PROOFS THAT DEVELOP INSIGHT: PROOFS THAT RECONCEIVE MATHEMATICAL DOMAINS AND PROOFS THAT INTRODUCE NEW METHODS

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There is a consensus amongst mathematicians and mathematics educators that proving is a fundamental activity that should play a prominent role in all mathematics classrooms. What is less clear is the exact role that proof should play, and how the construction and observation of proofs can be beneficial to students. To address these issues, some researchers have sought insight by investigating how proof is viewed by the mathematical community. In an influential article, de Villiers (1990) argues that proof serves numerous functions for mathematicians, including providing conviction that a theorem is true, explaining why a theorem is true, facilitating communication between mathematicians by presenting arguments in a common genre, enabling the discovery of new theorems, and systematizing mathematical theories. Many researchers have lamented that proof only plays a limited role in mathematics classrooms – establishing the truth of a theorem – and argue that proofs should also serve the function of providing explanation and facilitating communication, as they do in the mathematical community (*e.g.*, Hersh, 1993; Knuth, 2002).

Within the last few decades, many philosophers of mathematics have argued that the role of proof in providing conviction is often overstated (*e.g.,* de Villiers, 1990; Thurston, 1994; Rota, 1997; Rav, 1999; Dawson, 2006). de Villiers (1990) notes that mathematicians are usually convinced that a theorem is true before they attempt to prove it and are sometimes not fully convinced of the truth of a theorem after reading its proof. In a recent study, I asked eight research-active mathematicians whether they read a proof for correctness or hoped to get something more out of it. All eight emphatically answered their primary aim from reading a proof was to gain insight (Weber, 2008), a finding that corroborates the analyses of many mathematics educators (e.g., Hanna, 1991; Hersh, 1993). The goal of this paper is to illuminate what type of insight mathematicians gain from reading proofs. Of course, there is no single way that a proof may provide insight and several philosophers have analyzed the types of insights one might glean from a proof (*e.g.,* Mancosu, 2000; Avigad, 2006; Dawson, 2006). In this article, I contribute to this conversation by reporting the perspectives of nine prominent research-active mathematicians discussing what they hope to gain from reading the proofs of others.

Proofs that reconceive mathematical domains

In a recent study, I interviewed nine mathematicians about their experiences with reading mathematical proofs in their professional work and presenting proofs to students in their classroom. One specific question that I asked was, “What do you hope to gain from reading the published proofs of others?” There was some variation in their responses – some claimed one reason for reading a proof was to verify that the theorem was true while others specifically emphasized that they did not read a proof to obtain conviction of a theorem’s truthfulness. However, there was one commonality in all nine responses. All of the interviewed mathematicians claimed that reading published proofs provided them with new ways to think about mathematical problems that were of interest to them. These new ways of thinking were sorted into two categories – reconceiving mathematical domains and learning new proof techniques. I discuss the latter source of insight in a later section and describe how proofs can lead to new ways of conceiving mathematical domains here.

When mathematicians work within a mathematical domain, they usually have images of the concepts (in the sense of Tall & Vinner, 1981) that they are studying. In earlier papers, my colleagues and I stressed the importance that these images had in the construction and comprehension of proofs (Weber & Alcock, 2004, 2009; see also Raman, 2003). However, we did not discuss another use that these concept images could have – suggesting what mathematical assertions are likely to be true. The mathematicians that I interviewed argued that reading the proof of another enabled them to refine their intuitions and images about the concepts they were studying. In the excerpt below, I asked M3 what he hoped to gain out of reading the published proofs of others. In his response, he mentioned his purpose when reading a proof was to “look for insight”. I asked him to clarify what “insight” meant for him:

> M3: What you want to understand, among other things, is why the person who proved it thought it might be true. I have in mind that I was reading recently, that after reading the proof I still don’t understand why the fellow believed that it would be right. He proved it in a way that one would prove it if one expected it; *I mean would have done approximately*

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> *the same thing* … The statement isn't one that I would have immediately attempted to prove; it just didn't particularly look likely, and I wouldn't have … To find the proof will take a certain amount of time, you know what to do, but if you don't believe the statement then you go into it five steps and then it will break down, more or less, so what's the point? (emphases added)

Like M3, several mathematicians mentioned that when reading the proof of a theorem that they considered counter-intuitive or unlikely to be true, they would do so with an eye toward understanding why the author believed the claim was true. In the excerpt above, M3 describes one example (in which he was not able to obtain this insight) where the proof of a theorem he read was fairly routine, but he would not have invested the time to try to write this proof since his concept images suggested the theorem was not likely to be true. This tendency is also illustrated by the excerpt below, in which M8 describes his desire to understand how his productive colleagues are thinking.

> M8: If someone is proving a lot of interesting theorems, one thing that I try to get out of his proofs is I try to understand how he [sic] is understanding things. I want to know how he [sic] is thinking. If I can think in the same way, maybe I'll discover some interesting theorems too.

The issue raised above was also discussed in Thurston's (1994) essay on proof and progress in mathematics. Thurston claims that when he proved a significant result, what mathematicians most wanted and needed from him were his ways of thinking about the area of mathematics that enabled him to develop his proof; mathematicians valued his mental models more than the proof itself (p. 376). Thurston also notes that when a mathematician proves an important theorem, he or she usually makes a number of other important contributions in that area in the near future. He suggests the most important reason for this is that "a mathematical breakthrough usually represents a new way of thinking, and effective ways of thinking can usually be applied in more than one situation" (p. 372).

In some cases, the proofs the mathematicians read were used to radically alter the way they conceptualized the domain they were studying.

> M1: [discussing reading proofs] Now there are other proofs where you just don't pay attention to two thirds of the proof, and then an idea flashes forward and you just ignore the rest of the argument… you say "here's an idea, I didn't have that idea, that's an interesting idea". But not all of that is embedded in proofs. I remember a professor when I was undergraduate saying that one of the ideas of the 20th century was that functions could become points in a space. And I remember that a good deal more than the various proofs that he did that day in the classroom that may have illustrated that a little bit.

Many of the theorems in functional analysis cannot be justified using the representations of functions that students typically develop through their undergraduate mathematics curricula. Justifying – and in some cases, formulating – these more complicated theorems requires a new way of thinking about functions. One such way is thinking of functions as points in a functional space. The proofs observed by M1 introduced him to a powerful new way of thinking about functions that was indispensable for formulating and proving theorems in functional analysis.

Finally, at several points, the mathematicians in this study argued that reconceiving mathematical domains should be an explicit goal of instruction. In the excerpt below, M1 discusses how some proofs in linear algebra may require students to view vector spaces not just as arrays on which calculations can be performed, but more abstractly:

> I: Are there any proofs in your courses that you would consider a must-see?

> M1: In the introductory course where Linear Algebra means looking at vector spaces of R^n viewed as column vectors, then the notion of linear independence and dependence they find somewhat hard but they can convert it to an immediate computational task, namely taking your finite set of vectors, making a matrix out of it, and then calculating the reduced echelon form of that matrix. That's what happens roughly in the first half of the course. But the second part of the course, where we start talking about eigenvectors and eigenvalues, then the complexity is at a completely higher level, because now these are things that cannot be calculated by rational arithmetic. And so the key point one uses is that if you have a set of vectors, each of which is an eigenvector for a different eigenvalue of a matrix, then that set of vectors is linearly independent. So this is something that they cannot prove by the method they've used previously because they don't have those vectors given as column vectors and they can't apply this row-reduction algorithmic aspect, so they need an idea … So to me, in the Linear Algebra course, that's the first must-see proof … because that one to me is a point where they have to come up with a new way of looking at the subject and that the previous algorithmic approach is not sufficient to give them the answer.

> I: Oh, so that proof is introducing them to a new way of thinking.

> M1: A new way of thinking, absolutely. So that's why … in Linear Algebra, I consider that the first must-see proof.

The distinction between explanation and new ways of thinking

Mathematics educators have long argued that proofs should not only convince students that a proof is true but also explain why it is true (*e.g.,* Hanna, 1990; Hersh, 1993). Steiner (1978) defines an explanatory proof as one that "makes reference to a characterizing property of an entity or structure mentioned in the theorem, such that from the proof it is evident that the result depended upon the property" (p. 143). He operationalizes the notion of characteristic property by noting that the proof would fail if this property were not in place and, crucially, the proof can be deformed by holding the proof idea constant but substituting different characteristic properties to produce new theorems. Hanna (1990) introduced the mathematics education community to Steiner's construct and others have used this as an analytical tool for their research (*e.g.,* Reid, 1995), although most mathematics educators who use the term do not verify that the proofs they find to be explanatory can be deformed to prove new theorems as Steiner dictates. Some philosophers have questioned the utility of Steiner's definition because it does not apply to all proofs that mathematicians recognize as explanatory (Resnik & Kushner, 1987 [1]). Mancosu (2000) believes that this is because proofs can be explanatory in different respects.

I believe Steiner's notion of explanatory proofs is limited for the purposes of mathematics education for another reason. Steiner treats an explanatory proof as a property inherent in the text of the proof rather than an interaction between the proof and its reader. The issue of whom a proof is explanatory to is not considered in Steiner's definition or, indeed, in most philosophers' treatment of explanatory proofs. Mancosu (2000) suggests that such issues are beyond the scope of philosophy and belong in the domain of psychology or mathematics education. However, to mathematics educators who wish to use proofs to foster understanding, it is crucial to consider who is reading the proof; it is easy to imagine a proof that is explanatory to one student but not another and a good teacher cannot overlook this difference.

I offer the following student-centered definition of an explanatory proof. Mathematical concepts can be represented in a formal representation system where they are defined by a precise definition using logical notation, properties of the concept must be deduced from this definition, and inferences must be based on logical rules, definitions, theorems, and other permissible techniques. Proofs of theorems must be produced in such a formal representation system. However, one can also represent mathematical concepts in other representational systems. Raman (2003) provides the example of representing an even function formally as a function that satisfies the condition $\forall x \in R(f(x) = f(-x)$ and informally as a function whose graph is symmetric around the y-axis. One can reason about concepts and justify properties of concepts within these informal representational systems. Weber and Alcock (2009) refer to such systems as semantic representation systems and define reasoning within such systems to be semantic reasoning. Often, students and mathematicians will use this reasoning as a basis for constructing a formal proof. Weber and Alcock (2004) defined producing a proof in this way as a *semantic proof production.* I conceptualize a proof that explains as a proof that enables the reader of the proof to reverse the connection – that is, this proof allows the reader to translate the formal argument that he or she is reading to a less formal argument in a separate semantic representation system. Mancosu (2000) defines an important function of some explanatory proofs as "a reduction to the familiar" (p. 106). My characterization of explanatory proofs aims to capture this perspective with a focus on the individual who is reading the proof.

I argue that a proof that reconceives a domain of mathematics qualifies as a proof that explains, but it also does something more. It does not enable the reader to map a formal argument to an informal representation system that already exists in the reader's mind. Rather it helps the reader develop a new representational system for thinking about the mathematical ideas relevant to that proof. I believe this is what Thurston (1994, reprinted in Thurston, 1995) had in mind when he claimed that, after he proved a theorem, what mathematicians desired from him was not his demonstration that the proof was correct; rather, it was the mental models that he had developed that allowed him to construct this proof. They wanted to use his proofs to develop the representational systems that Thurston possessed.

The following example might make this point clear. Raman (2003) observed mathematicians proving that the derivative of an even function was an odd function. One mathematician's proof was based on the idea that since the graphs of even functions were symmetric with respect to the y-axis, the tangent lines to the graph at x and $-x$ would be mirror images of each other with respect to the y-axis. Therefore, the slopes of these tangent lines would have the same magnitude, but different signs. A proof that made this reasoning transparent could be a proof that explains, assuming that the reader of this proof already understood the standard graphical representations of even functions and derivatives. The key to the construction of this proof was to recall and coordinate conventional ways of thinking about the relevant concepts. In contrast, a proof that reconceives a mathematical domain offers the reader a new way of thinking about these concepts. Note that it is a direct consequence of these descriptions that a proof that reconceives a domain of mathematics for a student may merely be a proof that explains for a teacher and could conceivably be a purely formal demonstration for someone else that does not know the subject.

Proofs that introduce new methods

A second type of insight that proofs provided some participants was new methods that they could use to approach problems that were challenging to them.

> I: What do you hope to gain when you read [someone else's published proof]?
>
> M4: Okay. Two things. One is I would like to find out whether their asserted result is true... And that might help me, if it's something I'd like to use, then knowing it's true frees me to use it … More importantly, I want to understand the proof technique in case I can use bits and pieces of that proof technique to prove something that the original author hasn't yet proved.
>
> I: What do you hope to gain out of reading these proofs?
>
> M5: As a researcher, I want to understand the idea of the proof and to see if that idea could be applied elsewhere.

I: The point that you made about ideas is something that I've been hearing from your colleagues too. Can you elaborate on that?

M5: Sure. Sometimes when a mathematician answers a hard question, he has a new way of looking at the problem or a new way of thinking about it. As a researcher, when you see this, sometimes you can use this idea to solve problems that you are working on. Let me give you an example. We were having trouble showing bounds for approximation techniques on this space with an unusual norm. Someone realized that you could use this particular partial differential equation to find these bounds. This new idea made a lot of the other problems easier. The idea wasn't easy. It wasn't obvious at all that this partial differential equation was relevant. That was a great insight. But once we had the idea, it allowed us to approach questions that were inaccessible before.

I: So after this theorem came out, a lot of other theorems were proved using this idea?

M5: Oh yeah. But it doesn't always have to be big things … Sometimes when I read a proof, I get an idea that helps me get around a little thing that I was stuck with.

These mathematicians' comments are consistent with Rav's (1999) provocative essay arguing that proofs rather than theorems are the bearers of mathematical knowledge because proofs reveal methods and strategies that are applicable in multiple contexts. As an example, Rav claims the benefit of Euclid's proof that there are infinitely many primes was not the result, but the strategy of proving that a set is infinite by assuming the set is finite and then using the elements of this supposedly finite set to generate a new element of that set – a strategy used to prove there are infinitely many primes of the form $4k + 1$, for instance. Other philosophers have made similar claims (*e.g.,* Rota, 1997). For instance, Bressoud (1999) argues that challenging conjectures require mathematicians to refine their existing methods or create new ones. "The value of a proof of a challenging conjecture should be judged, not by its cleverness or elegance, or even its 'explanatory power', but by the extent to which it enlarges our toolbox" (p. 190).

Reading proofs to find new methods is also consistent with the fact that mathematicians often reprove theorems and value multiple proofs of the same theorem (*e.g.,* Avigad, 2006; Dawson, 2006), a practice that seems irrational if one believes the primary purpose of proving is to provide conviction. Dawson (2006) argues that benefits of a new proof of an old theorem include demonstrating the power of different methodologies and discovering new routes to the theorem. "As in mountaineering, being first to the summit is not the only worthwhile goal. It is also exciting to discover a new route to reach it" (p. 279). Similarly, Avigad (2006) contends new proofs of old theorems are valued because "praise for the proof can be read, at least in part, as praise for the method" (p. 107) and new proofs often reveal new methods.

Some mathematicians also mentioned that introducing students to new methods was an important reason for presenting proofs in their classrooms. In the excerpt below, M6 and I were discussing a proof that $\sqrt{2}$ was irrational and I asked him what he hoped students would gain from reading it.

I: What do you think a student would hopefully gain from reading this proof?

M6: [long pause] That's a really tough question. I haven't really carefully thought about this. I would say that you build up your family of examples of proofs, and you sort of have this foundation of a bunch of examples of proofs that you then get to draw on later on. You know, I'm not exactly sure if I can encapsulate exactly what's in this proof. Certainly proof by contradiction is such a thing, but then, you know, arguments based on parity of numbers, which are also really common … So that's another thing, another tool that illustrates … right.

His response indicates that a benefit of reading this proof was for students to be exposed to several reasoning methods within a proof, including proof by contradiction and reasoning based on parity of numbers, with the hopes that they could apply this in proofs they write in the future. When asked what it meant for students to understand a proof, nearly all participants mentioned having students apply the method used in that proof to prove a different theorem. In the case of the $\sqrt{2}$ is irrational, many participants believed a student who understood the proof could prove $\sqrt{3}$ is irrational.

An example of proof construction developing a new way of thinking

My colleague, Carolyn Maher, conducts longitudinal studies in which she videotapes students solving challenging mathematical problems and formulating justifications for their solutions. One example, described in Maher, Muter, and Kiczek (2007), illustrates how the search for and production of a proof led students to develop a powerful way to conceptualize combinatorics problems. The episode that I discuss occurred in tenth grade when five students – Michael, Ankur, Brian, Jeff, and Romina – were trying to determine how many pizzas could be formed if there were five toppings to choose from. Michael's classmates represented the pizzas by representing each pizza in a variety of ways with a subset of the integers 1 through 5, where each integer indicated the presence of one of the toppings. They attempted to list every possible pizza (in other words, list all possible subsets of the set {1, 2, 3, 4, 5}) and generated 31 pizzas, including a plain pizza with no toppings.

Rather than trying to count each of the pizzas, Michael developed a powerful, novel way of conceptualizing the problem. He represented pizzas using binary notation. Below is an excerpt of Michael discussing his ways of thinking with his classmates:

> I think it's 32. I'll tell you why … You know like a binary system we learned a while ago? … The ones would mean a topping; zero means no topping. So if you had a four topping pizza, you have four different

> places in the binary system … You've got four toppings. This is like four places of the binary system. It all equals 15.

Michael's idea was that one can represent each four topping pizzas as a string of four digits where each digit is a zero or a one. Each slot in this four-digit number is associated with a topping; a 1 in that location means that topping is on the pizza while a 0 means it is not. (For example, if the first location is associated with onions, 1000 would represent a pizza with only onions while 0111 would represent a pizza with all toppings except for onions). When Michael indicates "it all equals 15", this is because the binary number 1111 equals 15 in decimal notation. Like his classmates, he considers the plain pizza (represented by 0000) as a special case. If the plain pizza is included, you can add one to 15 to obtain 16 pizzas. Later in Maher's longitudinal study, Michael and his classmates were able to adapt Michael's binary notation to solve a number of other related combinatorics problems, some of which were very difficult (see Maher, Muter, & Kiczek, 2007 [2]).

This episode illustrates how the construction of proofs can allow students to reconceive mathematical domains and generate new proving methods. Michael and his classmates discovered they could recognize many combinatorics situations as sequences of binary strings, which enabled them to solve combinatorics problems more efficiently and notice connections between combinatorics problems with different surface features. They also learned a method for solving a particular type of combinatorics task – namely if there is a one-to-one correspondence between a string of n 0's and 1's and the objects you are attempting to count, then the total number of possible objects is the binary number represented by n 1's plus one to account for the binary number 0.

Discussion

The goal of this paper is to investigate what we might want students to learn from a proof by exploring how mathematicians gain insight from a proof. Similar to the results in Weber (2008), all nine mathematicians that I interviewed in this study desired insight from the proofs that they read. However, the insight they sought often went beyond explanation, at least in the sense that I have defined it in this paper. The mathematicians also sought ways to reconceive the domains they were studying and methods that might be useful for the problems they were researching. Importantly, they listed these benefits among the reasons that they presented proofs to students in their teaching. Further, section 5 illustrates how students can potentially gain these types of insights from the proofs they produce themselves.

This suggests that when proofs are presented in mathematics classrooms, it is important to go beyond explanation and justification. Hanna and Barbeau (2008) note that although many mathematics educators have cited Rav's (1999) paper on how the purpose of proof is to reveal new methods, these references are usually used in support of arguments downplaying the role of formalism in mathematics and emphasizing the dynamic social role in obtaining reliability. I concur with their suggestion that an important, but unexploited, purpose of proof in mathematics classrooms is to introduce students to new methods they can use in their own problem solving. Hanna and Barbeau (2008) provide illustrative high school proofs that might be used in this manner. Similarly, I also believe some proofs can help students refine their images of important mathematical concepts and domains. A complete discussion of how this might be done is beyond the scope of this paper, but areas where this may be considered are proofs that establish difficult geometry conjectures by representing geometrical situations algebraically as points and lines in the Cartesian plane, justifying curious arithmetic tricks using high school algebra, number-theoretic proofs that represent arbitrary integers as products of prime numbers, and representing combinatorics situations as binary strings. Proofs of this type are present in the high school and college mathematics curricula and, if exploited effectively, have the potential to expand the ways in which students represent fundamental mathematical ideas.

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Notes

[1] See also D. Sandborg's (1997) unpublished doctoral dissertation, *Explanation and mathematical practice,* Pittsburgh, PA, University of Pittsburgh.

[2] See also A. Powell's (2003) unpublished doctoral dissertation, *So let's prove it!: Emergent and elaborated mathematical ideas in the discourse and inscriptions of learners engaged in a combinatorial task,* New Brunswick, NJ, Rutgers University.

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