# Communications

## Holding together

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In response to my reading of some of the articles in FLM 32(3), I want to write about the theme of *holding together* what is seemingly disparate or even conflicting. It is so rare that an idea, emotional state, or interaction is just one thing. Upon examination, it is usually two, or three, or seven different things held together: I want to share six identical items among seven people so I am dividing, but I am also taking a fraction (one-seventh) of the six items, and I am also finding an equivalent ratio of the amount per one person given the amount per seven. Or, a little boy is so happy to see his father that he starts to throw everything around him, happiness and aggressive energy united in his small body. Or, I work with prospective secondary teachers on meanings of division because I want them to see how challenging these ideas are when pushed beyond whole numbers, because I want them to be able to recognize students' mathematical ideas, because I want them to know what they could aim for in helping students develop their mathematical ideas, because I want to set a foundation for work on ratios and rates. These are just quick examples, and not necessarily the best ones (I am heeding the editor's encouragement in 31(1) to present unpolished thoughts and responses). It is a somewhat scary thing to do that, so I hold together my trepidation and my desire to write.

In her application of the idea of mathematical caring relations to interactions between professional developers and classroom teachers, McCloskey (2012) demonstrates very well the holding together of her two roles in relation to the classroom teacher, Mrs. Garcia: professional development provider and witness-researcher in a teaching experiment. McCloskey experienced these two foci as competing at times, as she strove to support the teacher and also respect the goals of the research—a very natural experience. But, I wonder: is it possible for the holding together of two foci to feel more connected, less in competition?

I find I can do this with respect to what is often referred to as a divide between "the mathematics" and "the person." For me, "the mathematics" refers to my own mathematical ways of knowing (and perhaps also my mathematical ideals)—it can never refer to a view from nowhere. "The person" (student or teacher, the cared for) refers to my model(s) of that person's ways of knowing and being. My own mathematical knowing (a first order model, Steffe, von Glasersfeld, Richards, & Cobb, 1983) influences my developing model of the person with respect to their mathematical thinking and mathematical-pedagogical thinking (first and possibly second order knowledge, Steffe *et al.,* 1983). Likewise, my developing model of the person's thinking enriches and evolves my own mathematical knowing. So rather than see them as competing for attention or priority, I can start to experience them as intertwined, necessarily and deeply. Could something similar happen for the admittedly greater complexity of the roles and levels of models in a professional development setting, like the one McCloskey describes?

D'Ambrosio and Kastberg (2012) struck me as offering one response to McCloskey's recommendation that "mathematics teacher educators should make an effort to use research constructs about mathematics teacher thinking so that we can build better-informed models. In this way we can be better prepared to care for the mathematics teachers with whom we work" (p. 32). Although D'Ambrosio and Kastberg are focused on giving reason to prospective teachers, and not explicitly with mathematical caring relations interpreted anew, as is McCloskey, their article helps me think more deeply about what it means to develop "mathematical-pedagogical caring relations" with prospective teachers. To admit the legitimacy of views that differ from one's own—views that might even seem "wrong" in relation to expert knowledge in the field—and to honor those views with collaborative investigation seems to be at the heart of decentering, which is central to giving care.

D'Ambrosio and Kastberg's article also made me wonder about the trajectory of learning to give reason to those who are our students (or to develop mathematical caring relations, or mathematical-pedagogical caring relations). The authors say that they repeatedly failed to give reason to prospective teachers like Sheila and Dan. I do not know if I want to label as failure the process of starting on a path (to teach prospective teachers) and then making revisions (to rethink what it means to give reason to these teachers). We may not want to call our initial attempts at giving reason (or care) successful, but it is also true that we cannot know what it means to give reason (or care) in a situation prior to working in that situation for some time. So, D'Ambrosio and Kastberg made me want to think about making "room" for growth among mathematics teacher educators as they learn how to give reason to (care for) others, to hold care for prospective teachers together with care for self as a mathematics teacher educator.

And finally I get to Ernest (2012). The idea of holding together is one of relation, of being in relation. And if I see that as fundamental, then I would see Ernest's conclusion that ethics is the first philosophy of mathematics education as a perfectly reasonable one. However, I still must ask: why does there have to be a first philosophy? Why can't there be a holding together of epistemology and ethics, for example? Or a holding together of epistemology, ethics, and critical theory? This feels good to me as a mathematics educator, that I hold together multiple roots, that I do not try to choose between ethics and epistemology, but instead hold both in a close embrace (like holding a large, heavily scented, possibly unruly bouquet of wildflowers; or holding a small son who is both coiling to spring away from my arms while also pressing into them). It does not mean I get to ignore the bumps or bristles of holding together, or that holding together negates criticism. It does not mean that it is always possible to hold together two perspectives without contradiction, and it does not absolve me of aiming for


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coherence. But for me, the holding together of philosophies changes the conversation from what is first and last (again, a competition metaphor) to what it means to go deeply into the myriad of work that is mathematics educating.

With respect to this last point, Ernest and I hold quite different views on the power and efficacy of radical constructivism. Where Ernest sees "cognitive aliens" (p. 14), I see people *inter*actively building up perceptions and conceptions. I follow Steffe in viewing a scheme as an instrument of interaction (Steffe & Olive, 2010), where interaction includes both social interaction between people and the interaction of constructs within a person (Steffe, 1996). Where Ernest sees a theory that "denies the social and ethical foundation of human being" (p. 14), I see a theory that helps me and others to honor and investigate both of these foundations (*e.g.*, Confrey & Kazak, 2006; Hackenberg, 2005, 2010a, 2010b; Thompson, 2000).

Something that is liberating for me about radical constructivism, and that gives me freedom in investigating social interaction, is the very premise that another person will not necessarily think, feel, or perceive similarly to the way I do these things, and that even if they do seem to do these things similarly to me, I cannot know for sure: there is always some doubt because my knowing of another is based on mutual, negotiated interpretations with that person (Thompson, 2000; von Glasersfeld, 1995). I embrace that premise because it means that the way I or anyone else thinks, feels, or perceives is a view well worth "getting to know," but it is not the first or best view. For me this premise provides space and motivation for investigation, for curiosity. And, strangely enough, it provides, for me, a kind of peace. From there, I set forth on holding together my own knowing and my knowing of others, and I experience the joy that can come from holding together what can seem disparate or conflicting.

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# Offering and differing


### JEAN-FRANÇOIS MAHEUX


The last issue of FLM features an article by Wolff-Michael Roth (2012), in which he offers to develop a "phenomenological theory of learning" (mathematics). Mingling experience-based images and activity centered concepts, the article shows us how "affect" (in the deep sense of a capability to be influenced bodily, emotionally, cognitively, *etc.*) allows students to realize the underlying learning objective of a task without (the possibility of) aiming for it. At the core of this proposition, there is a difficult interweaving of phenomenology and cultural historical activity theory that presents a complication (from Latin *complicare* "fold together") I find useful to clarify to begin with.

Phenomenology, in Husserl's descent, takes intentionality as a first principle: in a bold move against metaphysics, phenomena are considered as objects of consciousness, things toward which we orient ourselves, observe, and reflect upon. This is not, however, the phenomenological perspective adopted here. Jean-Luc Marion reformulates Husserl's phenomenology situating *donation* as the core and truly immanent condition for consciousness and its objects to emerge. Marion critiques Husserl's intentionality because it hinges on the representations of experienced phenomena, which makes it impossible for consciousness to encounter anything radically new. The novelty of algebraic thinking, as opposed to arithmetical considerations, could be an example. Marion addresses this paradox (how can I intend to learn something I do not know?), situating donation and the capability to be affected as the conditions for consciousness, and its corollary of objects and intentions, to appear. To put it simply: learning mathematics is possible because students are affected by concepts or ways of doing that the teacher puts them in contact with, but which they only understand after they have experienced and become familiar with them.

Cultural historical activity theory (CHAT), on the other hand, gives societal motives an indisputable role in learning, because they create the socio-material conditions for a learner's action to take place. These actions are goal-oriented and contribute to a learner's development (*e.g.,* in mathematics) when they become functionally associated with a task's resolution. Can learning takes place if I solve a mechanical puzzle (such as a Rubik's cube) or an algebraic equation (*e.g.,* with a calculator) without knowing how my actions enabled me to do so? It is in the coupling of goals and motives—orientations in the learner's actions and in the socio-material setting—when the former finally meets the latter, that learning can be recognized.

The difficulty here lies in the fact that CHAT's approach to learning is an "external psychology," while phenomenology is interested in learning as an intimate, innermost happening. One conceptualizes development from the outside, attending only to what people's actions disclose from a third-person perspective, what they make available to one another or use as such. Mathematics is mathematical action, a historically developed observable practice, *recognizably* taking place within teachers' and students' transactions. The other is interested in the movement by which a possibility emerges for someone that is later recognized as, for example, a mathematics concept. It considers the object of learning and the way in which it is experienced from the first person's, the learner's perspective. Articulating CHAT and phenomenology in a coherent matter is thus a gruelling enterprise. Roth's article is one stepping-stone on this journey, coming and going between two theoretical trends to move forward, their opposing forces creating torque to push ahead these ideas, in an effort to (re)conceptualize how learning mathematics is both and simultaneously "an experience", and a societal practice.

## Provisions

For me, one possibility for this theory to be discussed, revised, augmented or even rejected within our field rests in the interrogation of its project to consider a "learning object" from the "actors' perspective", and to do so on the basis of "what the members to the setting make available to each other" (Roth, 2012, p. 15). My question is: can we really approach students' experiences of the mathematics they are wished to learn, and do so on based of what teachers and students observably do with/for one another?

The challenge here comes from the effort to bridge, combine, maybe even fuse first with third person perspectives: the "subjective" stance of a phenomenological investigation in which assumptions about the existence of a world external to the subject's experience are suspended, and the "objective" standpoint that embeds the subject in its socio-material environment. Can we consider mathematical objects to be the result of a learner's ongoing experience, *and all along* as the product of a societal endeavour realized in this learner's socio-material setting (including the school, the teacher, the task and the curriculum, the manipulatives, and so on)? How can or will such *coincidence* at the same time speak of the variety and the endurance of students' experiences and classroom lessons?

A "learning object" is something that can be thought both from the outside (through what the teacher sets up and what the students observably make of it), and from the inside (something I, as a learner, experience, and against which consciousness rises up). But under what conditions can we consider the *unity* of this object? Letting go of this quandary is difficult because with a focus on "consciousness", learning somehow has to be considered in terms of objects and intentions (consciousness is always consciousness of something). This appears to be even more inevitable in relation to a *teaching* situation in which some new thing is brought up for the students to learn. So much so that we might actually be better talking about a *phenomenological theory of being thought* (in the most noble sense of the term, not causal as in "making someone learn something," but as in showing, presenting, pointing out, saying "things" from which someone could learn).

Thinking in terms of learning objects, however, worries me. Objects are difficult to let go, difficult to consider as pretexts, or as human relations. Not that it cannot be done, but because this *language* somehow still involves (in today's socio-political and philosophical context) the possibility of (re)engaging mathematics education in an "object directed" perspective. Transitory, fugitive, after-the-fact objects are still objects, pieces standing against the fluent continuity of a background (*e.g., doing* mathematics). I worry about the prospect of breaking down mathematics into fragments to be experienced instead of focusing on *being mathematical,* on enriching one's aptitudes to play with/in mathematics, on enjoying mathematics in an artistic/aesthetic manner where pleasure, creativity, enchantment are envisioned. I worry about what Jim Neyland called the "scientific management" of mathematics education, always ready to sneak in, when I would like it to evolve from the idea of mathematics as a becoming. Not that it cannot be done within an activity grounded phenomenological approach…but I don't find its language particularly inciting.

## Offering and differing

To illustrate how such nuances could develop, consider the *gift* metaphor framing Roth's article, which invites us to consider teaching and learning in terms of donation. The image, powerful and stimulating, not only accounts for "everything that came before us", but is also welcoming to the challenge of "being here and now". I find it difficult to think in terms of donation, however, when it comes to teaching and learning; I am much more comfortable with the idea of *offering*. Francesco Varela, in his *Fragments for a Phenomenology of Organ Transplantation* (at www.oikos.org/ varelafragments.htm), recalls discussions of the concept of donation. A gift is something personal that implies reciprocity, obligations, and the act of receiving, all this being possible only in contrast with what is *not* given: the contraposition of the gift. In contrast, Varela suggests referring to such situations as *offering* because of its detachment from deontological-like responsibility, and its focus on safe keeping that does not objectify the offer as some "thing" to be preserved. Varela explains this by referring to Jacques Derrida and Jean-Luc Nancy. For these authors, at the heart of an offering is the disappearance of what could be a gift, which reminds me of Heidegger's distinction between the open nature of aesthetic "truth" (which he also calls an offering), as opposed to that of science: the cultivation of an already opened domain.

These ideas imply a significant difference from the perspective developed in Roth's article. What if teachers are not donors, but offerers? What if the offering is not that of objects to be comprehended, mastered, understood, and preserved, but consists in opening possibilities, hopelessly wishing they will be reached for and sustained? No need here for an object, and to wonder how it might be the same or different according to how we look at it, because the offer is that of a *relation* where, in a very concrete/practical way, *differing* is key. Using Varela's words, alterity and intimacy are no longer opposite (*in facto*), but recursively interpenetrate one another. It might be enough, then, to see (in Roth's example) that Mario says something, that Jeannie speaks in return, and that playing those differences they manage to bring about a mathematical understanding of the goblets and the piggybank story. After all, teachers and students doing mathematics *can* "engage with the task environment [and act] without knowing why" (Roth, 2012, p. 20).

In contrast with the proposed analysis, the data presented in the article speak to me about the problem of wanting so much to get the student somewhere (mathematically), rather than wanting to be with them mathematically. Mario and the others were presented with a problem aiming to lead them "step by step" to what seems to be a pretty specific understanding of a piggy bank story. They had little time, precise materials, clear hierarchized questions, a tightly organized classroom mode of functioning, and their "job" was essentially to get it. I am well aware that this kind of situation reflects the vast majority of mathematics education settings, and that all this has little to do with Roth's theoretical endeavour. But this is exactly where and why I am looking for a different route. In my view, theory *always* speaks of all the dimensions of people's experiences, although mostly by being silent about them. Here, a mathematical concept was offered, but the way of being mathematical accompanying this offering had little, it seems to me, that was ecstatic. We can still appreciate Mario and Aurélie and Jeannie as being mathematical together: when one points to the worksheet stating "I don't understand," there is an opening, an opportunity for mathematical activity to show. When the teacher steps in, orients the students to the question (turn 43) or breaks down the task up to a point where Mario actually can get going, she takes on the offering/opportunity to do something mathematical with him. However, the task's requirements provided them with little chance to heartily engage in doing mathematics in a rich, conceptual, game-changing way. The episode then becomes "evidence" for the importance of an environment mathematically offering as much as possible, to be serendipitously navigated by participants.

Perhaps some will not accept to follow me this far, because what I am saying is equivalent to refusing the concepts of teaching and learning mathematics as being worth considering. The *repetition* of mathematical activity already overcomes my concern for mathematic education in at least two ways. First because, as Derrida explains, in repetition lays the possibility for novelty, for reiterating also is altering, is differing, is invention. Second because I have other concerns, starting with a need to rethink what we do with/in mathematics education so that students and teacher *can* (no longer) undergo frustration, exasperation or despair. This is a question of *ethics*, also raised by Paul Ernest and Andrea McCloskey in the same issue. It is a topic Roth has also investigated, and to which he alludes at the end of his article, quoting Levinas:

> It means thinking about and theorizing learning in terms of exposure, vulnerability, which is an exposure to outrage, to wounding, passivity more passive than all patience, passivity of the accusative form, trauma of accusation suffered by a hostage to the point of persecution. (Levinas, 1990, p. 31)

As with most evocations of ethics in mathematics education, however, this elicitation is a bit disheartening, which might have something to do with the Judeo-Christian tradition toward suffering (from which there is no escape before death). My reaction is simply to wonder: why not take mathematical activity from the other side? Why not (even hopelessly) work as if suffering can be ended, and think in terms of mathematics as delight, as healing, the way we do with music for example? Why not find in passivity the origin of love, compassion, comfort, appeasement, relief? Why not shift from persecution to protection, as in offering oneself "in front" (*pro-*) of another "to cover" (*tegere*) her despite one's own vulnerability? And envision mathematics as an open field welcoming to differing, where the science of mathematics is never but *one* side of mathematical activity, and art the (an?) other.

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# From the archives

Editor's note: *The following remarks are extracted from a communication by David Wheeler (1982), published in* FLM3(1). *In it, Wheeler traces some of the evolution of his thinking about the idea of mathematisation (sometimes with an 's', sometimes with a 'z'). Despite (or, perhaps, because of) being published 30 years ago, his ideas retain their relevance; I leave it readers to find connections between them and some of the writing in the current issue.*

In some notes privately circulated in 1978 I drew attention to the difficulty of identifying acts of mathematization, but also to the significance of trying to do so.

> We can't observe mathematization—not even, I think, in ourselves when engaged in it. We mathematize without knowing it, only knowing the results, that we have "done the right thing", have acquired a skill or found a path or taken a fresh viewpoint. I am reminded (although mathematization is a more specialized operation) of the way in which we are able to find and utter all the words we want for a normal, easy, conversation without any need for awareness of the mechanism at all. The words come, string themselves in sequence, get uttered, and disappear again, without any awareness on our part of what we had to do at that moment, or what we had to learn sometime in the past to allow us to do it. If we are in a situation where we know that what we say may be very important for some reason, the fluency may desert us, and we find ourselves inhibited, "searching" for the right words, "inspecting" what we say just before we say it. We may become aware of the difficulty of what we normally take for granted, but we don't necessarily gain any new insight into the operation itself. In fact we may get frustrated and exasperated because we want to have more control over it and don't know how to exercise it…

…Putting it another way, all of us confronted with the outer manifestations of mathematical activity have met the difficulty of "getting inside it" (or getting it inside us) and have attempted to "read between the lines" or discover "what's really going on here". The formal face of mathematics generally hides, rather than reveals, the inner life—at least, until one has enough experience to be able to read its expression. A definition, for example, often covers up the real source of the awareness that "this will be worth pursuing", and a proof can mask the source of the conviction that a result is actually valid. In looking at mathematization we are, it seems to me, trying to get as close as we can to the phenomenology of the awarenesses and convictions that we experience when we are doing mathematics and which power the movements of our mathematical thought. We can try to raise the awarenesses and convictions into consciousness—become aware of our awarenesses if you like—and then we may be able to find a way of talking about them that will make sense of these experiences…

[…]

Perhaps of particular interest are the attempts I made to breakdown the concept of mathematization further, to try and articulate some of its constituents. The first attempt comes from a talk published in Mathematics Teaching in 1975.

In a crude attempt to make explicit the nature of mathematisation, I would include the following ingredients: the ability to perceive relationships, to idealise them into purely mental material, and to operate on them mentally to produce new relationships. It is the capacity to internalise, or to virtualise, actions or perceptions so as to ask oneself the question, "What would happen if…?"; the ability to make transformations—from actions to perceptions to images, from images to concepts, as well as within each category—to alter frames of reference, to refocus on neglected attributes of a situation, to recast problems; the capacity to coordinate and contrast the real and the ideal and to synthesise the systems of perception, imagery, language and symbolism. When these functionings are applied to pure relationships, detached from specific exemplars, the products will then be mathematics.

A second attempt, worked on at a meeting of the Canadian Mathematics Education Study Group in 1977, came out quite differently.

Although mathematization must be presumed present in all cases of "doing" mathematics or "thinking" mathematically, it can be detected most easily in situations where something not obviously mathematical is being converted into something which obviously is. We may think of a young child playing with blocks and using them to express awareness of symmetry, of an older child experimenting with a geoboard and becoming interested in the relationships between the areas of the triangles he can make, an adult noticing a building under construction and asking himself questions about the design, *etc*. We notice that mathematization has taken place by the signs of organization, of form, of additional structure, given to a situation.

I use these tenuous clues to suggest that mathematization is the act of *putting a structure onto a structure.*

Consider the experience of solving a problem, or mastering a new game. In each case there are moments when the whole situation or a part of it is suddenly *seen differently,* the perceptual difference marks a new stage in the mental structuration of the situation.

By the 1980 ICME-IV meeting there seems to be some lessening of confidence in "structure on a structure" and the addition of two new ideas, neither very convincingly worked out.

In order to help awareness of the activity of mathematization come to the surface, I propose the following clues to its presence:

*Structuration*
"Searching for pattern" and "modelling a situation" are phrases which grope towards this aspect. But our perceptions and thoughts are already structured; reality never comes to us "raw". So mathematization is better seen as "putting structure onto structure". Existentially, however, it seems more like discovery or restructuration since what we have brought into being seems new to us. The "eureka" feeling is an extreme case, marking the release of energy brought about by a new structuration.

*Dependence*
Mathematization puts ideas into relation and coordinates them; in particular it seeks to establish the dependence of ideas on each other.

*Infinity*
Poincaré points out that all mathematical notions are implicitly or explicitly concerned with infinity. The search for generalizability, for universality, for what is true "in all cases", is part of this thrust.

The latest triad received its inauguration at Sydney early this year.

(1) *Making distinctions.* This seems to be the fundamental mental action underlying the construction of mathematical sets and mathematical relations.

(2) *Extrapolating and iterating.* These are the main mental actions for producing new things out of old ones.

(3) *Generating equivalence through transformation.* This is the most powerful action of all since it generates stability (equivalence) out of flux (transformation).

[…]

I have quoted here only from myself. But I don't stake any personal claim to the ideas. Maybe recirculating them in the air will freshen them, and it, a little.

## References

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