Building Upon Student Experience in a College Geometry Course[1]

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This paper describes a geometry course[2] that has been developed with the belief that mathematics is a natural and deep part of human experience and that experiences of meaning in mathematics are accessible to everyone It is centered around a series of challenging and open-ended problems which invite students to draw upon their experiences and share their understandings The problems are designed to encourage students to ask "why?", to conjecture, to investigate viable solutions, and to construct their own justifications for their conclusions The problems, together with the instructional focus on multiple perspectives and meanings in geometry, allow students to concentrate on developing their own ideas, strategies and proofs in a non-competitive learning environment.

The course is currently offered to junior and senior level mathematics majors and prospective high school teachers at Cornell University It has evolved through classes and workshops held over the past 20 years with a variety of participants, including second graders, mathematicians, and mathematics teachers. Variations of the course, including particular aspects of the instructional approach and many of the course problems, have been adopted by mathematics educators at several other US colleges and universities.

All aspects of the course—the materials, the instructional philosophy, the underlying assumptions and expectations—have been shaped by students' ideas and responses to problems. We see the curriculum not as a finished product, but as a process: an approach to teaching and learning geometry which is strengthened, refined, and changed by students' perspectives and experiences. In keeping with the prominence of student voice in the development of the curriculum, we highlight students' reflections about their experiences in our discussion.

An invitation to explore

The course opens with the following expectations written to the students:

> In this course you will be presented with a series of inviting and challenging problems to explore. You should explore, either individually or in a small group, each problem and write out your thinking in a way that can be shared with others. Turn in whatever your thinking is on a question even if only to say "I don't understand such and such" or "I'm stuck here"; be as specific as possible. Feel free to ask questions In this way you will be able to actively develop ideas prior to passively reading or listening to comments of others When working on the problems you should be open-minded and flexible and let your thinking wander Some problems will have short, fairly definitive answers, while others will lead into deep areas of meaning which can be probed almost indefinitely. You should not accept anything just because you remember it from school or because some authority says it's good. Insist on understanding (or seeing) why it is true or what it means for you. Pay attention to your deep experience of these meanings. Everyone understand things in a different way, and one person's "obvious" solution may not work for you However, it is helpful to talk with others—listen to their ideas and confusions and then share your ideas and confusions with them.
>
> We will return your papers with comments about your solutions. Respond to our comments—use them as invitations to explore, to clarify your understanding of the problem, or to clarify our understanding of your solution. In the classroom we will share and discuss students' solutions This cycle of writing, comments, discussion continues on each problem until both you and I are satisfied, unless external constraints of time and resources intervene

Inviting and challenging problems

Problems 1 and 2 provide the initial context for the course expectations The notion of straightness developed in the problems is a basis for the rest of the course

> ***Problem 1. When do you say that a line is straight?***
>
> *Look to your experiences It might help to think about how you would explain straightness to a 5-year old* (or how the 5-year old might explain it to you!). *If you use a "ruler", how do you know if the ruler is straight? How can you check it? What properties do straight lines have that distinguish them from non-straight lines?*
>
> *Think about the question in four related ways:*
>
> 1. *How can you check in a practical way if something is straight—without assuming that you have a ruler, for then we will ask, "How can you check that the ruler is straight?"*
>
> 2. *How do you construct something—lay out fence posts in a straight line, or draw a straight line?*
>
> 3 *What symmetries does a straight line have? A symmetry of a geometric figure is a reflection, rotation, translation, or composition of them which preserves the figure. For example, the letter* "T" *has reflection symmetry about a vertical line through its middle, and the letter* "Z" *has rotation symmetry if you rotate it half a*

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revolution about its center.

4. *Can you write a definition of "straight line"?* [Henderson 1996b, p. 1]

Problem 2. What is straight on a sphere?

Imagine yourself to be a bug crawling around on a sphere. (This bug can neither fly nor burrow into the sphere.) *The bug's universe is just the surface; it never leaves it. What is "straight" for this bug? What will the bug see or experience as straight? How can you convince yourself of this? Use the properties of straightness* (such as symmetries) *which you talked about in Problem 1.*

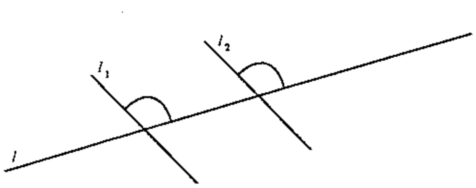
Show (i.e. convince yourself, and give an argument to convince others) *that the great circles on a sphere are straight with respect to the sphere, and that no other circles on the sphere are straight with respect to the sphere.* [Henderson 1996b, p. 14]

All of the course problems are real for students in the sense that they are not solved by simply applying an existing theorem or formula. Students use physical models, such as plastic spheres, paper-made cones and cylinders, and imagination, to search for the geometrical meaning behind each question. They formulate responses to the problems based on their investigations.

The problems are also "real" in their connections to human experience. For example, in problems 1 and 2, students are encouraged to focus on the intrinsic experience of straightness, rather than accept a formal definition they may remember from high school. The questions invite students to draw connections with their personal interests and to explore activities that rely upon a notion of straight. The mechanics of sailing, archery, and hanging wallpaper, the path of a painted ball, and a gymnasts' intuitive sense of balance in her surrounding environment, are among the many experiences students have turned to in formulating their own definitions of straight.

While in some problems the aim is to formulate a definition, in others, students are asked to conjecture and support their claims. *"What is the sum of the angles of a triangle on a sphere?"* and *"How would you explain 3-space to a person living in two dimensions?"* are two such examples. In most problems, students are asked their definitions and conjectures to construct proofs.

For example, students prove the Gauss-Bonnet Formula, which states that the area of a polygon on a sphere is equal to 2π minus the sum of the exterior angles. The students first consider the area of a triangle on a sphere. Then they investigate the connections between area and a notion of local parallelism, called *"parallel transport along a line"*, that is definable on all surfaces. In the figure, l_1 is a parallel transport of l_2 along l.



Parallel transport along l

Parallel transport has many applications in modern differential geometry, physics, and engineering. Students use the definitions and arguments they have constructed in these investigations to prove the Gauss-Bonnet Formula on a sphere.

All of the course topics are developed in a similar manner. Students construct definitions, propose and justify conjectures, and then use their definitions and earlier findings to prove selected theorems. Straightness on Cylinder and Cone, Triangle Congruence Theorems, Parallel Postulates, 3-Spheres in 4-Space, Dissection Theory, Geometric Solutions of Equations, Projections of a Sphere onto a Plane, Duality and Trigonometry, Isometries and Patterns, and Polyhedra are among the course topics.

Written assignments and dialogue: a focus on meaning and understanding

Students work on one to three problems each week and submit their responses to the instructor for written suggestions and feedback before the next class meeting. They are encouraged to respond to the instructor's comments and to continue working on each problem until it is completed. The final grades are determined by the number of problems completed. The distinction between learning activities and assessment activities is blurred in the course. The instructor's written feedback for a problem includes an assessment of the student's progress in her investigation. The following is an example of a typical written dialogue between a student and the instructor.

Problem 14. Angle-side-side (ASS):

Are two triangles congruent if an angle, an adjacent side and the opposite side of one triangle are congruent to an angle, an adjacent side and the opposite side of the other?

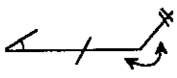
Example of students' work with instructor's comments:

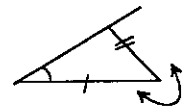
Instructors' comments:

Student's answer:

ASS is true on the plane because, as in the other cases, ASS uniquely determines the orientation and length of the second and third sides. I like to think of it this way: Imagine that you have two sides of fixed length, attached at one end-point. Also imagine that the second side is attached to the first with some sort of pivot joint, so that it is free to rotate around the point of intersection, while the first side is fixed:

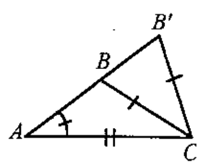
Draw the third side so that it joins the first side at a given angle and extends to infinity. Now rotate the second until it intersects the third side, but does not pass through it, i.e. no part of the second side "sticks out" on the other side of the third side. Allowing the second side to pass through would change the length of that





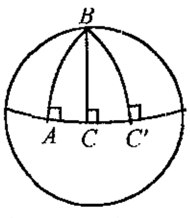
Behold:

$\Delta ABC \neq \Delta AB'C$



side of the triangle:

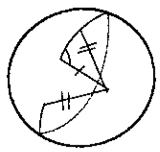
If the second side was not fixed at a particular length or could pass through the third side, then there would be an infinite number of intersections. Because the second side is of a fixed length, however, there is only one place that it can intersect the third side. This determines uniquely the length of the third side, and the orientation of the second side. Note: If the given angle is too large, or if the lengths of the other two sides are not appropriate, i.e., the first side is too long or the second side is too short, then the given ASS will not form a triangle at all.



$\Delta ABC \neq \Delta AC'C$

Not enough

For the sphere, problems can occur if the second side is "sufficiently" long. For any given ASS, there are certain combinations of angles and sides which can produce two different triangles:



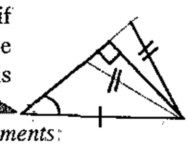
~~Limiting the third side to less than 1/2~~ great circle will rule out this second intersection, which caused the ASS to be false. As in the plane, there will be only one unique intersection possible between the third side and the second side.

Response to instructor's comments above.

I see now that dealing with ASS on any surface is quite a mess. I was drowning in a sea of triangles during my first write-up of the ASS case, so I didn't catch what would happen if you had the right combinations of sides and angles.

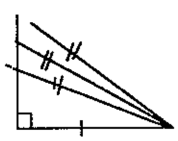
I've played with a new and improved model and discovered that the only time there is a unique side is if ~~the~~ angle between the second and ~~third~~ sides is ninety degrees:

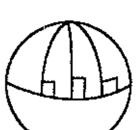
What if this angle is 90°?



Further response to comments:

If you look a little further, you can see that, on the plane, ASS works for right triangles. If you are given a triangle, you can see that given a sufficiently long enough second side (second side > first side), then there is only one intersection. On one side of the point, the second side is too short, on the other side is too long:





On a sphere, however, even right triangles don't work. (No, sometimes, see below) As the diagram that Eduarda drew on my papers shows, two sides and a right angle don't even determine a unique triangle, no matter what length:

[And then the student continues with a detailed coherent discussion of the spherical case.]

Multiple perspectives in the classroom

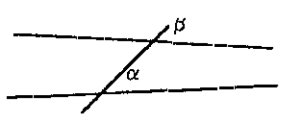
After most students have made significant progress on a particular problem, the instructor invites students to present their solution methods to whole class. Based on written submissions of solutions, the instructor carefully selects the presenters to include different solution methods and to ensure broad student participation. Students often use physical models to explain their solutions and to help others in the class understand their approaches.

As with the high school students in the spherical geometry course discussed in [Brink 1995], students in this course also develop many different definitions and arguments according to different viewpoints. For example, students first discover the need to assume a parallel postulate on the plane in response to the problem,

> *Show that if l_1 and l_2 are lines on the plane such that they are parallel transports along a transversal l, then they are parallel transports along any transversal. Prove this using any assumptions you find necessary. Make as few assumptions as you can, and make them as simple as possible. Be sure to state your assumptions clearly.*
>
> *What part of your proof does not work on a sphere?*

Many students choose to make their own assumptions rather than use usual Euclid's parallel postulate or Playfair's postulate. The following ar examples of some of the students' parallel postulates which they used in their proofs:

- A quadrilateral with three right angles is a rectangle (i.e. all four angles are right angles)
- If two triangles have two corresponding angles congruent then the third angles are also congruent.
- The sum of the angles of a triangle is constant.
- The sum of the angles of a triangle is 180 degrees.
- Every point midway between two parallel transported lines is a center of half-turn symmetry.
- If α is not congruent to β, then the lines intersect.



- Parallel transported lines are equidistant

Each of these postulates is a possible replacement for Euclid's parallel postulate. The students see that there is a need for some assumption and they see that many different assumptions are possible. In addition they are able to discover for themselves that not all the valid postulates are equivalent (for example, Euclid's parallel postulate is true on the sphere but Playfair's postulate is not). For more discussion of parallel postulates see the text [Henderson 1996b]. For more general discussion of multiple perspectives see [Henderson 1996a], the discussion at the end of this paper, and the students' comments in the next section

During students' presentations, the instructor does not comment on the validity of student methods, rather he encourages students to ask questions, try their best to see other people's viewpoints, and to make judgments for themselves. From time to time an innovative, yet incomplete, argument may be presented by a student; however, the course format allows the instructor to leave some issues not entirely resolved. The instructor conveys the belief that grappling with an idea for an extended period of time is a healthy part of the learning process, and that learning frequently stops when a complete solution is presented. He is attentive to the classroom dynamic and makes decisions regarding whether to allow more time for a particular issue to be explored further, or whether to steer away from a line of inquiry which he judges to be unproductive

One advantage to the instructor of allowing multiple perspectives is that he learns mathematics from the students Every semester students show him mathematics that he has never seen before. See [Henderson 1996a] for examples of this new mathematics and for a discussion of this phenomenon.

Students' comments about the instructional approach[3]

In course evaluations and self-evaluations of their learning, students commented about the instructional approach, their insights about mathematics and geometry, and their new and developing relationships with mathematics. Students liked the challenge of real problems They worked hard to develop their own solutions and in the end, took great pride in their work. As Ali[4] commented,

> Very little, if anything, was given to us We had to constantly work hard, and in the end, we were satisfied with the outcome and felt good about it This only caused us to take pride in the material and our own work, sparking interest and curiosity about what was beyond that particular problem. —Ali

Having to undertake writing assignments in a mathematics course was new for many students. However, most were thrilled about being asked what they thought about mathematical ideas Students appreciated the opportunity to express their ideas in their own words.

> ... what I liked the best about this course—I answered the questions in my own words. I certainly couldn't write the answers if I didn't understand the question or didn't know what was going on —Jane

> The method of writing up proofs of assigned problems and rewriting them until understanding is achieved, I feel, is a very effective method of teaching. I was forced to examine the problems on my own, and therefore gained a deeper understanding of each topic. —Diana

From time to time, students experienced frustration when trying to develop a convincing argument on their own. The frustration came from many sources, as Amy put it,

> There are a bunch of reasons for my difficulties and frustration, some of which are fundamental aspects of my personality, some are symptomatic of my last and most hectic semester, and some are results of my education in mathematics up until this point.
>
> One thing I always liked about math was that there was one right answer, and a finite number of different ways to go about getting that answer, and I always knew when I was finished because I had the answer. This course challenged all that, and I think that's a good thing —Amy

Like Amy, most students grew to appreciate the investigative process. In choosing their own assumptions, they came to feel like "mathematicians, capable of producing, reading, understanding, and perhaps even critiquing proofs and mathematical ideas, rather than just a student learning about other people's ideas" (Judy). Some considered the experience to be very similar to what mathematicians must have gone through when they first developed the course topics. However, with constructive feedback from the instructor, the students were never left completely in the dark

In the process of explaining and reflecting upon their ideas and strategies, students became more patient with themselves. They learned to step back, change an approach, and examine physical models when feeling stuck.

> I must admit that at times I was very frustrated with this course Often it took an extreme amount of time to come up with solutions to each individual question. However, I learned to be more patient Perhaps more importantly, I learned that if I am not as eager to find an answer, but rather to try to gain an understanding and to learn more about the mathematics, the easier it was in the long run to indeed find the solution to the problem —Alissa

Students were willing to spend a good deal of time pondering a particular problem partly because assessment is an integral part of the instruction The focus of assessment was the development of one's own mathematical ideas. This non-competitive, non-intimidating format encouraged students to explore and to be creative. Ed's comments reflect a sentiment expressed by many students:

> I liked how the course is based on the completion of a number of problems They challenged me without producing a large amount of stress I really feel that I gained significant understanding from the course-knowledge that is solid; not memorized facts.
>
> I look back and cherish the long hours spent on all those problems. This isn't always true in my other classes. MATH 451 gave me ways to think about mathematics Things I could eventually understand and master Concepts which could be argued and proven to others I have had many conversations about math since enrolling in this class. —Ed

After spending long hours on a problem, students were anxious to see how the same problem could be solved by others with different approaches. Even though it could be a challenge to make sense of a method that is totally different from one's own, students appreciated the opportunity to broaden their own perspectives and to see the creative aspect of mathematics

> By hearing other approaches, my understanding of the problems deepened because I was able to look at the problem in different ways . I also realized how different people can be confused by

different parts of problems . It was very useful to have lively discussions about the struggles we were having understanding geometry. —Jenny

The opportunity for broad student participation in a mathematics class was new for many students. For some, the experience led to insights into their past assumptions about their peers' mathematical ability. As Susan commented,

. . in this class, and this would be my favorite part, everyone could participate. In previous math classes that I have had, there was always a certain group that understood everything. But in this class, people would get up in front of the board, people who I didn't really think knew much about the class, and put up an awesome proof. —Susan

Broad participation in the course was achieved partly because of the instructor's belief that experiences of meaning in mathematics were accessible to everyone. This belief was communicated well in the way the instructor handled class discussion. As Yolanda described it,

He listens to everybody. He encourages us to participate, and he says that he is learning from us. Something really important is the confidence that he has in us. He patiently and silently waited. He did not even interrupt me. That was very significant because he had confidence in my work. That is caring. Why does he want everybody to explain their own method and he does not summarize them? Because in this class students are the protagonist, and the teacher is a guide. —Yolanda

Students' insights about mathematics and geometry

Most students described the sharp contrast between the geometry course they took in high school and this particular course.

When I took geometry in high school, I thought it was neat—a series of puzzles I could figure out. I liked how precise and exact it was, but I wasn't sure how much use it was in the real world—you usually don't see diagrams like the ones in geometry classes in the real world. —John

During the course, students were asked to examine their surroundings, and most importantly, to look for meaning behind the "given truth." It was a mind blowing experience for many students to realize that a lot of the things they had taken for granted on the plane were no longer valid on a sphere. For example, on a sphere there are at least two straight paths connecting any two given points. When two points are antipodal to each other, there can be an infinite number of straight lines passing through both. With this fundamental difference, students start to appreciate many geometrical qualities of planar surfaces, rather than simply take them for granted. They begin to ask more "why?" and "what if?" questions.

As a result of these geometrical experiences, students' views of mathematics also changed. They started to see the subjective and human aspect of mathematics, in which creativity and imagination play important roles.

Your class was the realization of the idea of how creative math is, how it is not the complicated arithmetic shoved down our throats all through school and labeled "mathematics." Math is a creative process, a development of a system of thought. It is not mysterious magical formulas understood in only the most esoteric circles but accessible to the public. —Mary

This course confirmed my belief that math is to be an arena for imagination and experience. What a relief to find an environment which promotes, supports and nurtures such a concept. Rather than stifling expressions of imagination, Math 451 invites them. Now, ideas are not just held, they are tasted, savored and digested into my being. —Gila

Many students were also happily surprised to discover that the world of mathematics includes more than numbers:

I think one of the greatest changes in viewpoint is that I have now seen that Math does not equal number crunching. This is important because it shows that math in general and geometry in specific can be approached from a reasoning/intuitive standpoint rather than purely from an analytical/numerical standpoint. I was surprised how much you could do without numbers! Obviously geometry is independent of measurement but somewhere in the process of being taught math, everything turns into numbers and you lose sight of the beauty as well. —Kent

Students' insights about themselves and mathematics

Two ideas stood out in the students' self-evaluations of their learning in the course. First, many students expressed that through this course they could now understand or "see" mathematics and experience the joy of discovering mathematics. As Daisy and Rick described,

How strange to go from thinking math was all known and nothing new; to discovering math, new math (to me at least) every week. It didn't feel as though I was just learning knowledge that someone else "solved" hundred of years ago. this is a wonderful feeling. I would have to say this was the best part of the course for me, the most interesting and exciting. I can imagine what I believed only people who lived centuries ago felt like. I never would have believed I could do that and I did. —Daisy

It gave me the feeling that I was actually discovering properties. In this way, the course made math seem more like a newly born, exciting subject rather than an old and boring one. —Rick

Several students reported that what they learned in the course made good topics for conversation with friends and family. The problems enabled many to see their physical surroundings with new insights.

I would find myself discussing problems with Kristen, my girlfriend, or Steve (my partner in Blue Light escorts) because it made a good topic for conversation on some of those long nights, or just asking questions to my apartment mates on poster boards on the walls of my apartment. This helped me get a good perspective from non-geometrical, non-mathematical bound people (believe it or not it is possible to learn from them). —Avery

Never before did I feel that mathematics is everywhere. Absolutely everywhere—curvature of line, slope, paths on the Arts Quad, the great blue sky, myself, etc. —Robert

The second theme that stood out in students' reflections about their own learning was that for many, the course had renewed their interests toward mathematics. Hal, a mathematics senior, described his experience,

> This course has helped to rejuvenate my enthusiasm about math in general. It allowed me to puzzle through some problems which cannot be addressed in terms of an algorithm or a pre-established theorem. My brain feels quite a lot bigger, which is nice. —Hal

The course helped many students increase or regain confidence in their ability to learn mathematics. Renewed confidence frequently came from the experience of solving a problem which was new or looked difficult at first. However, in most cases it did not come without struggling with one's own beliefs about mathematics and about what it means to do mathematics. Yolanda's comments capture that struggle and the new way of experiencing mathematics that she found through the course:

> It took me several weeks to realize that I was having problems with my philosophy of what mathematics should be. I tried to prove the first problems but the ideas that came to my mind did not seem to be mathematics; like when I said that an angle was like peeling a banana, or when I looked at symmetries using children games. To me that was not mathematics. On the other hand, I was contradicting myself. I "believed" that mathematics should be constructed by human beings, by the students, if we want it to be meaningful. But that was only an idea. I realized that after all my courses in mathematics, after teaching mathematics, I had never experienced mathematics. Mathematics was now something alive, not something that a dead person (or a living mathematician) created but something I was creating from my own and unique experience. —Yolanda

Some insights gained by the instructor through teaching this course

Insights about definitions and axioms: There is a story (sometimes associated with social constructivism) that in mathematics we construct many possible definitions and axioms and then we "need" to socially agree on one definition. This is not the experience of the students in this course. Most semesters they came up with three or more definitions of "straightness" and "angle" and the seven versions of the parallel postulate mentioned above. Rather than settling on one we glorified in the diversity and richness that these notions gave us. It seemed natural to hold and explore the complexity and interrelations. We were comfortable with the collection of definitions and axioms and our sense of connections among them. It was not so much that the differing definitions and axioms contradicted each other, but rather, that they enriched and supplemented each other and pointed out differing points of view and aspects of our deep experiences of "straightness", "angle", and "parallel". For more discussion of this issue refer to [Pimm 1993] and the introduction to [Henderson 1996b].

Issues about consistency: There is a common notion that it is important in mathematics to be consistent at the level of notation, definitions, and axioms. However, it is an empirically observable fact that neither mathematicians nor mathematical texts are consistent with one another. For example, nine different definitions of angle were found in plane geometry texts in the Cornell Library. There is no agreement across calculus texts as to whether the function $f(x) = 1/x$ is continuous or discontinuous. Even within a single text, consistent notation is not, in general, possible because different notations have differing experiential power in different contexts. (See, for example, the notion of derivative in almost any calculus text.) To shackle ourselves to this sort of consistency would be foolish and limiting.

But: What is the consistency of mathematics? Is is all arbitrary? "Arbitrariness" is a dangerous notion. There may be more than one possible starting point, but that does not mean that the starting points are arbitrary. Differing contexts and differing points of view bring with them a demand for differing starting points. If the choice is arbitrary, then that does away with the need for any discussion. In many discussions about mathematics there is an easy slide from "if it is not absolute" to "then everything is completely relative/arbitrary". In many of these discussions there does not appear to be any middle ground that might allow choice among alternatives, or allow the decision not to choose. In the geometry course we chose not to choose but rather to hold onto the complexity of the multiple definitions and axioms. Our understanding was enriched by seeing other points-of-view, but there was a demand (requirement) that a point-of-view must be meaningful. For example, each person was asked to validate their parallel postulate by using it to prove a theorem and the diversity of postulates led to an enrichment of everyone's own meaning of the theorem.

Learning from the students—Challenges and Rewards: The two important instructional components of the course, the written students/instructor dialogue and the whole-class discussions, made it possible to help students develop powerful mathematical knowledge based on their experiences with the problems. However, it is a constant challenge to understand the students' mathematical insights. Often students' ideas are tentative and fragile and the students lack the self-confidence to push forward their ideas. But, also, the instructor must also overcome obstacles that interfere with listening to and hearing the students' ideas. (See also, [Henderson 1981] and [Henderson 1996a], for discussions of the development and history of the approach.)

When I started using this teaching method twenty years ago I felt threatened when I could not understand a student's questions or explanations—after all I was the expert in geometry. Gradually, after much persistence from the students, I began to realize that my own ways of understanding had deafened me from hearing alternatives. This process is also a very rewarding experience. As I learned to listen more and more to my students I discovered that I was learning much mathematics from them. At first I was surprised—How could I, an expert in geometry, learn from students? But this learning has continued for twenty years and I now expect its occurrence. In fact, as I expect it more and more and learn to listen more effectively to my students I find that a larger portion of the students in the class are showing me something about geometry that I have never seen before. Furthermore, I discovered that I am learning more (percentage-wise) from those students who differ from me (a White man) in terms of gender and race. (See [Henderson 1996a] for examples of the mathematics learned from the students and a discussion of the percentage data.) This is very encouraging, since the underlying premise of the course is that every person who needs some part of mathematics in order to understand some aspect of their experience can grasp that part of mathematics in a very short time.

Notes

[1] The authors wish to thank David Pimm and David Wheeler for their helpful comments.

[2] This course was developed and taught by Henderson during the period 1974-1995. Lo is a Visiting Scholar and participated in the course as a student. Gaddis was a student in the course and assisted Henderson in the writing of a text based on the course [Henderson 1996b]. Gaddis currently teaches the course described here and other courses using the same approach at Buffalo State College. All three continue to work together evaluating and further developing the approach and materials.

[3] This analysis was based on 97 student self-evaluations from 1991-1993.

[4] Students comments are used here with their permission, but all names have been changed.

Discussion

[Brink 1995], Jan Van Den Brink, 'Geometry Education in Midst of Theories', *For the Learning of Mathematics*, 15 (1), pp. 21-28

[Henderson 1981], David Henderson, 'Three Papers', *For the Learning of Mathematics*, 1 (3), pp. 12-15

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[Pimm 1993], David Pimm, 'Just a Matter of Definition', *Educational Studies in Mathematics*, 25(3), 1993, pp. 261-277. (Essay review of Raffaella Borasi's book *Learning mathematics through inquiry*.)

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