# Functions Today and Yesterday

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*"The development of the function concept has revolutionized mathematics in much the same way as did the nearly simultaneous rise of non-Euclidean geometry. It has transformed mathematics from a pure natural science—the queen of the sciences—into something vastly larger. It has established mathematics as the basis of all rigorous thinking—the logic of all possible relations." C.B. Boyer [1946]*

### Introduction

We tend to think that the function concept in school mathematics is something new—a result of the "new maths"—but it was already noticed, if perhaps not implemented explicitly, many years ago. For example, Godfrey wrote in 1912:

> The fact is that we have been teaching functionality for years, whether we have realised it or not. Every schoolboy now learns to plot graphs; this is nothing but the study of functionality in its visible form.

However, although the function concept in school texts is certainly some hundred years old, it has undergone a considerable change in that time. Thus in textbooks from the end of the 19th century until the middle of this century, a function was considered as a change, or as a variable depending on other variables. The following definition from 1888 [Hight, 1968] is typical.

> Two variables may be so related that a change in the value of one produces a change in the value of the other. In this case the second variable is said to be a function of the first.

Note that the "variable" is the "function"—if $y = f(x)$, then $y$ (or $f(x)$) is the function; $f$ does not have a separate existence. The function concept is also associated with numbers only. For example, in the seventh year book of the National Council of Teachers of Mathematics published in 1932, the following definition can be found [Hight, 1968]

> Any mathematical expression containing a variable $x$, that has a definite value when a number is substituted for $x$, is a function of $x$.

There is no doubt that these definitions were a reflection of the state of the function concept then currently in use in higher mathematics. Similarly there is no doubt that the changes which occurred from about the middle of this century were influenced by the rise of abstract algebra, of which Bourbaki was the most famous proponent. In higher mathematics the function came to be defined as a special type of subset of the cartesian product of two sets. A function was no longer something associated purely with numbers, nor was the (dependent) variable identified with the function. The formal set definition was far too abstract for school mathematics, but its influence was felt at school level. In almost all school curricula the function is now defined by two sets $A$ and $B$ (not necessarily numerical), with a rule which assigns exactly one member of $B$ to each member of $A$. For example, in SMSG [1960] in the U.S.

> Let $A$ and $B$ be sets and let there be given a rule which assigns exactly one member to $B$ to each member of $A$. The rule, together with the set $A$, is said to be a function and the set $A$ is said to be its domain. The set of all members of $B$ actually assigned to members of $A$ by the rule is said to be the range of the function.

Or in SMP [1966] in the U.K.

> A relation is a connection between members of two sets or members of the same set.... A mapping (relation) is a function, if each member of the domain has only one image.....

In the majority of textbooks the definition is similar to one of the two above.

The "modern" definitions are more abstract than the old ones, and the relation between variables, which was emphasized in the old definitions, goes almost unnoticed in the new, although this aspect is implicitly included in the new definitions.

### Advantages and disadvantages of the new definition

As happens with almost every curriculum innovation, the modern definitions of a function had their supporters and objectors. The following points are some of those used in the debate.

— In the sciences as well as applied mathematics, the function is conceived as a relationship between variables, whereas the set definition is not used. However, there is no reason not to teach precise definitions in the context of mathematics, just because they are not used in other fields. [Read, 1969] Moreover, it is accepted that a sound understanding of the function concept is needed in order to build additional mathematical concepts in later courses. [Buck, 1970; Khinchin, 1968]


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— The historical development of the function definition, and the development of the student, may provide an argument against the teaching of the new definition. It may be better for the student to follow the historical development and encounter the function concept first as a relationship between variables and only after that to learn the new definition. [Malik, 1980]

— Another reason for postponing the new definition is that it is not essential until the study of analysis and topology. Since only a small percentage of school students eventually study these topics, the set theoretic definition can be postponed to the beginning of these courses, while at the elementary level an easier definition could be taught. [Malik, 1980]

— The old definitions allowed only numerical functions, whereas the new set definition includes non-numerical functions and transformations. But must functions used in junior high school, and even in high school, are numerical functions, so maybe the new definition is not needed at this level.

Most of the arguments for and against the new definition which we have cited are based on a mixture of theoretical, mathematical, historical and pedagogical considerations. But the situation today is that the set definition has been taught in schools for about 25 years, and no one disputes the central importance of the concept, whatever the arguments about its definition. The natural question, therefore, arises: do the students "understand" the new definition? In other words, can we inject some "objective" facts into the pedagogical aspects of the argument, whether the new definition is a good or bad thing—or, more likely, how much of a good and how much of a bad thing? Thus the purpose of the study to be described was to investigate how junior high school students understand the concept of function and what are their difficulties and misconceptions (if such exist).

**"Understanding" the concept of function**

A "general understanding" of the concept of function includes, of course, many aspects, such as being able to use the concept in fields other than mathematics, together with the use of the concept in different contexts within mathematics itself. There is no hope of doing justice to all aspects in a single study, and as implied by our introduction, we were particularly interested in understanding at the beginning of the students' learning of the function concept—i.e. at the definition end, rather than with later applications. Thus the study to be described restricts itself to junior high school students and certain components of the understanding of the function concept which we shall detail.

In order to place these components in their proper context, we first analyse the stages the students pass through (or should pass through) when they learn explicitly about functions. First they learn that a function is composed of three sub-concepts: domain, range and rule of (many-one) correspondence. Then they learn that functions can be represented in several forms, such as arrow diagrams, verbal, graphical and algebraic representations. In each, the three sub-concepts take on appropriate forms, as shown in Table 1.

**Table 1**
**Representations of a function and its components**

| sub-concept \ representation | Verbal | Arrow diagrams | Algebraic | Graphical |
|---|---|---|---|---|
| Domain | verbal or mathematical notation | a curve enclosing the members of the domain | verbal or mathematical notation | the horizontal ($x$) axis or parts thereof |
| Range | verbal or mathematical notation | a curve enclosing the members of the range | verbal or mathematical notation | the vertical ($y$) axis or parts thereof |
| Rule of correspondence | verbal | arrows | formula | a set of points in the coordinate system |

Then they learn (in most cases in junior high school) that the same function can be represented by each of the above representations, so they have to learn to translate a given function from one representation to another, dealing with the three sub-concepts and with two representations simultaneously.

This essentially completes the initial study of functions in general; the students then go on to study linear and, towards the end of ninth grade, quadratic functions. In this context, we were concerned with one further point.

Many functions, especially in applications, are not given in the neat and precise form of domain, range and rule of correspondence, but rather implicitly by constraints (observations), e.g. a few pairs of corresponding elements in the domain and range. These constraints in many cases do not "define" one function but a family of functions. For example, a function "defined" by the contraints $f(-1)=5$, $f(2)=9$ and $f(6)=11$ has to "pass" through the points $(-1, 5)$, $(2, 9)$, $(6, 11)$, and the number of functions satisfying these constraints is infinite.

The topic of a function "defined" by constraints is not really touched upon in the usual curriculum materials. Therefore, as something of an enrichment to the main part of the study, we decided to include questions on this topic. It would seem reasonable to assume that with a sound understanding of the function concept, as outlined above, a student would be able to apply his knowledge successfully to this relatively unfamiliar area. Also, because the constraints are often concerned with the "behaviour" of the function, such questions can be used as an indicator of whether, from the newer set definition, the student still manages to grasp the essential features inherent in the older "variable" definition.

According to the above, and taking into account that a "good understanding" has two stages: the passive (and easier) one—such as classifying, identifying, etc.—and the active (and more complicated) one—such as doing something, giving examples, etc.—we chose to concentrate on the following components in the understanding of the function concept

*Components in the understanding of the function concept*

I a. The ability to classify relations into functions and non-functions.
 b. The ability to give examples of relations which are functions, and of relations which are not.

II. a. For a given function, the ability to identify preimages, images and (preimage, image) pairs.
 b. The ability to find the image of a given preimage and vice-versa.

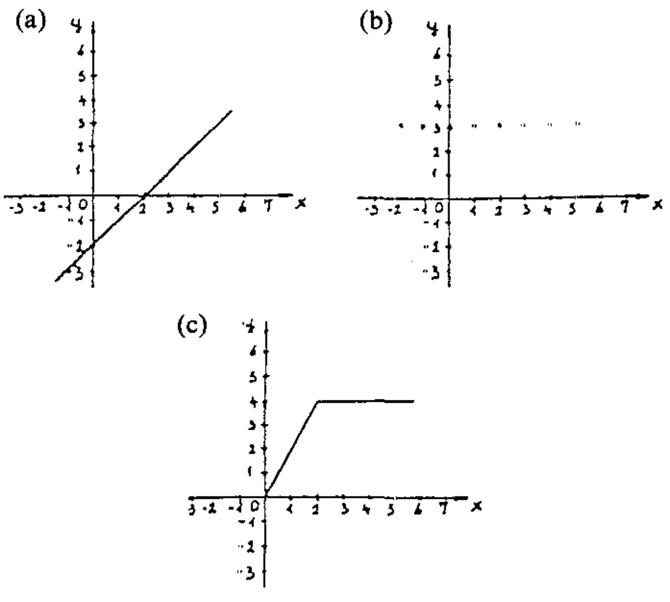
III a. The ability to identify identical functions
 b. The ability to transfer from one representation to another.

IV a. The ability to identify functions satisfying some given constraints.
 b. The ability to give examples of functions satisfying some given constraints.

**Procedure of the study**

In order to investigate how students understand the above components of the function concept, we wrote a large variety of problems, restricting ourselves to the graphical and algebraic representation of numerical functions.

The problems were given to 9th graders (ages 14-15) after they had studied the relevant part of the curriculum. Since there was a large variety of different items, many of them open problems, only a small number of students answered each item (in all, some 400 students were involved). However, the multiplicity of items allowed us to identify common concepts which caused difficulty, independently of the particular context of each item. Thus the problems were designed to identify student difficulties and also to suggest the causes of these difficulties. For example, in each of the following three cases, students were asked to find the algebraic representation



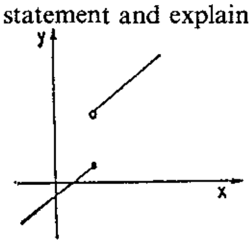

Item (a) was designed to check the basic ability to "move" from the graphical to the algebraic representation. If this "familiar" case was answered correctly, but there was failure in item (c), we could assume that the procedure was known, but the piecewise function caused difficulties. Item (b) was inserted to see if students would pay attention to the domain of the function and to the fact that the function is constant

We bring the results obtained, together with some sample problems, according to the components in the understanding of the function concept listed above.

**Results**

Ia *The ability to classify relations into functions and nonfunctions*

1. Indicate the correct statement and explain.
   a. The relation is a function.
   b. The relation is not a function.




2. Indicate the correct statement and explain
   a. The relation is a function.
   b. The relation is not a function.

$$f:\left\{\begin{matrix}\text{real}\\ \text{numbers}\end{matrix}\right\}\rightarrow\left\{\begin{matrix}\text{real}\\ \text{numbers}\end{matrix}\right\}$$

$$x\rightarrow\begin{cases}1/(1+x), & x\neq -1\\ 0, & x=-1\end{cases}$$

When the relations were given in graphical form most students correctly distinguished those which were functions from those which were not. Some difficulty was experienced when the relations were given algebraically, especially when the relation was a constant function or defined piecewise, as in item 2. In response to this item about half the students said that the relation is not a function, because "every preimage has more than one image." This "reason" recurred for other piecewise functions, and it seems that students did not realize that different rules of correspondence applied to different parts of the domain

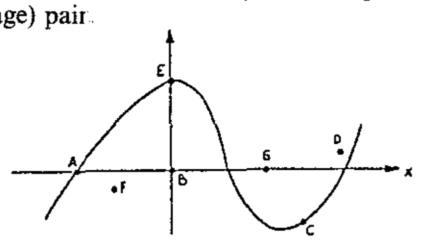
Difficulties with piecewise functions were found also by Vinner [1979] who investigated the function concept among older students (15-17 year olds) after they had studied the formal definition. He found that students believe that a function must have the same rule of correspondence over the whole domain, otherwise two or more functions are involved

Ib. *The ability to give an example of a relation which is a function, and one which is not*

Students performed this task reasonably well. Some of the students had difficulties caused by confusion between many-one, and one-many relations, especially when the arrow diagram representation was used.* It is noteworthy that about half the students gave examples graphically.

IIa. *For a given function, the ability to identify preimages, images and (preimage, image) pairs*

3. For each of the given points and the function represented by the graph, decide whether or not it represents a preimage, image, (preimage, image) pair.




4. Given the function $f$

$$f: \begin{Bmatrix}\text{natural}\\ \text{numbers}\end{Bmatrix} \rightarrow \begin{Bmatrix}\text{natural}\\ \text{numbers}\end{Bmatrix}$$

$f(x) = 4x + 6$

a. Which of the numbers 2, −1, 0, 11.5, 1267 can be a preimage of $f$?

b. Which of the numbers −2, 10, 8, 46, 4006 can be an image under $f$?

c. Which of the following ordered pairs (5, 26), (0.5, 8), (2, 10) is (preimage, image) pair of $f$?

Explain your answers.

5. Given the function $g$

$$g: \begin{Bmatrix}\text{real}\\ \text{numbers}\end{Bmatrix} \rightarrow \begin{Bmatrix}\text{real}\\ \text{numbers}\end{Bmatrix}$$

$g(x) = 4$

a. Which of the numbers 2, −5.5, 9.07, 0 can be a preimage of $g$?

b. Which of the numbers 12, 0, 4, 3.3 can be an image under $g$?

c. Which of the following ordered pairs (2, 8), (0, 4), (0, 0), (19, 4) is (preimage, image) pair of $g$?

Explain your answers.

* Although we restricted ourselves to the algebraic and graphical form of representation, the students were not so restricted when appropriate.

In the graphical representation the results indicate that students understand that points on the curve (e.g., C in problem 3) represent (preimage, image) pairs of the function and points not on the curve (F, D) do not. However students had considerable difficulty in locating preimages and images, not realizing that they are located on the relevant axis.

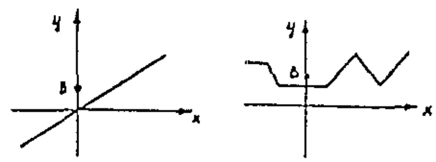
Points located on the curve and also on the axis (A, E) caused a lot of trouble. Problems 4 and 5 investigate the same aspect but in the algebraic representation. In problem 4, the rule of correspondence is a familiar one, but the domain may be expected to cause trouble. In problem 5 the domain and range are familiar but the rule of correspondence is constant. Student responses, together with their explanations, indicate that most of the students used the correct procedure to identify a preimage—they checked whether the number belongs to the domain. Mistakes that occurred seem to be due to uncertainty about what the set of natural numbers includes (for example, if zero is included in $N$). In order to identify if a given number is an image of $f(x) = 4x + 6$ (problem 4) three operations are required—to check if the number belongs to the range, to calculate the "preimage", and to check if this "preimage" belongs to the domain. For example, the "preimage" of 8 is 1/2, which does not belong to the domain, a point unnoticed by the majority.

Only a few students worked all three steps correctly. About half the students only checked whether the number belongs to the range. Although in the case of the constant function, all the students had to do was to check if the image obtained was 4, only about a quarter of the students actually did this.

In order to identify whether a pair of numbers represents a preimage and image of a given function, again three operations are required—to check if the first number belongs to the domain, if the second belongs to the range, and if the numbers together are a (preimage, image) pair. The students had difficulty and, for most of them, one or two of the three operations was all they could manage.

IIb. *The ability to find the image for a given preimage and vice-versa*

6. For each of the graphs of the functions given, mark the elements of the domain which are preimages of the point B, in the range



7. For the function $f$,

$$f: \begin{Bmatrix} \text{real} \\ \text{numbers} \end{Bmatrix} \to \begin{Bmatrix} \text{real} \\ \text{numbers} \end{Bmatrix}$$

$f(x) = 4x + 6$

complete the following:

| | |
|---|---|
| $f(2) = \square$ | $f(\square) = 10$ |
| $f(0) = \square$ | $f(\square) = 8$ |
| $f(-(1/2)) = \square$ | $f(\square) = -26$ |
| | $f(\square) = 0$ |

When the function was given in graphical form, more than half of the students did not realize that preimages are located on the $x$-axis and images on the $y$-axis, a difficulty already noted above.

Many of them thought that the preimages or images are on the curve, corresponding to the given points on the axes

When the function was given in algebraic form, the students could find the image for a given preimage, but had some difficulty when they were asked to find the preimage for a given image, apparently due to the technical manipulations required, since success decreased with the complexity of the calculations Again, as already noted, students had difficulties when the function was constant. Marnyanskii [1969] also reported difficulties with the constant function.

IIIa. *The appreciation that the same function can be represented in several forms, and the ability to identify identical functions*

8. Given the function $f$

$$f: \begin{Bmatrix} \text{natural} \\ \text{numbers} \end{Bmatrix} \to \begin{Bmatrix} \text{natural} \\ \text{numbers} \end{Bmatrix}$$

$f(x) = 4x + 6$

For each of the following, decide whether or not it describes a function identical to $f$, and explain

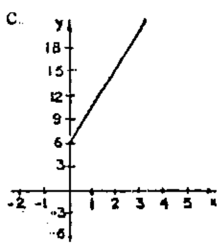
a. $g: \begin{Bmatrix} \text{real} \\ \text{numbers} \end{Bmatrix} \to \begin{Bmatrix} \text{real} \\ \text{numbers} \end{Bmatrix}$

$g(x) = 4x + 6$

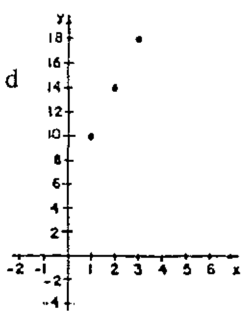
b. $g: \begin{Bmatrix} \text{natural} \\ \text{numbers} \end{Bmatrix} \to \begin{Bmatrix} \text{natural} \\ \text{numbers} \end{Bmatrix}$

$g(x) = 2x + 3$

c.



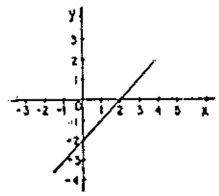

d




In response to other items, it was clear that students understand that a function can be represented in several forms, and most used the algebraic or graphical form, or both, to represent a given function However difficulties occurred when students were asked to identify functions identical to a given function, as in problem 8. In section (a) of problem 8, where the only change is in the domain and range, less than half the students gave the correct answer. Section (b) did not cause much trouble In section (c) about half the students gave the correct answer, the others neglected the domain and range. It seems that many students think that a function is defined by the rule of correspondence only, which is a relevant point in the debate between the old and new definitions. There was little success with section (d), apparently caused by the change of units. Other incorrect reasons given included "the points must be connected".

IIIb. *The ability to transfer from one representation to another*

9 Find the algebraic form of the function shown in the graph, specifying its domain and range.




10. Draw the graph of the function

$$h: \begin{Bmatrix} \text{real} \\ \text{numbers} \end{Bmatrix} \to \begin{Bmatrix} \text{real} \\ \text{numbers} \end{Bmatrix}$$

$$h(x) = \begin{cases} 4, & x \neq 2 \\ 1, & x = 2 \end{cases}$$

This topic was difficult. Less than a third of the students responded correctly to problem 9, although the graph was familiar. Less than a quarter performed correctly on problem 10.

In order to check if a difference exists between the two kinds of transfer, from graph to algebra, and from algebra to graph, we included items which asked for the two kinds of transfer for the same function. We found that when the function was "familiar" as in problem 9, algebra-to-graph was easier than graph-to-algebra. This would seem to accord with the different kinds of manipulations involved in the two kinds of transfer When the function was "unfamiliar", difficulties were manifested in both kinds of transfer.

Students' answers to problems in this component give further evidence of their neglect of domain and range. Only one student drew the graph of the following function correctly.

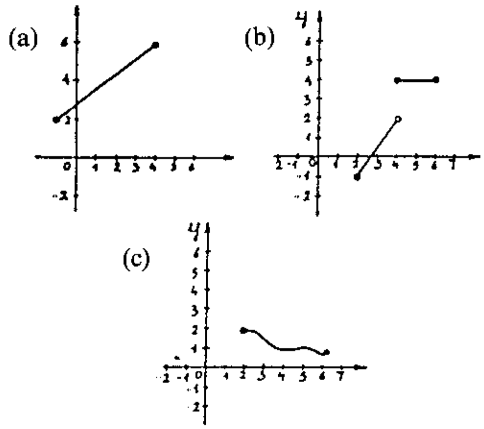
$$f: \begin{Bmatrix} \text{natural} \\ \text{numbers} \end{Bmatrix} \to \begin{Bmatrix} \text{natural} \\ \text{numbers} \end{Bmatrix}$$

$f(x) = 3.$

Most answers (subconsciously) replaced the natural by the real numbers. In those problems where the graph-to-algebra transfer was required, students were also explicitly asked to specify the domain and range of the function. Nevertheless few actually did so

IVa. *The ability to identify functions satisfying some given constraints*

11. Indicate the graphs which represent a function whose domain is $\{x \mid 2 \leq x \leq 6\}$ and whose range is $\{y \mid -1 \leq y \leq 4\}$.

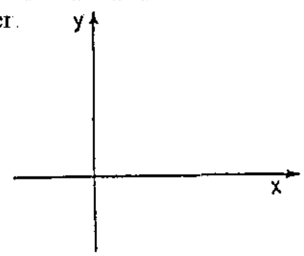



(This item included another six options on the questionnaire used in the study.)

Students had difficulty in understanding that the set of images may be a subset of the range (c). They also had difficulty with piecewise functions (b).

IVb *The ability to give examples of function satisfying some given constraints* [See also Markovits et al, 1983].

12. a. In the given coordinate system, draw the graph of a function, which increases over part of the domain and is constant over the remainder.




b The number of different such functions that can be drawn is

i) 0
ii) 1
iii) 2
iv) more than 2 but less than 10
v) more than 10 but not infinite
vi) infinite.

Explain your choice

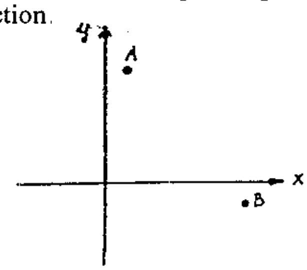
13. a. Give an example in algebraic form of a function defined from the real numbers to the negative number.

b. The number of different examples of such a function is

i) 0
ii) 1
iii) 2
iv) more than 2 but less than 10
v) more than 10 but not infinite
vi) infinite.

Explain your choice.

14. a. In the given coordinate system, draw a graph of a function such that the coordinates of each of the points A, B represent a preimage and the corresponding image of the function.




b. The number of different such functions that can be drawn is

i) 0
ii) 1
iii) 2
iv) more than 2 but less than 10
v) more than 10 but not infinite
vi) infinite.

Explain your choice.

The constraints were of three kinds:

— properties the function has to satisfy, (problem 12),
— constraints on the domain and range (problem 13),
— points the function has to "pass" through (problem 14).

The constraints were similar in both the graphic and algebraic representations. Each problem had two parts. In the first the students were asked to give an example, and in the second to specify the number of possible functions and to explain their choice.

When the constraints were attributes the function had to satisfy, in the graphical representation most of the students drew correct functions and said that the number could be infinite. In the algebraic form correct answers were obtained only when the constraints allowed familiar functions as examples. For example, when the function increased over part of the domain and decreased over the other part, the parabolic function was given. But when the function increased over part of the domain and was constant over the remainder, the students failed to find an example This is in line with the difficulty already mentioned: the students do not see a piecewise function as a single function. When the constraints were on the domain and range students did not pay attention to them in the graphical representation and again had difficulty in seeing that the set of images may be a subset of the range. This may also be one of the reasons that in the algebraic representation they invariably used complicated functions instead of using the constant function Another reason is the general difficulty they have with the constant function.

Points the function had to "pass" through caused a lot of trouble, a difficulty which increased with the number of points. In the graphical representation, students drew mostly linear functions or functions composed of straight lines, and not many appreciated that the number of different functions is infinite * In problem 14, half the students said that the number is infinite. The other half said that only one function can be drawn, because "through two points there is only one straight line".

In the algebraic representation, the students could manage somehow when one or two points were given, but when a linear function did not suit (for example, the function had to "pass" through (3,4), (6, 9), (8, 13)), only one student gave a correct answer, and that in graphical form.

**Discussion**

In order to draw any conclusions from the above results, we need to integrate and classify the observations. In the first place it should be clear from a perusal of the sample items that many of them are not standard, and hence the level of understanding that we are measuring may well be considered quite high for junior high school students. Therefore we are not really interested in the students' overall success but rather in the types of difficulty they encountered

i Whatever the particular nature of the question, three types of function caused difficulty: the constant function, a function defined piecewise, and a function represented by a discrete set of points.

ii There was a general neglect of domain and range, whether attention to them was explicitly required by the question or only implicitly.

iii Both in the algebraic and the graphical form, the concept and representation of images and preimages was only partially understood.

iv The variety of examples in the students' repertoire of functions was limited both in the graphical and algebraic form, but more especially in the latter.

v Transfer from graphical to algebraic form was more difficult than vice-versa, and both were conditioned by the limited repertoire noted in the previous point.

vi "Complexity" of technical manipulations inhibited success

vii When examples of functions were required, there was an excessive adherence to linearity.

viii Many of the above difficulties were clearly in evidence in the questions on functions defined by constraints

Most of the above difficulties (but certainly not all) are peculiar to "modern" curricula in the sense that they would hardly have arisen in more traditional programs Perhaps only the difficulty related to technical manipulations and the excessive adherence to linearity would have been expected (and were more than likely also present) in traditional programs Moreover, the students seemed to have a general understanding of the new definition in the sense that they were reasonably successful in distinguishing between functions and non-functions and could give examples of each.

Thus we would suggest that the essential difficulty in the use of the new definition is that it is explicitly composed of many more components than previously—domain, range, rule of correspondence, preimage and image—and that the function concept has been given a much wider context If we accept this view then it is not really surprising that there may be more difficulties—the real argument would seem to be whether the whole exercise is worthwhile.

The set definition has the advantage of being in harmony with other apsects of the present school curriculum. In addition, because of its more general form, it can be gradually introduced in non-numerical contexts (e g., transformations in geometry). Thus if we can remove or overcome some of the problems detailed above, the use of the modern definition would seem to be worthwhile.

The difficulty of the number of components would seem to be in line with present learning theories regarding student memory Therefore we would suggest the downgrading of some components—in particular, to place relatively little emphasis on questions of domain and range. For example, items such as question 3, 4 and 5 above would almost entirely disappear from the curriculum, except for the best students, with more emphasis on items like questions 6 and 7. We also suggest the addition of "constraints" questions such as 12, 13 and 14, since these questions probe whether students grasp the "variable" aspect of the function, and discussion of them with students can bridge the gap between the old and the new definition. Also, this aspect of functions is most important in applications.

In a historical perspective, these are still early days in the use of the set definition of functions, and we believe that many of the other difficulties noted above can be overcome if we are willing to study them further, analyse carefully what exactly we expect of a student who understands (as illustrated above), and experiment with various treatments. We have some evidence in support of this belief. In line with our general policy of not just doing educational research but harnessing it to the day-to-day business of curriculum development and implementation, we followed

*Karplus [1978] found that students frequently used linear interpolation when solving problems involving graphs.

Continued on page 28