria and Vygotsky's experiment showing how grouping concepts develop with education is an example.

In the same paragraph Radford and Roth add:

> Learning (development) is the equivalent of the process in which the alien (culture) is transformed and recognized, opening up a new space of action, reflection, and transformation (Radford, 2012, 2016). That is, as learning and developing individuals, we find ourselves in the paradoxical situation: "it is *qua alienus*—foreigner and other—that man is not alienated (Levinas, 1978, p. 99). (p. 377)

The first point they make in this latter quote indicates that 'internalisation' or 'appropriation', terms we use for the transfer from the social plane, the intersubjective, to the individual plane, the intrasubjective, refer to an active process of transformation, not some kind of wholesale transfer from one to the other. That 'individualisation' process is not to be seen, however, as a mind acting separately from the body and from that social knowing; each person is part of the world being individualised. Vygotsky was a Spinozist, as Roth (2017) shows from those final thoughts and writings just before Vygotsky's death, and Radford and Roth extend that position into research studies in classrooms.

The point being made by them when referring to Levinas is that when we face each other, whatever the circumstances of our separate past histories, we are saying "I see you and you see me. We are humans in the same world encountering each other right here and now". This attitude, though developed in his work from the 1930s, was deepened and enriched post-holocaust, post the Second World War. An example Levinas gives, one that fascinated him, is from his reading of Vasily Grossman's book *Life and Fate,* about the siege of Stalingrad. In a particular scene, a Russian woman who has starved and suffered a great deal sees a German officer in a line with others. She is about to throw a brick at him when, instead, she gives him some of her bread. This is the face-to-face encounter, Buber's 'I-thou', contrasted with "I-It".

Levinas' approach is an interpretive phenomenology, developing from Husserl but in direct contrast with Heidegger. In their study of mathematics classrooms Roth and Radford (2011) demonstrate a methodology that develops from Levinas's phenomenology and attempts to represent the lived experience of each child as they work in their ZPD towards individualisation. I attempted a similar study (Lerman, 2001), though without the theoretical background. I tried to show how the communication between two students in their mathematics classroom, whilst working as a pair to solve mathematics tasks, could be seen as a product of the 'social' of their classroom. One student was set up by the teacher as the more knowledgeable other, though my analysis suggests the situation was not as simple as it seemed to the teacher. My analysis was neo-Vygotskian, in the sense that I took into account factors in the communication between the two, whilst assessing the nature of the ZPD created in the activity, that would not have been taken into account in Vygotsky's times. Another feature of the interpretive phenomenology, picked up in Meira and Lerman (2009), is our proposal that the ZPD created between the nursery teacher and the young child came about because of the loving relationship that had developed between them over time. The child's development that occurred in the incident we analyse would likely have been missed in many other circumstances in classrooms.

In summary then, those of us engaged in working on the ZPD are properly making that work relevant to time and place, to historical, cultural and social settings. In this we are doing the same as Vygotsky himself. Perhaps what should run through all our developments of work in the ZPD is a taking into account of the philosophical and ethical principles that drove Vygotsky, even though their realisation will be different across time and place, and work to use them to inform our methodologies. We should recognise, however, as shown in the debate between Williams and Radford and Roth, that interpreting those philosophical and ethical principles is no straightforward task.

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