A Study of Children's Strategies for the Comparison of Numerals

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This paper describes a research [Bessot, Comiti, 1986] which was presented to the 9th Conference of the Psychology of Mathematics Education Group during the sessions of the PRINCIPLES FOR THE DESIGN OF TEACHING Working Group.

The main themes of discussion during these working sessions were:

— the design of lessons and short lesson sequences, types of situation, modes of pupil activity, teachers' interventions, manipulating didactical variables;

— designing for long term learning: choice of contexts, structures and tasks.

As A. BELL wrote when he introduced this working group, "Some prominent questions are the relations between realistic and more artificial concrete contexts: the way in which the teaching focuses on specific misconceptions; the way in which the didactical variables such as number size, question structure, mode of communication, are manipulated by the teacher to promote the desired learning, and how the pupil becomes aware of the correctness or otherwise of his responses" [Bell, 1985]

This contribution describes an example of French research in the didactics of mathematics — see Vergnaud [1982] for a good review of current French theory and methodology in the didactics of mathematics — related to child strategies for comparing numerals. The research presented here concerns a classroom teaching situation which was designed with two aims in mind, namely, to bring out the number concept as it exists in the minds of "Cours Elémentaire l°année" pupils (C E.1) — which is equivalent to Grade 2 in the U.S.A. — and to modify these representations i such a way as to improve their grasp of numeration. The focal points of the study are the conditions under which children's notions of number appear and change, i.e. the "situation didactique" which enables the ideas of number and numeration to appear and to be modified in accordance with the learning goals in question

Aims and theoretical framework

The teaching situation that we studied involved pupils in the C.E.1. The three main aims of this research were as follows:

*****_to make use of an epistemological approach to the study of numeration._ The notion of number is the end product of a long process. During this process adaptation to a variety of contexts, cultural factors and personal involvement all play important parts. [El Bouzzaoui, 1982]. The study of numeration in French primary schools was not distinguished from that of numbers until 1970's. First 1 was learned, then 2, then 3, ..., 9, then 10, ..., and so on. It was only in the 1970s that it became apparent that numeration could itself be an object of study. However, during this period it was considered in isolation: the study of different bases, the study of base ten, the designation of units, tens, hundreds. The understanding of numeration was not taken beyond this. It was not used to give meaning to other notions and, by and large, was only used in solving problems relative to numeration. Such an approach does not help the learner to learn either the concept of numbers or of numeration. Numeration must be distinguished from numbers Clearly, numeration plays a role as a way of designating numbers both in number construction and in counting, but it also plays a role as a way of comparing and ranking numbers. Numeration-systems are codes which allow numbers to be designated or described and certain number properties to be represented in writing.

**to create a teaching situation that would oblige pupils to use numeration rules as a tool for solving a problem.* The meaning of a mathematical notion is given by the types of problems for which it constitutes the most efficient - i.e. the most reliable and the "cheapest" tool for solution.

**to favor feedback, the control of the situation by the learner himself, and the elimination of dependence on outside help.* A pupil's behaviour and the type of controls he may have over the solution he produces strongly depend on the possible feedback loops in the situation. When the learner is confronted with a situation in which the falsity of a solution implies certain consequences, he will search for some degree of proof and even sometimes reconsider his own knowledge before producing a definitive answer.

Brousseau's "Théorie des situations "[Brousseau, 1980], was used as a basis for the organization of the desired learning process The cornerstone of this theory is the analysis of the various functions which knowledge can fulfil. According to Brousseau, these functions are as follows:

1) to enable decisions to be taken in a context where immediate effects are possible — what one might

refer to as *real time decision making*; Brousseau describes contexts in which knowledge plays this role as "situations d'action";
2) to enable information to be communicated in what Brousseau describes as "situations de formulation";
3) to provide a means of proving or refuting hypotheses; the corresponding contexts are known as "situations de validation";
4) to act as a cultural baseline for a group or society: this involves conventional definitions of the status of knowledge in "situations d'institutionnalisation".

Description of the activity
The teacher chooses two different numbers n_1 and n_2. She writes each number on separate cards. She gives to A the card upon which n_1 is written. She gives to B the card on which n_2 is written. A and B are mutually unaware of the number on the other's card. A and B play against each other.

The game consists in finding out, in the shortest possible time, which of A or B has the larger number. There is no talking during the game. A and B each have a message-sheet and only one pencil. A asks B a question in writing and B poses a written question to A. The only forbidden question is, "What is your number?". The questions are only exchanged when A and B are both satisfied with the questions that they have written. The exchange of answers is also simultaneous. After examining the answers that they have received the players ask, if necessary, further questions, and so on, until one of them claims to know "who has the larger number". He then has to explain in writing to his partner the grounds for his claim. The partner indicates his agreement, or otherwise, with the explanation that he has received and, in the latter case, writes down the reasons for his disagreement. Once the learners have come to an agreement they compare their two message-sheets and confront their reasoning, with the teacher acting as referee. Finally, the unknown number is supplied to each partner. This is done by turning over the card. The game is played N times

Example of a message sheet.

Players' name ________________

Partners' name ________________

QUESTIONS	*ANSWERS*

Specificities of this situation
The experiment took place between the 15th of November and the 15th of December 1983. By this time in the school year C.E.1 teachers had already carried out a review of numeration and had begun the systematic teaching of the comparison of written numbers. The learners, therefore, had already been required to solve problems involving the comparison of two or more numbers which they had produced and written themselves. The game differs from more conventional situations in several ways:

*The game allows numbers to be described or designated; its communicative aspect was deliberately built in, to force learners to communicate written information concerning numbers (*situation d'action*);

*Each learner knows only his own number and is not allowed to know his partner's number. Consequently he has to obtain information about it. In order to do this he has to set up hypotheses that enable him to pose relevant questions in the light of his current representation of the concept of number. A list of questions that allow him to compare the unknown number to his own is thus drawn up. These questions become an object of reflection as the game goes on, the aim being to find the larger number as quickly as possible. The learner is thus obliged to organize and to make explicit number properties implicitly used before (*situation de formulation*);

*The learner is in a position to organize his own time, as the exchange of questions and answers between A and B occurs only when both are satisfied with the questions that they have written down. The game finishes only when one of the two claims that he knows who has the larger number and produces an explanation which satisfies his partner.

*The game makes the learner responsible for the validity of any claim made. He is obliged to give reasons to convince his partner and, in the last resort, the two partners must confront his claims with the actual numbers involved (*situation de validation*).

Theoretical analysis of the situation
This theoretical analysis is concerned essentially with *epistemological considerations concerning the activity*. The arguments involve considerations that could be quantified: for example, cost, reliability, optimal strategy, and so on. In this analysis we attempt to throw light on different phenomena which may occur. In particular we seek to identify, via a study of the situational variables, the major classes of strategy and the conditions which produce them. The validity and the efficiency of these strategies will depend on the values given to the variables. What actually happens in a specific context of the experimentation (for example a C.E.1 class) is then interpreted in the light of the range of potentialities produced by the preliminary analysis. The significance of the observations will be established with reference to what could have occurred but did not. A number of different factors may come into play in the elaboration of winning strategies by the "Cours Elémentaire" pupils: the identification of the properties of numbers; the manipulation of several of these properties and, when necessary, the taking into account of their use in conjunction with each other; also the desire to provide the opposing team or player with as little information derived

from the opponents' questions and answers

These three factors are not of the same kind. The first two are linked to the solution of mathematical problems. They allow potential solution strategies to be determined. The third factor, however, is to do with interactive and psychological aspects of game playing, factors which oblige A to take B's strategy into account .

The properties of natural numbers which can be employed to solve the problem may be, for example:

*Properties of written numeration, number of digits, numbers included in the tens place, the hundreds, the significance of position, etc.

* Properties deriving from the total order over N and particularly from the fact of preceding a certain number or following a certain number.

As for the strategies allowing a C.E.1 learner to successfully make the required comparison, it can be predicted that they will fall into two major classes. Firstly, strategies based on the search for the unknown number n_2 in order to compare it to the known number n_1; strategies of this sort enable the pupil to boil the problem down to the classic situation of the comparison of two given numbers. This strategy class is coded as F (F stands for FIND). Secondly, strategies based on the search for properties which are sufficient to allow the comparison to be deduced without the value of n_2 being known; these are coded ~F.

Constraints and variables in the problem situation

The chosen problem situation involves a certain number of variables. Some of these are held constant; we call these fixed variables: situational constraints. Others are manipulated: these active variables are called situational variables.

What are the constraints? In the game A always has to play *against* B and not with him. This device prevents questioning that would provide too much information to the opposing player about the unknown number. A written communicative situation was adopted in order to obtain a written record of the sequence of questions and answers, a written interaction between the strategies of A and B, since each learner was thus obliged to read and answer his partner's questions, and this opened up the possibility, for each player, of going back over what had already been said and thus gaining new insights and restructuring.

What are the variables?

*The numbers n_1 and n_2. These may (or may not) belong to the same numeric field. Similarly, they may, or may not, have the same number of digits. If they have the same number of digits, they may, or may not, begin with the same digit, or with the same two digits... and so on. The numeric field from which they are chosen may be the field with which young learners are most familiar (that is to say 1 or 2 digits). Alternatively, it may be a field in which they have never made calculations in a class situation (for example, numbers with 4 digits)

*The composition of A and B. There is the option of making one learner play against another learner or, alternatively, one team against another team. This allows certain types of interaction to be avoided or, conversely, developed; for instance, learner-cooperation within one team in order to formulate a mutually satisfactory question. Thus, the option of assigning to A and B a group of learners working in cooperation will diversify the interaction. Such diversification will encourage the production of questions and also their confrontation and the development of a collective strategy. If on the other hand, the game is played between two individual players, the interaction between A and B will only encourage the elaboration and the evolution of strategies if the opposing players are not at the same stage. Such an option, therefore, implies selection in the formation of the learner-pairs, a selection based on a preliminary analysis of the knowledge-systems of each learner.

*The number of games, *N*, played by the same learners A and B. Quite clearly, the larger this number is, the more opportunity there is for each learner to think about, to encounter and thus to elaborate new strategies. *N* can either be fixed, or alternatively, the decision to continue or to stop can be left to A and B.

Assigning different values to the various variables heightens the incidence of, not only variation in the strategies adopted, but also adaptation and adjustment. As developed by Brousseau in his theoretical *situation didactique* model, the choice of variable values enables situational sequences to be organized in such a way as to produce the required learning.

Predicted child strategies

There are two major strategies: that wherein the learner attempts to find the unknown number, and that in which the learner asks questions so as to order the two numbers without actually finding out what the unknown number is.

Let us look at the first approach. The pupil tries to find the unknown number by asking questions of the "Guess-what" type. For instance, "Is 35 your number?", followed by, "Is 49 your number? ... and so on. This approach will be referred to as F. GUESS. But, as this is pure guesswork and pure chance and is likely to fail, the child will then proceed differently. He will begin asking questions to delineate a certain area before trying to find out the unknown number. He will, for example, ask questions such as: "Is your number larger than 50?". This strategy will be referred to as F. DEL. Such an approach requires the manipulation and interpretation of successive delineations of limits. Such manipulation may be difficult for many pupils as soon as the numbers n_1 and n_2 fall beyond the numerical range that they are familiar with. Another way of proceeding to delineate the area in which it is possible to find the unknown number consists of putting questions like: "Has your number got 2 digits?" or, "Does your number begin with 3?" or, "What is your hundreds digit?"

This way of delineating does not use the same type of knowledge as the previous one. After a certain number of unsuccessful attempts and interactions with the opposing players' questions, some pupils may contrive to proceed more systematically, e.g. by first asking how many hundreds, and then, how many tens, a partner has got. This constitutes a real strategy for, after a certain number of tries, the pupil is sure to find the unknown number and, consequently be able to compare it with his own. Such an approach will be referred to as F. SYS.

An example may help to make this clear. The following F. DEL procedure uses the properties of written numeration:

$n_1 = 35$ (known); ($n_2 = 221$)

1. Does your number have two digits? — No
2. Does your number have one digit? — No
3. Does your number have three digits? — Yes
   Is it 100? — No
   Is it then 124? — No

(Example of domain restriction following the response to question 3)

4. Is the hundreds' digit 1? — No
5. Is the hundreds' digit 2? — Yes
   Is your number 200? — No
   Is your number 215? — No

(Example of the extension and systematic organisation of questioning following the response to questions 5)

F. SYS

6. What is the tens' digit in your number? — 2
7. What is the ones' digit? — 1

Let us now look at the second approach. This approach makes it possible for a comparison to be made without actually specifying the unknown number. That is to say, the player can situate his partner's number n_2 with reference to his own. This can be done with just one question "Is your number larger (or smaller) than n_1?" This strategy is referred to as SITI. (SIT from the word "situate" and I from "immediately"....)

Here is an example with n_1=35, n_2=72;

SITI

Is your number larger than 35? — Yes

Your number is larger than mine.

This strategy is optimal as long as only one game is played. However, if a second game takes place, this is no longer so if the partner uses the information that is thus supplied to him to deduce n_1. Let us look at an example of what could be a second game played by the same partners with n_1=412, n_2=123;

first player		second player
Is your number larger than 412?	(No)	Has your number 2 digits? (No)
I have the largest number because mine is 412.		You have the largest because I have 123 and you 412.

Thus, it can be predicted that replaying the game will lead to the rejection of the SITI strategy in favour of strategies which are similar but more complex: for example, strategies which require the manipulation of several unequal values and which make it possible to situate n_2 in relation to n_1. These strategies are referred to as SIT. Here is an example with n_1=35 and n_2=72.

SIT

Is your number larger than 20? — Yes

Is your number larger than 100? — No

Is your number larger than 80? — No

Is your number larger than 40? — No

Your number is larger than mine

It is also possible to try and obtain information on the written transcription of the unknown number. That is to say, it is possible to make the required comparison by using the rules of written numeration. We have termed this strategy RUL. Let us look at an example.

RUL

How many digits does your number have? — 2

Is your tens digit larger than 3? — Yes

Your number is larger than mine.

Although the questioning is of the same type as that used in the F. SYS strategies, it reveals a modification of the level of knowledge. The strategy requires that not only the relationship between the writing of the numbers and the order of the numbers should be established but also that the relationship should be operational since it is used for making deductions. A summary of the different predicted strategies is shown in TABLE 1

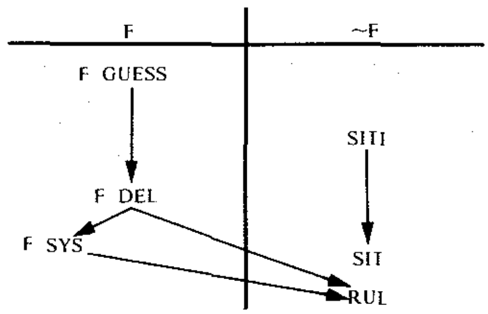

Table 1.

The experiment

The state of knowledge of the learners prior to the experimental sequence.

Prior to the experiment, during the early part of the school year, the class teacher had covered the following areas in the field of numeration: exercises in different bases, activities employing the use of a lexis referring to the designation of digits in numbers of up to two and three digits, comparison of one, two or three-digit numbers, reading and writing of numbers, exercises on numerical sequences. In order to diagnose the state of knowledge of the learners after this preparation and before the experimental sequence, we asked the teacher to give the children a pre-test. This was an individual test in the form of a written questionnaire.

The questionnaire covered the comparison of two numbers of two or three digits and also the designation of the tens digit in three-digit numbers: 21 out of 24 learners were able to supply a digit referring to the tens and construct a number with a given "tens digit". 19 out of 24 were able to compare two written numbers without error. But only 9 of the pupils were able to justify their answers. This shows the difficulty that children have in formulating the properties of numbers even when they have already used them for the purposes of comparison and confirms the results obtained by Bednarz and Janvier [1982] in their study of children's comparison of two numerals.

Timescale of experiment

November 14th, 1983: Introduction and use of the experimental set-up, (video equipment, 8 observers) during an ordinary class.

Period 1: November 17th (90 min.)
Presentation of the problem situation by the teacher.
Game: A and B are groups of three learners each.
(n_1, n_2) are assigned the values of (30,9), (42,77).

Period 2: November 24th (90 min.)
1. Recapitulation of the problem situation by the teacher
Game: A and B are the same groups of three learners
(n_1, n_2) = (128, 49), (137, 215)
2. Modification of A and B: The learners are now divided into two half-classes. The teacher explains how cooperation and decision-making is to be organized within each half class group.
Game: A and B are the two half-class groups, (n_1, n_2) = (147, 86).

Period 3: December 1st (75 min.)
The teacher talks over the difficulties which arose in the functioning of the half-class groups during period 2. She reminds the learners of their respective roles and the aims of the activity.
Game: A and B are the same half-class groups, (n_1, n_2) = (523, 412)
This game is ended by a discussion involving the whole class

Period 4: December 9th (90 min.)
Modification of A and B: The learners now play against each other
Game: A and B are reduced to one learner each.
(n_1, n_2) = (361, 246); (8, 125); (71, 54); (284, 236).

Period 5. December 16th (90 min.)
Recapitulation phase during which the knowledge which is the object of the teaching is consolidated.
Game: A and B are kept at one learner per group
(n_1, n_2) = (82, 293); (598, 603); (1085, 856); (725, 781); (2846, 3524); (3182, 3159).

The actual experiment was carried out in two phases: a start-up phase introducing the game (Periods 1 to 3) and a second phase where numeration was appropriated as a tool (Periods 4 and 5).

Start-up phase
This phase is comparatively long (3 periods). This had been predicted. Clearly, at the start, the range of possible questions is infinite insofar as any question, with the exception of "What is your number?", is possible. Furthermore, the learner does not have any basic strategy with which to approach the problem. This means that he is faced with an unfamiliar situation in which it is he who must make the decision as to which question to pose. The purpose of this phase is to allow a partial restriction of the situation so that each learner has an opportunity to develop an initial strategy. (This is not, in fact usually the strategy which pupils are intended to acquire.) The restriction of the task, which corresponds to a reduction in the learners' state of uncertainty, is obtained by fixing the variables on the one hand, and, on the other, by teacher action.

In order to encourage the generation of questions, at the beginning the numbers n_1 and n_2 were chosen from a numerical field which was familiar to the learners, i.e. that of numbers less than a hundred (Period 1). The field was then widened to include numbers less than three hundred. Number pairs were chosen for which questions about their size enabled a comparison to be made. Furthermore, one pair was chosen for which the use of this property did not allow any conclusion to be drawn. It was thus necessary to have recourse to other properties.

In order to encourage the discussion and the dissemination of questions, A and B were initially groups of three learners in free cooperation (Periods 1 and 2), followed by two half-class groups (end of Period 2 and Period 3) during which cooperation and decision-making were under teacher control.

For groups consisting of three pupils, the lay-out of the table was as in Figure 1: the crosses show the position of the learners, the circles indicate the position of observers. A_1 plays against B_1.

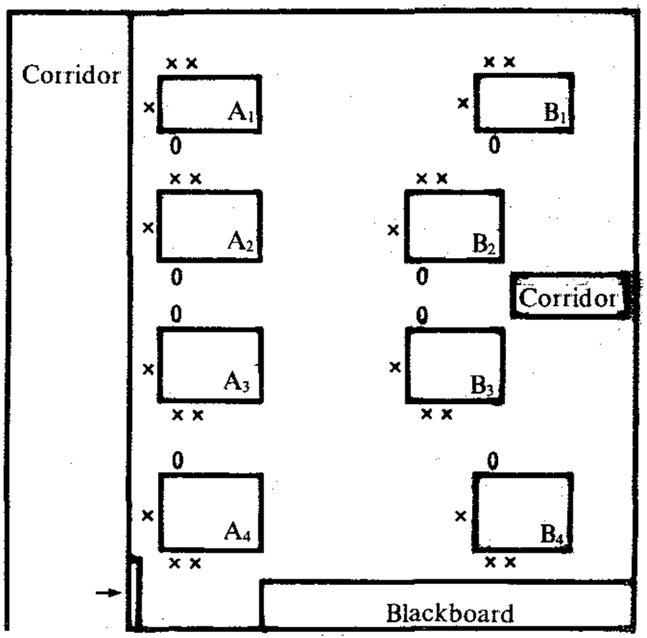

Figure 1

All the A_is were given the same number, as were all the B_is. Each group had only one message sheet and only one pencil (no eraser was supplied). There was one observer per group and one video camera. The class teacher transmitted the message sheets from A_i to B_i, and she also mediated during the validation phase.

The first half-class consisted of pupils from group A, and the second consisted of pupils from group B. The table lay-out was unchanged. Each pupil was given a clean sheet of paper and a pencil. The same number n_1 was given to each pupil in the first half-class. The class-teacher explained the modification in the rules of the game. Each pupil wrote one question and then the half-classes separated in order to choose the best question. (B remained in the classroom, A went out into the corridor.) In order to facilitate the management of the discussion the teacher nominated a leader and a scribe for each group. The leader had to make sure that every pupil read his question to the others. The scribe wrote down the selected question and handed it to the teacher. When each half-group had come to an agreement as to the question to be selected the pupils returned to their places in the classroom and the teacher wrote up the two questions on the blackboard, together with the answers given by each team. Each pupil then wrote down the second question, the half-classes regrouped ..... and so on. The game came to an end when one of the groups wrote down which had the highest number and gave written reasons why. The teacher then organized a class discussion based on the two lists of questions written up on the blackboard.

The numeration assimilation phase

During period 4 the pupils played one-to-one. Pairs were decided on after examination of the methods used during the start-up phase: children who haven't yet hit upon a stable procedure are pitted against those who have. Cooperation is no longer possible and each player has to carry on alone with questioning to the end of the game. The teacher only intervened if one of the children ran into too much difficulty or if conflicts arose, notably when there was disagreement with respect to the rules of the game.

In Period 5 the teacher pointed out that using the rules of written numeration was the best way to win and the children were then made to play the game in pairs once again.

Results

Major underlying misconceptions concerning number

The examination of the message sheets and the analysis of the encodings used by the pupils indicate that, in whatever category the strategy falls, *the underlying major misconception concerning number is a global one* which can be described as follows: a number is seen as belonging to what one might call a "family". For example, 26 belongs to the 20 family, 351 belongs to the 300 family, i.e., those numbers between 299 and 400. It is this conception that accounts for the RUL. 1 and F. SYS1 strategies.

RUL. 1: The learner tries to define the family category as accurately as possible and the order with respect to the families (the two-hundred family comes before the six-hundred family, for example) is used to deduce the order with respect to the numbers. (Two hundred and fifty-eight is smaller than six hundred and twelve.)

F. SYS1: In this strategy the learner works out the family category, but only to narrow down the range of possibilities for the unknown number; first of all by its size, next by the largest family of which it is a member, and so on until all that is left to do is to go over the ten numbers of the remaining family. For example: a three-digit number is identified as in the family two hundred and then in the two hundred and fifties. All that remains to do is to ask, "Is it two hundred and fifty-one, two hundred and fifty-two .....?" and so on. We will now examine one example illustrating the two strategies RUL 1 and F. SYS1.: Laurent is playing against Yannick (period 4):

Laurent ($n_1 = 54$)		Yanick ($n_2 = 71$)	
Has your number got 3 digits?	N	Is it a 1-digit number?	N
Has your number got 2 digits?	Y	Is is a 2-digit number?	Y
Are the tens 40s?	N	Is it in the 30s?	N
Is it in the 30s?	N	Is it in the 40s?	N
Is it in the 10s?	N	Is it in the 80s?	N
Is it in the 90s?	N	Is it in the 10s?	N
Is it in the 20s?	N	Is it in the 50s?	Y
Is it in the 70s?	Y	Is it 59?	N
You have the highest because		Is it 58?	N
mine has 5 tens and yours 7.	Y		

The rules of position numeration were used by only two pupils (Guillaume and Veèronique) in order to solve the problem and, it should be added, not without some difficulty in the encoding. Here is an example

Game 1

Veronique ($n_1 = 246$)		Aurelia ($n_2 = 361$)	
Has your number got one ten?	N	Have you got 200?	N
Has your number got 2 digits?	N	Have you got in the 30s?	N
Has your number got 3 digits?	Y	Have you got in the 100s?	N
Has your number got 3 hundreds?	Y	Have you got in the 600s?	N
You have got the highest number		Have you got in the 200s?	Y
because you have got 3 hundreds			
and me 2 hundreds	Y		

Game 4

Veronique (236)		Aurelia (248)	
Has your number got 3 digits?	Y	Have you got 333?	N
What is your hundreds' number?	2	What is your tens' number?	3
What is your tens' number?	4	What is your number of	
I have the smaller number		units?	6
because I have 3 tens and you 4	Y	What is your tens' number?	3

The examination of Veronique's message sheet indicates that she uses the rules of numeration even though she asks the best questions only in the last game of this Period. We will call this strategy RUL. 2. Similarly the corresponding search procedure for the unknown number will be called F SYS2.

Overview of individual procedures and their modifications up to the end of Period 4
Message sheet analysis enabled the lists of questions generated by the groups to be linked to the types of procedure which had been defined a priori. Moreover, thanks to detailed observation of individual behavior within each group, it proved possible to assign a procedure type to each child: within a given group in any given period, questioning evolves and one procedure becomes dominant. Either this procedure is put forward (or accepted) by a child, in which case it is assigned to him, or it is resisted by proposing questions associated with a different set of procedures and, in this case, it is the latter type which is assigned to the child.

Diagrams showing the series of procedures types through which each child evolved from Period 1 to Period 5 are given below. A crossbreak of pupils versus procedures during these periods is also presented.

DEVELOPMENT OF PROCEDURES

*PATH to RUL 2: 13 pupils
Change from RUL 1 to RUL 2 in Period 5: 5 pupils

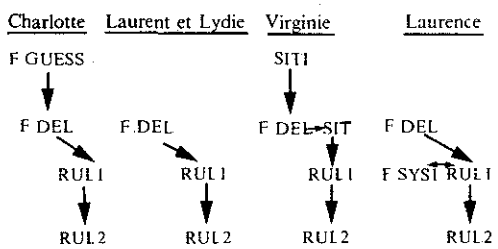

Figure 2

Change from F. SYS to RUL 2 in Period 5: 5 pupils

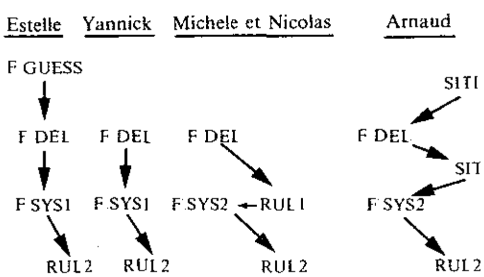

Figure 3

Change to RUL 2 (others): 3 pupils

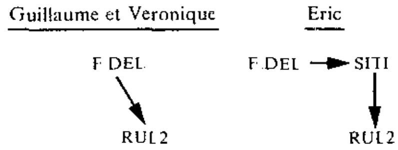

Figure 4

*PATH to F. SYS2: 9 pupils
Change from F. DEL to F. SYS in Period 5: 5 pupils

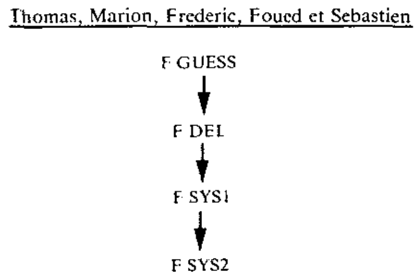

Figure 5

Change from RUL 1 to F. SYS2 in Period 5: 2 pupils

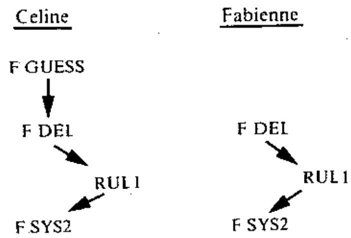

Figure 6

Change to F. SYS2 (others): 2 pupils

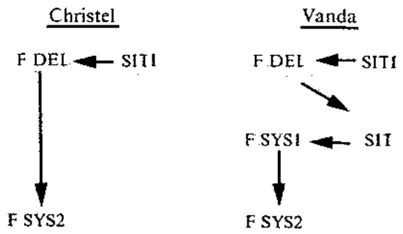

Figure 7

Other paths (*impasses*): 2 pupils

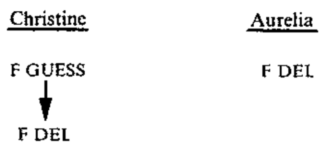

Figure 8

Procedures by Period (1, 2, 4 & 5)

procedure type	none	F GUESS	F.DEL	F.SYS 1	F.SYS 2	SITI	SIT	RUL 1	RUL 2	total
Period										
1	2	7	14	0	0	1	0	0	0	24
2	0	3	14	0	0	0	3	4	0	24
4	0	0	8	5	0	0	1	10	2	24
5	0	0	2	0	9	0	0	0	13	24

Table 2

It would be wrong to think that this communication situation was chosen to reduce the teacher's role. Quite the contrary. In particular, the teacher plays a crucial part in the consolidation of the knowledge system that is being created. This can be seen during the recapitulation sessions, the synthesis sessions, and in the running of the class periods. It will also be seen in the final period when the teacher decides to build on the global concept which is shared by the majority of pupils, and also on Veronique's "discovery" (when she makes a direct request for the n_2 tens' digit), in order to consolidate the numeration of position as an efficient problem solving tool. This part of the sequence is very important. The following section deals with the way in which the teacher manages the recapitulation session.

Teacher: We are going to play a game that you all know
Lydie: The one with the highest number?
T: I'm not going to explain the game again. Before we begin, give me a number with three digits
Laurent: 180. (T. writes it up on the blackboard).
T: Now, in this three-digit number I'd like to know where the hundreds digit is.
Pupils: Lydie, Marion didn't know that the other day.
Thomas: It's the one at the end, on the left. It's a 1. (T. underlines *1*80)
T: And what's that one? (Pointing to the 8)
P: The tens.
T: And that one? (Pointing to the 0)
P: The units
T: Now, give me another three-digit number
Sebastien: 140
Marion: 300. (T. writes up 300 on the blackboard)
T: Well, what can you tell me about the three?
Lydie: It's the hundreds' digit.
T: And what's the tens' digit?
Marion: The tens, that's a nought.
T: And what about that one? (Pointing to the right hand 0)
P: The units.
T: Last time Veronique found a system for going faster. She asked "What's your tens' digit?". Now, what do you think about that?
Marion: It's good, because you can answer it.
T: Suppose you've got 180. What do you answer?
Laurence: 8
T: Are you allowed to ask for the tens' digit?
Lydia: No.
P: Yes, you are
T: Yes, you are allowed to ask for it. What question are you not allowed to ask?
P: What's your number
T: What other question may you ask?
Arnaud: What's your hundred's? What's your units?
T: Yes, that's allowed.
Fabienne: But it's all going to be over so quickly!
F. and A.: That's what I'm going to do.
T: What have you go to know before you ask these questions?
Laurent: If the number has got hundreds in it or not
Vanda: If it's got one or two digits.
T: You've got to know if there is one or two or three digits. We can start now.
Lydie & Virginie: I'm going to win in three goes.
T: The game is to know who has got the highest number.

This consolidation of the formulation and use of the rules of written numeration transforms the game. Henceforth, players have to use specific numeration codes (number digit designation) and the rules for comparing two numbers based on these codes to find out who has the largest number. In this new context winning no longer depends on the type of question used but rather on factors such as seeking out the unknown number or not.

What happens during Period 5?

It is clear from Diagrams 2 to 8 and from Table 2 that the pupils are quick to adopt the new tool It involves a move towards the most economical strategies (that is to say, the fastest), whether for finding the unknown, number (F. SYS2) or for comparing it directly to their own (RUL 2).

Summary and conclusion

Our research brings out the existence of a dominant overall idea of number in these C.E.1 children's minds. This notion of number stems from early learning of numeration but also shows that the elaboration of specific numeration codes is still going on.

The process analysis undertaken shows that the nature of the knowledge in question, i.e. numeration, underwent change. Children not only established a link between number writing rules and number orders but they made this link operational in the situation created, since it was used for purposes of anticipation and deduction. This change in knowledge levels was brought about in the classroom using the fixed and active variables identified at the theoretical analysis stage. This theoretical analyais and a posteriori process analysis also showed the important of certain teacher decisions. These concerned:

— The various permitted interactions that enabled most children to start playing on the basis of their current notions;

— The important role of the start-up phase, i.e. the stage where the teacher devolves the problem to her pupils;

— The point at which the rules of numeration are to be consolidated, thereby enabling this resource to be deployed as a winning strategy in the game.

Bibliography

Bednarz, N. & Janvier, B. [1982] The understanding of numeration in the primary school *Educational Studies in Mathematics*. Vol 13, 35-57

Bell, A. [1985] Principles of the design of teaching Proceedings of the Ninth P.M E. Conference

Bessot, A , Comiti, C [1986] Un élargissement du champ de fonctionnement de la numération: étude didactique du processus *Recherches en Didactique des Mathématiques* Vol 7.1

Brousseau, G. [1980] Problémes de didactique des décimaux *Recherches en Didactique dès Mathématiques*. Vol 1 1, 11-60 et Vol 2 1, 37-127

El Bouzzaoui, H. [1982] Etude des situations scolaires des premiers enseignements du nombre et de la numération. Thése de 3me cycle, Universitésde Bordeaux

Vergnaud, G. [1982] Quelques orientations théoriques et méthodologiques des recherches françaises en didactique des mathématiques *Recherches en Didactique des Mathématiques*. Vol 2.2, 215-232

The point to grasp is how closely the growth of consciousness is related to the growth of intellect. The two are not synonymous, for the growth of consciousness has much wider implications — but the link with intellectual growth is none the less intimate and profound. If the intellectual powers are to develop, the child must gain a measure of control over his own thinking and he cannot control it while he remains unaware of it. The attaining of this control means prising thought out of its primitive unconscious embeddedness in the immediacies of living in the world and interacting with other human beings. It means learning to move beyond the bounds of human sense. It is on this movement that all the higher intellectual skills depend.

The process of moving beyond the bounds of human sense is unnatural in the sense that it does not happen spontaneously. The very possibility of this movement is the product of long ages of culture; and the possibility is not realized in the life of an individual child unless the resources of the culture are marshalled in a sustained effort directed to that end.

But in another sense the movement is not unnatural at all — it is merely the fostering of latent power.

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