# STAGES IN CONSTRUCTING AND COORDINATING UNITS ADDITIVELY AND MULTIPLICATIVELY (PART 2)

CATHERINE ULRICH

In Part 1 of this two-part article (Ulrich, 2015), I introduced a framework for how students develop their ability to construct and coordinate arithmetical units and described the first two stages in that framework, ending with the construction of composite units. In Part 2, I characterize the final two stages in the framework. I then discuss the relationship between the development of additive and multiplicative reasoning within this framework. In particular, I conclude that the types of coordinations we associate with additive reasoning serve as a harbinger of the ability to make related multiplicative coordinations, and that key constructs, such as iterable units, underlie the fluent use of both kinds of reasoning.

Recall that the first stage—the initial number sequence (INS)—is characterized by the interiorization of an arithmetical unit of one, and the second stage—the tacitly nested number sequence (TNS)—is characterized by the construction of composite units (units greater than one). Eventually, the student will use the unitizing operation to reprocess arithmetical units of one and construct *iterable units.* If a student abstracts out the same sense of "adding one more" in a series of arithmetical TNS units, the units become interchangeable. The effect on the student's numerical meanings is subtle, but important. The meaning evoked by a counting number, like 14, will include a strong sense of 14 sequential counting acts for a TNS student, whereas, when working with iterable units, the sequentiality can be evoked, but the primary meaning is a co-occurrence of 14 identical units. This allows the student to further curtail the operations needed to assimilate a composite unit: the student can be merely aware of one unit that is iterated fourteen times instead of being aware of a sequence of fourteen distinct countable units. This efficiency in working with composite units allows students who have constructed iterable units to interiorize their coordination of two levels of units and form an assimilatory construct for the coordination of two levels of units. This is equivalent to assimilatory units coordination and Hackenberg and colleagues' second multiplicative concept (MC2 [1], Hackenberg & Lee, 2015; Hackenberg & Tillema, 2009).

## Coordinating three levels of units

The construction of iterable units portends a variety of more powerful ways of operating (see, for example, Steffe & Olive, 2010) because it decreases the information students must attend to when operating with composite units. In particular, it allows students to reorganize their TNS into an *explicitly nested number sequence* (ENS). In an ENS, for reasons that will become clearer in the upcoming sections on the development of additive and multiplicative reasoning, students are better able to simultaneously attend to two numbers in their number sequence. Whereas in a TNS, a child is aware of the fact that 7 is less than 16 and can determine that they count 9 times to get from 7 to 16, an ENS student can actually think about 7 as a subset of 16 and become aware of the remainder of 16 as representing how much bigger it is than 7. Because students can more explicitly reflect on the nested nature of their number sequences and the additive relationships between these numbers, the number sequence is called the *explicitly nested* number sequence.

Going back to students' development of quantitative complexity, the increased efficiency of working with iterable units allows ENS students to attentionally bound operations on their composite units in order to coordinate three levels of units in activity (see Steffe & Olive, 2010). In particular, in an example given in Part 1, a TNS student evaluated $7 \times 4$ by imagining counting with composite units of 4, seven times. For an ENS student, the act of counting drops out to some extent, and the student can focus on the resulting seven (composite) units of 4 units each. By attentionally bounding those seven sets and equating them with 28, the student can reprocess 28 as being made up of 7 (composite) units of 4 while still being aware of 28 as the composite unit consisting of 28 singleton units. This ability to attentionally bound and reflect on the three levels of units in a composite of composite units, as in this example, is an example of *coordinating three levels of units in activity.*

The difference between coordinating three levels of units in activity and coordinating two levels of units can often be seen more clearly in tasks involving fractional (non-integer) comparisons. Consider how these different levels of coordination would play out in the following mathematical situation (Steffe & Olive, 2010): given a bar that is 6 units long and a bar that is 24 units long, a student who is coordinating with two levels of units would know that this is 6/24 of the long bar. However, such students can often not give another name for such a fraction, instead giving answers such as 4/24 and 1/24 to describe the 1/4 multiplicative comparison. Such students may be aware that the small bar fits into the long bar 4 times and know that "one fourth" is an appropriate name for a length of which you would need four copies to form the whole. However, until the original composite units of 6 and 24 can be unitized and further coordinated, the students remain focused on their actions of counting by six as opposed to re-constructing 24 as made up of four groups of six. Therefore, the idea that six was one fourth of 24 does not occur to them independently. When they do try to focus on the four iterations, the number of iterations becomes conflated with the length of the iterated bar, to yield the answer of 4/24. In contrast, students who can coordinate three levels of units in activity would use the result of their four iterations of the 6-bar to reprocess 24 as made up of four groups of six. They could then independently determine that the 6-bar is one fourth of the 24-bar.

The generalized number sequence (GNS) results from the reprocessing of these complex coordinations so that they are assimilatory. At this point, the student can begin the process of recursively utilizing coordinations of three levels of units to fluently operate in situations involving four levels of units, five levels of units, *etc*. Hackenberg and Lee (2015) characterize the construction of their third multiplicative concept (MC3) similarly as the ability to "take three levels of units as given and flexibly switch between three-levels-of-units structures" (p. 206). Taking three levels of units as given results from the assimilatory nature of a composite unit of composite units and the ability to flexibly switch between three-levels-of-units structures corresponds to the recursive application of that assimilatory structure. Therefore construction of MC3 and GNS co-occur.

The recursive coordination of two levels of units can be used by ENS, and even TNS, students to make sense of similarly complex situations, but because they have to mentally unpack and repack the coordinations more frequently than their GNS peers, they will work less fluently and more conflations will be evident. Many areas of middle school mathematics, such as operations with improper fractions and algebraic equations involving composite fractional relationships, are inappropriately complex for students until they have constructed a GNS or learned to assimilate situations utilizing three levels of coordinated units.

## Relating additive and multiplicative coordinations

At the end of her chapter on rational numbers and proportional reasoning in the *Second Handbook of Research on Mathematical Teaching and Learning,* Lamon (2007) lists questions where further research is needed. The first of these questions was, "What are the links between additive and multiplicative structures?" (p. 662). The constructs in the theory of unit construction and coordination help us understand these links. The first step in addressing this question is to analyze the difference between an additive and a multiplicative comparison.

### Additive versus multiplicative comparisons

Some version of the Flower Problem is often used to illustrate the difference between additive and multiplicative reasoning:

> Two weeks ago, two flowers were measured at 8 inches and 12 inches, respectively. Today they are 11 inches and 15 inches tall. Did the 8-inch or 12-inch flower grow more? (from van de Walle, Karp & Bay-Williams, 2013)

| Number sequence | New types of units | Multiplicative concept | Arithmetic coordinations |
|---|---|---|---|
| Pre-numerical | Perceptual items, figurative items | None | None |
| INS | Arithmetical units | None | None |
| TNS | Composite units | MC1 | Double counting: use one number sequence to assimilate another in order to count on or solve an additive comparison task. |
| | | | Coordination of two levels of units in activity: use one number sequence to explicitly count composite units in order to solve multiplicative comparison tasks. |
| ENS | Iterable units of 1 | MC2 | Assimilatory coordination of two levels of units: use one number sequence to explicitly count composite units in order to solve multiplicative tasks. |
| | | | Coordination of three levels of units in activity: coordinate a composite unit, the number of times that composite unit is applied, and the resulting product. |
| GNS | Iterable composite units | MC3 | Assimilatory coordination of three levels of units: coordinate a composite unit, the number of iterations of that composite unit, and the resulting product. |

*Table 1. Correspondences between unit types and various frameworks.*

There are two correct answers to this problem, depending on whether you want to look at their growth in some common unit of measure, like inches, or with respect to their initial height. These two approaches will lead you to answer that the flowers grew the same amount, 3 inches, or that the 8-inch flower grew more because it grew to 11/8 of its original height, while the 12-inch flower grew to only 15/12, or 10/8, of its original height. The first approach is reflective of additive reasoning and the second approach is reflective of multiplicative reasoning. While additive reasoning is appropriate in this situation, it is often inappropriately applied by students who seem to be generally limited to additive reasoning.

Note that in this situation, additive reasoning relied on making additive comparisons of the start and end heights of each flower, while multiplicative reasoning relied on making multiplicative comparisons of the start and end heights of the flowers. Furthermore, students had to assimilate the situation as requiring either an additive or a multiplicative comparison. First, I will discuss what mathematical operations are involved in making these two kinds of comparisons.

If a student wishes to compare the magnitude of some number, say 24, to the magnitude of some smaller number, say 8, the student must use the smaller number to operate on the larger number, regardless of whether the student is making an additive or multiplicative comparison (see Figure 1): the student must use the 8 as a unit that is being measured off of the 24. In the additive case, the remainder is then enumerated in units of one. In the multiplicative case, the remainder continues to be enumerated in terms of units of 8 in order to enumerate the original number, 24, in terms of units of 8.

In conceptualizing an additive or multiplicative relationship, even in the simplest case of making a comparison that will involve only whole numbers, the student must be able to simultaneously reflect on the two numbers that are being compared and their role in the comparison. That is, the student needs to be aware that she is trying to figure out how many times bigger 24 is than 8 by measuring off how many 8s would make up 24 in order to construct a multiplicative relationship between 8 and 24. In order to construct an additive relationship between 8 and 24, the same student would need to be aware that she is trying to figure out how much bigger 24 is than 8 by measuring off 8 from 24 and enumerating (by 1s) the difference between their magnitudes in order to construct an additive relationship between 8 and 24. This ability to simultaneously reflect on two quantities and the result of using one quantity to operate on the other is what underlies both additive and multiplicative coordinations. Although the smaller number is being used as a unit more explicitly in a multiplicative coordination, the student must have unitized and reversibly disembedded the smaller number in order to construct an additive coordination as well. Nonetheless, the complexity of the additive coordination is less, because the student does not have to keep track of further operations of the smaller number on the difference, as in the multiplicative comparison.

## The development of additive coordinations

In middle grades work, additive reasoning is often used as a foil for multiplicative reasoning without an explicit characterization of what constitutes additive reasoning. For example, Lamon (2007, 2012) utilizes the term exclusively to stand in for the absence of multiplicative reasoning. Further characterization is not missed in her work because she is focused on the development of multiplicative, not additive reasoning. However, in order for us to see links between additive and multiplicative structures, I need to delve more deeply into gradations in additive reasoning.

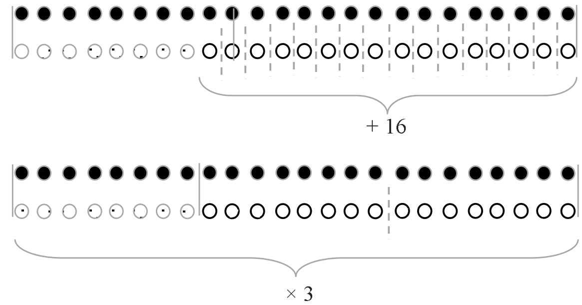


*Figure 1. Additive versus multiplicative comparisons of 24 and 8.*

The very first unit coordinations students do are in the interests of either determining a sum or difference. From the student's perspective at the INS level, the focus is always on units of 1. Therefore, from an INS student's perspective, he is never forming a relationship between numbers when finding a sum or difference, but, rather, describing a delimited counting activity (what number word do I end at if I start at 13 and count 7 more times?). A TNS student can use monitoring and double counting to make, what is to the observer, an *additive comparison.* However, we must differentiate between the question we, as adults, are answering when we solve this problem, "How much more is 20 than 13?" and the question that a student is answering. A TNS student is tacitly aware of 13 and 20 as composite units that represent nested subsequences, but such students are generally limited to understanding the additive comparison in terms of a transformation of one number into another. At no point, does the student necessarily think of 20 as made up of the union of 13 and the determined difference, 7. The number 7 is instead like an adjective describing the action of counting on or adding. The fact that 7 is describing the counting activity rather than representing a quantity on par with 13 and 20 is represented by the having the "+" in the rectangle with the 7 for the first two diagrams in Figure 2. The arrows indicate that the sequential act of counting is still central to the child's understanding of the situation.

Once students have constructed iterable units of an ENS and a GNS, indicated by the identical dots in the second and third diagrams in Figure 2, the resulting efficiency of assimilating the situation allows students to begin utilizing a reversible *disembedding operation* to attend to the embedded composite unit, 7, as a quantity in its own right that can be compared to the containing unit, 20, while still maintaining the ability to reflect on the embedded structure. In particular, the student could be simultaneously aware of both the embedded unit, the containing unit, and be able to reflect on, attentionally bound, and possibly unitize the difference that quantifies the additive comparison of the two composite

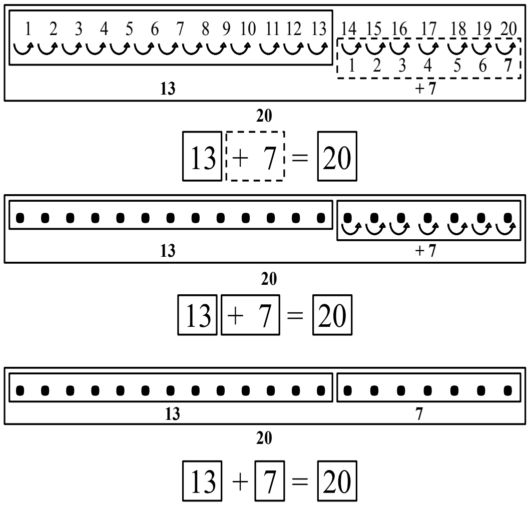


*Figure 2. Transformational and additive approaches.*

units. The solid rectangles around the 7 in the last two diagrams in Figure 2 represent this ability to attentionally bound the difference, while the removal of the "+" from the rectangle in the final diagram represents the unitization of the difference to form a quantity on a par with the original quantities 13 and 20 and additively coordinated with them. This simultaneous ability to attend to the 13 and 20 as quantities in their own right and their embedded relationship is what makes the construction of a true additive relationship possible. While a TNS student might start off with an awareness of the 13 and interpret the 7 in terms of the numbers 13 and 20, the TNS student would lose track of the 13 in the actual counting on activity.

The power of a disembedding operation [2] is seen when ENS and GNS students utilize strategic reasoning to determine related sums and differences. For example, if a student were determining the difference between 58 and 75, an ENS student could potentially utilize the easier number fact that 75 – 60 is 15. The student would realize that 15 enumerates the subsequence {61, 62, … , 75}instead of the subsequence {59, 60, 61, … , 75}. Because his disembedding operation is reversible, he could re-embed the enumerated subsequence, recognize that its lower bound, 60, is two more than 58, so that he needs to increase the length of the second subsequence by 2. (Alternatively, the student could have utilized 75 – 55 = 20 equivalently.) In contrast, an advanced TNS student would be able to adjust what he or she is counting up to (utilizing 78 – 58 = 20 and then decreasing by 3 because he has counted 3 too many). However, that student would have great difficulty in mentally adjusting the number he started counting from (Ulrich, 2012). This inability to adjust the lower bound of the difference represents his inability to re-embed the difference into the original number sequence once he starts enumerating it.

At the GNS level, composite units themselves become iterable. This iterability allows a more fluent decomposition and recomposition of composite units resulting in even more complex strategic reasoning. In fact, students at this stage could independently reason out our standard algorithms for subtraction. In the case of finding the difference between 58 and 75, for example, they might reason: "58 is 50 and 8, so first I'll take away an 8 from 75. 75 is 60 and 15, so I'll get 60 and 7 leftover. Now I'll take away the 50 and I'm left with 10 and 7, or 17." The ability to completely disregard the sequential order of units in addends is reflected by the lack of any arrows in the third diagram in Figure 2. Also note that, at this point, the 13, 7 and 20 are all composites of iterable units that are in an additive relationship. Such a student would, for example, see as necessary that 13 + 7 = 20 implies 7 + 13 = 20. There would be no need to explicitly teach such a student how to utilize the commutativity of addition, nor would such a student rely on patterns to pseudo-empirically abstract that property of addition. Based on findings from two teaching experiments (Ulrich & Phillips, 2015; Ulrich, 2012) such assimilatory additive relationships are within reach of both ENS and GNS students, although they are not trivial for ENS students to interiorize.

## The development of multiplicative coordinations

In her classic chapter on the development of multiplicative reasoning, Confrey (1994) describes an intuitive sense of exponential growth, which she calls *splitting*. Whether or not her splitting construct is separate from and can precede the construction of assimilatory composite units or results from the construction of iterable units is still debated (see Steffe & Olive, 2010). However, regardless of the relationship between splitting and composite units, a general development of multiplicative reasoning will necessarily leverage both. In discussing the development of multiplicative coordinations, I will focus on the role of units and unit coordinations.

Similarly to the discussion of additive coordinations, I will distinguish between solving a multiplicative task and constructing a multiplicative comparison. As soon as students can use their number sequences to count how many times they apply a composite unit, they are able to solve a multiplicative task, such as, "How many 5s are in 45?". However, as with an additive comparison, the student is not constructing a multiplicative structure involving the two given numbers, but rather is describing their counting actions, "I had to count by 5 nine times to get to 45."

ENS students might count to find the answer, 9, in a similar way that TNS students would, but they can use that information to reconceptualize 45 as decomposed into nine groups of 5. This is a multiplicative structure, and it is due to the fact that the ENS students can disembed a sequence of 5 from the sequence of 45, iterate it 8 more times to get a sequence of nine 5s and then re-embed their result into the original sequence of 45 to equate 45 singleton units with 9 composite units of 5. That is, they can simultaneously attend to the 5s and the 45 and the cardinality of the set of 5s in 45. However, there are two constraints to this structure. The first is that their composite units of 5 are not iterable, which means that they are thinking about nine equivalent, but not identical, composite units of 5 in sequence, which reduces the efficiency of the mental structure. The second constraint is that the structure is not stable or anticipatory so that the student will have to build it up each time to make sense of or operate on a composite of composite units.

GNS students have reprocessed these multiplicative structures as assimilatory. They can therefore, for example, conceptualize factors of numbers before carrying out the multiplication or division necessary to determine the particular factors. They can also assimilate a situation as representing a multiplicative relationship before constructing that relationship. This assimilatory multiplicative relationship leads to the kind of immediate multiplicative reasoning needed for proportional reasoning.

### The relationship between additive and multiplicative reasoning

Now that I have considered the mental accommodations students must make to develop additive and multiplicative comparisons, I will address the consequent relationships between the development of additive and multiplicative reasoning across the stages of unit construction and coordination. Solving what, to the observer, is an addition problem involves monitoring counting on activity, which the operations underlying an INS would make possible. Solving what, to the observer, is a multiplication problem, involves monitoring (or counting) successive applications of a composite unit, which the operations underlying a TNS make possible. However, solving addition problems or multiplication problems and making additive or multiplicative comparisons are not equivalent, as discussed earlier.

Recall that, while solving an addition problem requires monitoring counting on activity, making an additive comparison involves developing the goal of enumerating an indeterminate subsequence. Making an additive comparison does not happen spontaneously [3] until the TNS stage. In fact, the actual construction of a multiplicative or additive comparison involves the ability to conceptualize the two numbers being compared as quantities in their own right, which requires the construction of composite units. Because this construction happens at the TNS stage, neither additive nor multiplicative comparisons would be constructed before this stage. Even then, it is not necessarily straightforward for students to construct these comparisons.

Until the construction of iterable units at the ENS stage allows for disembedding, the child will be very focused on his or her action, meaning that their conceptualization of the numerical comparison will be more a description of transformations than an awareness of a quantitative comparison. Once disembedding is possible, students can construct an additive structure or multiplicative structure, in that they can be simultaneously aware of the two quantities being compared and how the comparison relates them. Therefore, both types of comparison can emerge at the TNS stage, but multiplicative comparisons will occur mainly “in the moment” and will not become fluent until the ENS stage.

The primary difference between an additive and a multiplicative comparison stems from the fact that when making an additive coordination, the student can disregard the composite unit after using it as a starting point, thereafter determining the difference in terms of units of one. The fact that a difference is measured solely in terms of units of 1 and does not require further use of the composite units allows additive comparisons (differences) to become assimilatory quantities at the ENS stage. A multiplicative comparison involves an extra layer of complexity, in that the student must keep track of iterations of a composite unit. Keeping track is possible at the ENS stage, but the process will not become assimilatory until the GNS stage. Therefore, multiplicative comparisons can themselves be unitized and coordinated with the other quantities “in the moment” for ENS students, but this process will not become assimilatory until the GNS stage.

As summarized in Table 2, at each benchmark for additive or multiplicative reasoning—solving a task, constructing a quantitative comparison, constructing a structure (a comparison as a quantity in its own right that contains records of its coordinations), and assimilating with that structure—the multiplicative analog requires more complexity and therefore develops later than the additive analog. However, additive reasoning is never far ahead of multiplicative reasoning. Increased power with additive reasoning can be seen as a harbinger of multiplicative reasoning.

## Conclusions

The stages of unit construction and coordination represent a

| Number sequence | Multiplicative concept | Additive reasoning | Multiplicative reasoning |
|---|---|---|---|
| INS | None | Solves addition problems. | None. |
| TNS | MC1 | Constructs additive comparisons between two quantities as a description of a transformation. | Solves multiplication problems “in the moment”.<br>Constructs multiplicative comparisons between two quantities as a description of a transformation. |
| ENS | MC2 | Constructs additive comparisons as an assimilatory quantity. | Constructs multiplicative comparisons as quantities “in the moment”. |
| GNS | MC3 | | Constructs multiplicative comparisons as an assimilatory quantity. |

*Table 2. Summary of the development of additive and multiplicative reasoning.*

way of characterizing mathematical thinking that focuses on the nature of the units a student can use to make sense of a mathematical situation and the ways in which the students can coordinate these units. These stages have been characterized in the past in terms of Steffe and colleagues' number sequences (*e.g.,* Steffe & Cobb, 1988), multiplicative concepts (*e.g.,* Hackenberg & Lee, 2015; Hackenberg & Tillema, 2009), and levels of units coordination (*e.g.,* Hackenberg & Tillema, 2009; Steffe & Olive, 2010). In these two articles, I have explained and related terms from these various frameworks. Further, I have identified some of the major steps in this process: the development of arithmetical units, composite units, iterable units of one, and iterable composite units. The increase in the efficiency of mathematical thinking available with each of these unit constructions allows for more sophisticated coordinations of units: the composite unit opens the way for double counting and a coordination of two levels of units in activity; iterable units make possible the assimilatory coordinations of two levels of units and coordinations of three levels of units in activity; and the construction of iterable composite units allows the assimilatory coordinations of three levels of units.

This attention to the nature of the units and unit coordinations that a student uses also gives insight into the development of additive and multiplicative reasoning, as outlined in the previous sections. Specifically, some additive tasks can be solved with arithmetical units alone, but it is not until the construction of composite units that students can reliably solve missing addend tasks and multiplicative tasks. Even at this stage, though, the students are not able to reflect on their operations to abstract out quantitative relationships until the construction of iterable units and the ensuing construction of the disembedding operation. The construction of iterable units and the disembedding operation are sufficient for the interiorization of additive structures, but multiplicative structures do not become assimilatory until composite units are reprocessed as iterable. The ability to assimilate with multiplicative structures is essential for the multiplicative reasoning we hope to instill in the middle grades.

This analysis of additive and multiplicative reasoning implies that while multiplicative tasks require more complex thinking than corresponding additive tasks, there are often some underlying mathematical constructions that are necessary for both. For this reason, it is possible that complex additive tasks that involve, say, assimilation of a situation involving additive comparisons, will serve to develop mathematical operations such as disembedding and reflection on composite units that will serve as a good foundation for the development of later multiplicative reasoning.

## Notes

[1] Three multiplicative concepts were developed by Hackenberg and colleagues (Hackenberg & Lee, 2015; Hackenberg & Tillema, 2009) in order to describe the nature of the multiplicative units coordinations students can construct in activity and use as assimilatory constructs. MC1 corresponds to the construction of a single multiplicative coordination in activity and is introduced in Part 1 (Ulrich, 2015).

[2] Throughout the article, *disembedding* is meant to refer to a reversible operation that allows a student to attentionally bound part of their counting sequence, operate on it, and then reestablish its relationship to the original sequence. This use of the term is in contrast to a less common, more general use of the term to mean, roughly, experiential or attentional bounding (see Steffe & Cobb, 1988).

[3] By *spontaneous* I mean without direct instruction on an appropriate procedure given certain contextual clues. An example of the type of direct instruction that can be given for additive comparisons: "Whenever I ask 'how much more', you need to put the smaller number in your head and use your fingers to keep track as you count all the way up to the bigger number." INS students can be taught to arrive at the correct answer in this way.

## References


Confrey, J. (1994) Splitting, similarity, and rate of change: a new approach to multiplication and exponential functions. In Harel, G. & Confrey, J. (Eds.) *The Development of Multiplicative Reasoning in the Learning of Mathematics,* pp. 293–332. Albany, NY: State University of New York Press.

Hackenberg, A. J. & Lee, M. Y. (2015) Relationships between students' fractional knowledge and equation writing. *Journal for Research in Mathematics Education* **46**(2), 196–243.

Hackenberg, A. J. & Tillema, E. S. (2009) Students' whole number multiplicative concepts: a critical constructive resource for fraction composition schemes. *Journal of Mathematical Behavior* **28**(1), 1–18.

Lamon, S. J. (2007) Rational numbers and proportional reasoning: toward a theoretical framework for research. In Lester, F. K. (Ed.) *Second Handbook of Research on Mathematical Teaching and Learning,* pp. 629–667. Reston, VA: NCTM.

Lamon, S. J. (2012) *Teaching Fractions and Ratios for Understanding: Essential Content Knowledge and Instructional Strategies for Teachers* (3rd edition). New York, NY: Routledge.

Steffe, L. P. & Cobb, P. (1988) *Construction of Arithmetical Meanings and Strategies.* New York, NY: Springer.

Steffe, L. P. & Olive, J. (2010) *Children's Fractional Knowledge.* New York, NY: Springer.

Ulrich, C. (2012) *Additive Relationships and Directed Quantities.* Unpublished doctoral dissertation, University of Georgia, Athens, GA.

Ulrich, C. (2015) Stages in constructing and coordinating units additively and multiplicatively (part 1). *For the Learning of Mathematics* **35**(3), 2–7.

Ulrich, C. & Phillips, N. (2015) The growth of additive reasoning with second differences. In Che, S. M. & Adolphson, K. A. (Eds.) *Proceedings of the 42nd Annual Meeting of the Research Council on Mathematics Learning.* Las Vegas, NV: Research Council on Mathematics Learning.

Van de Walle, J. A., Karp, K. S. & Bay-Williams, J. M. (2013) *Elementary and Middle School Mathematics: Teaching Developmentally* (8th edition). Upper Saddle River, NJ: Pearson.