Some Remarks on Understanding in Mathematics*

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The present paper is composed of two parts The first deals with the notion of understanding in general. It is proposed to conceive of understanding as an act (of grasping the meaning) and not as a process or way of knowing The notion of meaning is also discussed. Relations between the notions of understanding and epistemological obstacle are found; it is argued that understanding as an act and the act of overcoming an obstacle can be regarded as complementary images of the same mental reality Further, a categorization of acts of understanding is presented, inspired, on the one hand, by the philosophies of Locke and Dewey, and, on the other, by the UDSG model for learning mathematics [Hoyles, 1986, 1987] Finally, a method for elaborating epistemological studies of mathematical concepts, inspired by the philosophical hermeneutics of Paul Ricoeur, is suggested In the second part of the paper this method is tried on the example of the concept of limit of numerical sequence, which the author has studied elsewhere from the point of view of epistemological obstacles [Sierpinska, 1985_1, $1987_{1,2}$].

I WHAT DOES IT MEAN "TO UNDERSTAND?" GENERAL CONSIDERATIONS

1. Questions related to the meaning of the notion of understanding

"Understanding" is a fairly common word in mathematics education. Several uses of the word can be distinguished. In the practice of teaching, "Do you understand?" is quite often just another way of saying: "May I go on?". In research, "understanding" is sometimes assumed to be a well-defined notion and appears as an ideal to be attained by the students. The main goal of elaborating teaching designs, projects, new software and textbooks is to promote better understanding of the subject matter in students Sometimes understanding is the goal of learning (as in "from doing to understanding", as formulated by Celia Hoyles [1987]). Sometimes it is its precondition (as in the phrase "learning without understanding", cf. Rosnick [1980]; Hejnỳ [1988])

Other researchers regard understanding as a method of study, like in the so-called "interpretive approach" to research in mathematics education (Carr *et al.* [1986] Chapter 3: The interpretive view of educational theory and practice).

But understanding can also become an object of study in mathematics education [Byers, 1985; Gagatsis, 1984, 1985; Davis, 1980; Duval *et al*., 1984; Hejnỳ, 1988; Herscovics, 1980, $1989_{1,2,3}$; Krygowska, 1969-1970; Matz, 1980; Minsky, 1975; Skemp, 1978; Sierpinska, 1990]. The questions we ask are: how do children come to understand mathematical notions? Why don't they understand (what are the obstacles to understanding)? What is understanding? What does it mean to understand (a mathematical notion, a mathematical text, a mathematical discourse)?

In this last question, "understanding is no more a common word. It becomes a scientific term that calls for being clarified, made precise.

This is not an easy task. The notion of understanding has been a constant concern of philosophers, but from Locke and Hume to Dilthey, Husserl, Bergson, Dewey, Gadamer, Heidegger, Ricoeur, Schütz and Cicourel, views and contexts have varied enormously.

The situation is not much better in mathematics education, and one aim of Skemp's famous paper on "Instrumental and relational understanding" [1978] was to draw the attention of researchers to this fact. Since 1978 a lot has been done to clarify the idea of understanding in mathematics by Skemp himself, Herscovics, Bergeron, and others. However, can we say that there exists a consensus concerning the following questions?

- Q1. Is understanding an act, an emotional experience, an intellectual process, or a way of knowing?
- Q2 What are the relations between understanding and: knowing, conceiving, explaining, sense, meaning, epistemological obstacle, insight?
- Q3. Are there levels, degrees or rather kinds of understanding?
- Q4. Are: understanding a concept, understanding a text, understanding a human activity and its products, different concepts, or are they just special cases of the general concept of understanding?
- Q5. What are the conditions for understanding as an act to occur?
- Q6. What are the steps of understanding as a process?
- Q7 How do we come to understand?
- Q8 Can understanding be measured and how?

In the sequel, mainly questions Q1-Q3 will be considered. These considerations will further serve as a starting point for proposing a method of epistemological analysis of mathematical concepts.

2. Is understanding an act, an emotional experience, an intellectual process, or a way of knowing?

Understanding can probably be regarded as an act as well as a process, and the choice between the two is not a matter for

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argument but of focus. We have all experienced those sudden illuminations when the solution to what we thought an unsolvable problem appeared clearly and plainly before our astonished eyes* Reports of such experiences in famous mathematicians, scientists and poets can be found in Hadamard's "Essay on the psychology of invention in the mathematical field." [1964] But, for Hadamard, these stories were just proofs of the importance of unconscious work for mathematical (and other) inventions Poincaré was suddenly illuminated by the solution of a problem when he was not consciously thinking about it. But he had spent a lot of time on unsuccessful attempts, analysing the problem and trying different solutions.

If we focus our attention on this long period of, first, conscious and then unconscious mental work, then the idea we ourselves make of understanding is that it is a process which can be rewarded, eventually, by a moment experienced as an "illumination"

In psychology one of the key problems is how we understand what we are told — how we understand the information which is communicated to us by our interactions with other people or by media. Understanding a sentence in our mother tongue is always very quick (if we admit that recognizing a sentence as being incomprehensible to us is also a kind of understanding) so it may be regarded as an act even in a child The rapidity of understanding is not a discriminating property. What counts is the quality or "level" of understanding. This level changes with the growth of knowledge, the complexity and richness of its structure. As we focus on the changing level of understanding, we think of understanding as a process and not as an act: "The continuous development of cumulated knowledge, stored in our memory system, strongly influences the way new information is assimilated. It also strongly discriminates between the coding of information in a child's memory and the coding of the same information by an adult. In a child, new concepts must be built up in empty spaces. The initial stage of construction of a data bank is necessarily linked with huge amounts of information memorized mechanically. *Understanding is achieved slowly*, along with the accumulation of properties of objects, examples and development of concepts concerning relations between classes of concepts. At the beginning, concepts in the memory are generally only partially defined and weakly related to other stored information. In later years when the resources of information are rich and organised in a data bank built on an elaborate system of criss-crossing connections, the character of learning changes New concepts can be assimilated mainly on the basis of analogies with what is already known. The main problem lies in incorporating the new concept into the existing structure. When the relation is established, all the previous experience is automatically included into a fuller interpretation and understanding of new situations." [Lindsay, 1984, p 438]

However, the quality of understanding need not grow with age; understanding does not depend solely on the richness of accumulated experience, information, and highly elaborate structuralisation of the "data bank". Lindsay and Norman [op cit.] speak about a mechanism which explains the existence of epistemological obstacles in our ways of knowing: "Very seldom does an adult meet with something completely new, unrelated to his or her conceptual structure (...) Even if the received information is in obvious contradiction with previous experience, his or her conceptual structure, which constructed such a complicated system of interrelations, stands against any revision. And thus an adult prefers to reject inconsistent information or change its meaning rather than rebuild the system of his or her convictions." [ibidem, p. 439]

Understanding is an act in Gestaltpsychologie where the influence of the idealistic conceptions of Husserl and Bergson can be clearly felt. Understanding is thus an act of mind which consists in a direct perception of the "essence des choses". This act "is not prepared by a preparatory analysis of relevant relations between elements of a problem situation. These relations are perceived directly, like the sensory properties of objects." [Tichomirov, 1976, p 45]

In the frame of Husserl's theory of the intentionality of meaning (Bedeutungs-intention), mental acts of understanding a sign are directed towards some object; this object is called the *meaning* of the sign. Meaning is an ideal object, i.e. it belongs neither to the physical nor to the mental reality (The existence of ideal objects is justified as follows: take, for example, number 4 True as well as false statements can be made about this number. E.g. $4=2^2$ is a true statement. Truth is in conformity with reality. Therefore if something true can be said about the number 4, it must belong to some reality. This reality is neither physical nor mental Therefore it must be some third kind of reality, let us call it the domain of ideal or abstract objects) [Mała Encyklopedia Logiki, 1988, p 233].

Neopositivism in philosophy and behaviorism in psychology define understanding as a kind of reaction to stimulus (so it is an act rather than a process). In understanding concepts, the stimulus is the name of the concept. The word "meaning" has no designation in any reality (even ideal reality) because it is not a name even though its grammatical form gives this impression It cannot be considered outside of expressions like "x has a meaning" or "x and y have the same meaning". The last expression means that x and y stimulate the same behaviour [Mała Encyklopedia Logiki, 1988, p 234].

Understanding as an experience has been considered by Dilthey not in the context of understanding concepts but in that

*I myself have experienced such illuminations with respect to the notion of understanding First, when I identified "understanding" as an object worthy of study Of course the notion of understanding was always there in my research on epistemological obstacles; an epistemological obstacle was an obstacle to understanding But it was in the background, as a common term, tacitly admitted as a clear and unproblematic everyday concept And then, suddenly, it became the "figure" not the "ground".

A second illumination came after long discussions I had with Ewa Puchalska on "levels" "degrees" and "measuring" of understanding in connection with Herscovics and Gagatsis' papers It came to me that understanding a concept could be measured by the number and quality of epistemological obstacles related to it that one has overcome.

In both cases the illumination came after a period of mental trouble or conflict. In the first case I had strong difficulties in describing epistemological obstacles in two girls, Kate (10) and Martha (14), confronted with problems of equipotence of infinite sets [Sierpinska, 1989_3] Kate seemed to have no epistemological obstacles but obviously did not understand Martha had to fight with many epistemological obstacles but I did not hesitate to say that she understood the problem The second case was concerned with the mismatch between my conviction that understanding is a two-valued function (either you understand a concept or you do not) and the results of the existing research on understanding in mathematics.

of human activity and its products, i.e. in the context of theories of humanistic interpretation. Humanistic interpretation is the attribution of sense to human activity (and its products). According to Dilthey, this attribution of sense is made by means of an experience called "understanding" (Verstehen). This sense is a value which, in its turn, is the goal to be attained by the activity. Dilthey conceives of understanding as purely intuitive and even preconceptual: it is not based on establishing logical connections between the given phenomenon and its sense, but rather in grasping the phenomenon and its sense together directly [*Filozofia a Nauka*, p. 265, 408-411; Dilthey, 1970; Krasnodębski, 1986, p. 75]

Were we concerned only with understanding mathematical concepts we might disregard Dilthey's point of view and forget about his theories. But as we are interested in understanding mathematics in the context of classroom interactions between teacher and students, where the student has to grasp not only the meaning of concepts but the sense of the teacher's and his/her own activities as well, understanding as a method of humanistic interpretation is not to be neglected.

For Dewey [1988; first published in 1910], "to grasp the meaning", "to understand", "to identify a thing in a situation where it is relevant", all mean the same. All these expressions define "the fundamental *moments* of our intellectual lives [op cit. p. 152]". Therefore Dewey seems to be conceiving of understanding as an act. However this is not the intuitive and preconceptual act of Dilthey; at least, not all acts of understanding are of this kind [see p. 7] In its more sophisticated forms, understanding is a result of a thinking process and in fact the goal of all knowledge: "All knowledge, all science endeavours to grasp the meaning of objects and events and this process always consists in stripping them of their individual character as events and discovering that they are parts of a bigger whole that explains, clarifies and interprets them, thus providing them with meaning [p. 152-153]". This way, explanation, which Dilthey opposed to understanding, becomes for Dewey a means to understanding.

Understanding and explaining are even more deeply reconciled in the conception of interpretation (of discourse or text) presented by Ricoeur [1989]: "For the purposes of a didactical presentation of this dialectic of explaining and understanding as phases of a specific process, I propose to describe this dialectic as the passage, first, from understanding to explaining, and then from explaining to comprehending. At the beginning of this process understanding is a naïve grasping of the meaning of the text as a whole. By the second stage, as comprehending, it is an elaborate way of understanding based on explanatory procedures. At the beginning understanding is making a guess; at the end it becomes consistent with the notion of 'appropriation', which we characterized above as a reaction to a kind of distance, a strangeness, which is the result of the full objectivization of the text. In this way explaining appears as a mediator between two phases of understanding. Explanation considered outside a concrete process of interpretation is but an abstraction, a product of methodology [op cit. p. 160-161]"

So the process of understanding starts with a guess which we further try to justify and validate. In the course of validation the guess may be improved, changed, or rejected. The new guess is then subjected to justification and validation. The spiral process continues until the thing to be understood is considered to have been appropriated

However, in this complex dialectic, understanding is again an act. On the other hand, explaining is a process: "I assume that while, in the process of explaining, we develop a series of statements and meanings, in the course of understanding, a chain of partial meanings are related and composed into a whole in a single act of synthesis [op.cit p. 157]".

Ricoeur's model is concerned with the interpretation of literary texts. This is not seen in the excerpts we have quoted above. The specificity lies in the way Ricoeur conceives of the procedures of explanation [op cit. 161-179]. If we let the explanatory procedures be a variable in the model, it generalizes to a model of understanding any text. It is probably not as easy to generalize from "text" to any "discourse" (verbal or written) because, while in the interpretation of a text the validation of the guess is made on the basis of the same material (the text is reread and analysed), in the spoken discourse the validation of the guess is developed in the course of a dialogue in which new pieces of discourse are introduced and have to be understood. It would be even more difficult to modify this model so that it covers the understanding of (mathematical) concepts, because the understanding of a concept is not normally reached through reading a single text. It demands being involved in certain activities, problem situations, dialogues and discussions, and the interpretation of many different texts. Let us keep, then, from Ricoeur's model just the general idea of the dialectic between understanding and explaining, starting with a guess and developing through consecutive validations and modifications of the guess. Presented in this way Ricoeur's model strikes us with its similarity to the Lakatosian model of development of mathematics through a chain of proofs and refutations [Lakatos, 1984].

I propose, then, to regard understanding as an act, but an act involved in a process of interpretation, this interpretation being a developing dialectic between more and more elaborate guesses and validations of these guesses

3. What are the relations between understanding and knowing?

Skemp [1978] defines "understanding" in terms of "knowing". "Instrumental understanding" means "to know how", and "relational understanding" means "to know not only how but also why". In the article, "instrumental" and "relational" are qualifications not only of understanding but also of thinking, mathematics, teaching and learning. They are used as names of styles.

Can we speak of styles of understanding if we conceive of it as an act?

The words "understanding" and "knowing" are often closely associated in literature. Do they mean the same?

Under the title "An essay concerning human understanding" John Locke discusses the notion of knowledge, its different "sorts" and "degrees".

Dewey [1988] distinguishes between two kinds of understanding, and says that in many languages they are expressed by different groups of words: "some denote direct appropriation or grasping of meaning, others a roundabout understanding of meaning; for example: *gnōnai* and *eídēnai* in Greek, *noscere* and *scire* in Latin, *kennen* and *wissen* in German, *connaître* and *savoir* in French; in English the corresponding expressions are: *to be acquainted with* and *to know of* or *about*. Our intellectual life consists in a particular interaction of these two kinds of understanding [p. 154]".

Thus in spite of conceiving understanding as an act, Dewey, like Skemp, defines kinds of understanding by ways of knowing.

How can we explain this?

Perhaps understanding is an act, but this act brings about a new way (or style) of knowing. Understanding as an act appears in expressions like: "Oh, I understand now!", or: "Oh, I see!". Understanding as a way of knowing (manière de connaître) in, e.g., "I understand it this way ..."

If we stick to conceiving understanding as an act, we may say that Skemp classifies acts of understanding according to the styles of knowing they produce. And Dewey classifies acts of understanding into direct (which he further calls apprehensions) and indirect (comprehensions: those that have to be consciously prepared). These different kinds of acts of understanding lead to different ways of knowing: gnōnai, noscere, kennen, connaître, to know; or: eídēnai, scire, wissen, savoir, know that.

4. What are the relations between "understanding", "sense" and "meaning"?

"Sense" is often used as a synonym of "meaning", but let us consider the following two sentences:

(a) "I know what it means now"

(b) "It makes sense to me now".

For the boy in Skemp's article [1978], multiplying length by breadth to get the area of a rectangle was obviously a sensible activity, although he was unable to grasp the meaning of the rule. He knew *why* he was multiplying: in doing so he got all his answers right, and this certainly is a highly valued goal of an activity. He might also have used multiplication "because the teacher said so". To satisfy the teacher is another goal worthy of effort in a young student's life.

On the other hand, knowing the meaning of a procedure does not imply its making sense for us.

The main difference between sentences (a) and (b) is that (a) refers to something objective (the meaning), and (b) tells us about a subjective feeling of the speaker.

Perhaps we should not be satisfied as teachers with our students "understanding" their tasks only in this subjective sense, but certainly all that we ask the students to do should make sense to them.

But "sense" may also have an objective meaning; for example, when we ask: "In what sense are you using this word?" The explanation is normally given by an example of a more common use of this word: a sentence in which this word is used.

The sentence gives *sense* to the word by placing it in a structure which defines the function of the word.

The structure of the sentence *is the sense* in which the word is used. But the sentence also *refers* to something, denotes something, states something, that can be true or false in some reality. And it is the sense of the sentence *together with* its reference that make the meaning of the word.

While sense is considered within the language, reference transcends it and forces us to decide ontological questions.

The principle that the meaning of names should never be considered outside sentences, as well as the distinction between sense (*Sinn*) and denotation or reference (*Bedeutung*), are attributed to Frege [1967; cf. Ricoeur, op. cit. p. 89]. His ideas have been developed and formalized in logical semantics by Church [Mała Encyklopedia Logiki, p. 233], but the above presentation of the duality of sense and reference in meaning is based on Ricoeur's interpretation of "Über Sinn und Bedeutung" from the point of view of philosophical hermeneutics [Ricoeur, op. cit. p. 89-91].

Ricoeur defines the sense of a sentence as an answer to the question: what does the sentence say? Reference tells us what the sentence is about. Let us consider the following sentence to see better the difference between its sense and its reference: "The sum of internal angles in any triangle is equal to two right angles".

The structure of the sentence is as follows:

	is equal	
the sum of internal angles in any triangle ↗		↖ to two right angles

Sense. The sentence states the equality of two objects.
Reference. The sentence is true in the reality of certain ideal objects — triangles, defined in a theory called Euclidean geometry (to be distinguished from non-Euclidean geometries). A triangle is ... An angle is ..., etc.

5. An idea of a method for the epistemological study of mathematical concepts

Ricoeur's considerations have a methodological value: the distinction between sense and reference is directly linked with the way he discriminates between semiotics and semantics. They can also inspire the search for a method of elaborating epistemological analyses of mathematical concepts.

Suppose we start from the informal language of mathematics. Let us find words and expressions used in defining, describing, working with the concept we are analysing. Let us then find sentences which are the senses in which these words and expressions are used. Then let us seek the references of these sentences. And then seek relations among all these senses and references.

This analysis can lead us to a description of the meaning of the concept in question (at a certain level, depending upon the degree of analysis we have made). Understanding the concept will then be conceived as the act of grasping this meaning.

This act will probably be an act of generalization and synthesis of meanings related to particular elements of the "structure" of the concept (the "structure" being the net of senses of the sentences we have considered). These particular meanings also have to be grasped in acts of understanding.

What are these acts? Are they always syntheses and generalizations? Maybe there are some other kinds of acts of understanding. We shall deal with these questions in section 7.

6. What are the relations between the notions of understanding and epistemological obstacle?

All our understanding is based on our previous beliefs, prejudgements, preconceptions, convictions, unconscious schemes of thought. Claiming that we can do without these or are able to get rid of them is an epistemological daydream [cf. Descartes, Dilthey, Husserl]. However if we discover that our understanding is erroneous we then use ugly names for the same kinds of things, calling them myths, prejudices, misconceptions, preconceived opinions, intellectual habits.

All these are ways of knowing.

We know things in a certain way. But the moment we discover there is something wrong with this knowledge (i.e. become aware of an epistemological obstacle), we understand something and we start knowing in a new way. This new way of knowing may, in its turn, start functioning as an epistemological obstacle in a different situation. Not all, perhaps, but some acts of understanding are acts of overcoming epistemological obstacles. And some acts of understanding may turn out to be acts of acquiring new epistemological obstacles.

A description of the acts of understanding a mathematical concept would thus contain a list of the epistemological obstacles related to that concept, providing us with fuller information about its meaning.

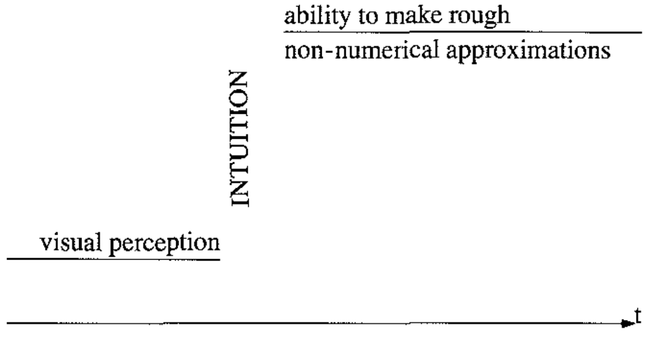
In many cases overcoming an epistemological obstacle and understanding are just two ways of speaking about the same thing. The first is "negative" and the other "positive". Everything depends upon the point of view of the observer. Epistemological obstacles look backwards, focussing attention on what was wrong, insufficient, in our ways of knowing. Understanding looks forward to the new ways of knowing. We do not know what is really going on in the head of a student at the crucial moment but if we take the perspective of his or her past knowledge we see him or her overcoming an obstacle, and if we take the perspective of the future knowledge, we see him or her understanding. We cannot adopt the two perspectives at the same time. Still, neither can be neglected if the picture is to be complete. This looks very much like complementarity in Niels Bohr's sense [cf. Heisenberg, 1989, p. 38; Otte, 1990]: overcoming an epistemological obstacle and understanding are two complementary pictures of the unknown reality of the important qualitative changes in the human mind.

This suggests a postulate for epistemological analyses of mathematical concepts: they should contain both the "positive" and the "negative" pictures, the epistemological obstacles and the conditions of understanding.

7. Are there degrees, levels, or kinds of understanding?

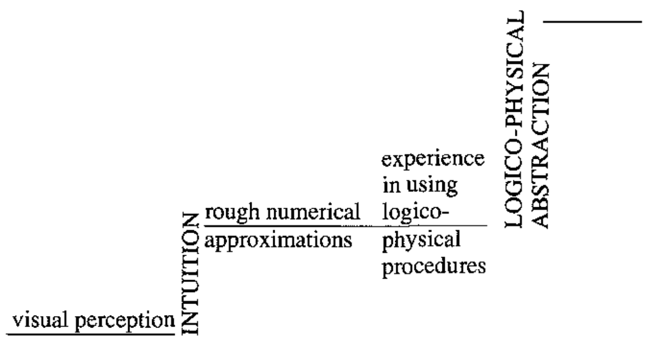
The Herscovics-Bergeron model for understanding mathematical concepts (1989_2) distinguishes three "levels". Two of these levels can be regarded as categories of acts of understanding. The third seems to be rather a kind of knowledge. The two categories of acts of understanding are: intuition and logico-physical abstraction.

Intuition, or "intuitive understanding" as the authors name it, of number is defined as a "global perception of the notion at hand" which arises from "a type of thinking based essentially on visual perception" and results in an ability to make rough non-numerical approximations.



Acts of understanding which belong to the category of logico-physical abstraction consist in becoming aware of logico-physical invariants (e.g. conservation of plurality and position), or of the reversibility and composition of logico-physical transformations, or in generalization (perceiving the commutativity of the physical union of two sets).

These are all acts of understanding and not ways of knowing. However, the reason why they have been divided into two such categories does not lie in the specificity of these acts themselves but in the levels of knowledge from which these acts sprang. Visual perception is what gives birth to "intuition"; rich experience and complex mental operations are required to produce the awareness of logico-physical invariants, the reversibility and associativity of logico-physical transformations, not to mention their generalization. This "rich experience" is named "procedural understanding" and constitutes the third level in the discussed model. Therefore what is classified here, in fact, are the levels of children's mathematical knowledge, not their acts of understanding.



In his "Essay concerning human understanding", John Locke speaks about "degrees" of knowledge. He mentions three degrees, two of which resemble Descartes' types. They are: "intuitive knowledge" (immediate perception of agreement or disagreement between ideas); "demonstrative" or "rational" knowledge (when the mind does not perceive the agreement or disagreement between ideas immediately but after the intervention of other ideas, i.e. proofs); sensory knowledge (knowledge of the existence of particular external objects). Intuitive knowledge is irresistible and certain. Rational knowledge is acquired with pain and attention

Although Locke speaks about "perceptions" as acts, this again is a classification according to the level of intellectual effort that is needed to produce such a perception.

But Locke also speaks of "sorts of knowledge", and this resembles a classification of acts of understanding.

For Locke, "knowledge" is the perception of "connexion and agreement or disagreement and repugnancy" of any of our "ideas". He distinguishes four "sorts" of this "agreement and disagreement" The first he calls "identity or diversity" because "it is the first act of the mind to perceive its ideas and to perceive their difference and that one is not the other", as in: "blue is not yellow". This act of knowing might be called IDENTIFYING IDEAS AND DISCRIMINATING BETWEEN IDEAS It might also be useful to distinguish these as two different sorts of acts of understanding.

The second of Locke's sorts of knowledge is "relation" or "perception of relation between two ideas", as in: "two triangles upon equal bases between two parallels are equal" This is important, Locke says, because "without relations between distinct ideas there would be no positive knowledge". Let us call this sort: FINDING RELATIONS BETWEEN IDEAS

The third sort of knowledge might be called: DISCOVERING PROPERTIES of a complex idea: "co-existence or necessary connexion" in Locke's terminology. This appears in saying, e.g., that "gold is stable", or "gold is resistant to fire", or "iron is susceptible to magnetic influences"

The fourth sort of knowledge is concerned with "the actual real existence agreeing with any idea", as in "God is". Let us call this: FINDING RELATIONS WITH REALITY

Locke's distinctions remind us of models comprising generalizations and discriminations mentioned, e g , in Dewey [1988], and developed in mathematics education by Hoyles [1986].

According to Dewey (and in this he is a forerunner of Piaget), concepts are not abstracted from sensory impressions; the child does not develop the concept of "dog" by abstracting from characteristics such as colour, size, shape, etc , but starts from IDENTIFYING one dog it has seen, heard, touched. Then it tries to transfer its EXPERIENCE with this single object onto other objects, anticipating some characteristic ways of behaving (this, in fact, is GENERALIZATION). Cats become "small dogs", horses are "big dogs". Then comes DISCRIMINATION, distinguishing between properties characteristic of dogs and non-characteristic of dogs. SYNTHESIS does not consist in a mechanical accumulation of properties but in the "application to explaining new cases with the help of a discovery made in one case [op.cit., p. 164-165]".

Experiencing, identifying, generalizing, discriminating, synthesizing, applying, are, according to Dewey, the crucial moments of concept formation but, except for experiencing and applying perhaps, they look like good candidates for the important acts of understanding.

In Hoyles [op.cit.] a model for learning mathematics is presented which is very similar in spirit and terms to Dewey's: "USING — where a concept/s is used as a tool for functional purposes to achieve particular goals; DISCRIMINATING — where the different parts of the structure of a concept/s used as a tool are progressively made explicit; GENERALISING — where the range of applications of the concept/s used as a tool is consciously extended; SYNTHESISING — where the range of application of the concept/s used as a tool is consciously integrated with other contexts of application [op.cit. p. 113]".

8. Categories of acts of understanding

Let us synthesize Locke's, Dewey's, and Hoyles' ideas and try to generate a categorization of acts of understanding a mathematical concept:

> IDENTIFICATION of objects that belong to the denotation of the (or: a) concept (related to the concept in question), or: identification of a term as having a scientific status; this act consists in a sudden perception of something being like the "figure" in the Gestaltist experiments.
>
> DISCRIMINATION between two objects, properties, ideas, that were confused before.
>
> GENERALIZATION consists in becoming aware of the non-essentiality of some assumption, or of the possibility of extending the range of applications.
>
> SYNTHESIS is grasping relations between two or more properties, facts, objects, and organizing them into a consistent whole

Of course, the necessary condition for all these acts to occur is experiencing, using, and applying: "If we behave passively towards objects, they remain hidden in the confused blotch which absorbs them all [Dewey, op.cit. p. 159]"

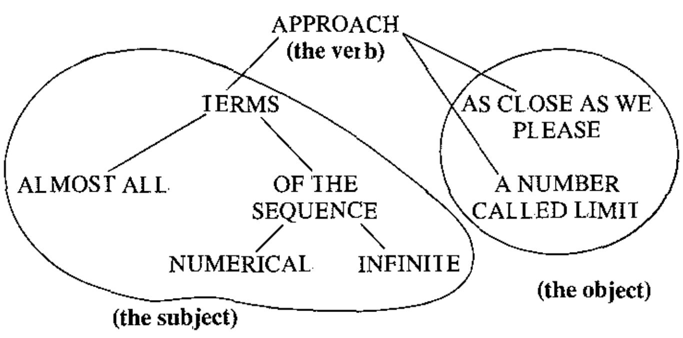
II. WHAT DOES IT MEAN TO UNDERSTAND THE CONCEPT OF A CONVERGENT NUMERICAL SEQUENCE? an example of an epistemological analysis of a mathematical concept

Of the many sentences that can be formulated about a convergent sequence, let us choose this one:

> Almost all the terms of the infinite numerical sequence approach as close as we please to a number called its limit

This sounds a bit artificial but has the advantage of containing all the elements of the *definiens* in the definition of convergent sequences

Let us first consider the logical structure of this sentence:



This structure defines the SENSE of the sentence. It says that something approaches something in a certain way.

Let us now see if the four categories of acts of understanding we have identified above will suffice to produce an epistemological analysis of the concept of convergence of numerical sequences.

The predicate states something general about the subject which points to something individual. "The subject (...) identifies one and only one object. The predicate, on the contrary, points to some quality or class of objects, or type of relation or type of activity [Ricoeur, op.cit. p. 78]" In the case of our sentence the verb, APPROACH, seems to point to an activity; however, an understanding of the concept of limit will lead to the perception that, in fact, it defines a type of relation. But, in saying this, we enter the area of the REFERENCE of the sentence.

Let us then consider the question: what is the sentence about?

1. The subject: *terms of infinite numerical sequence*

The subject refers to the world of infinite sequences. Hence, the first step towards understanding the notion of convergent numerical sequences (abbr. CNS) must be to discover the world of infinite sequences. The first infinite numerical sequence encountered by the child is most often the sequence of natural numbers. Becoming aware that one can count on and on for ever is probably the first act of understanding what an infinite sequence can be. Later, other infinite sequences come as well: odd and even numbers, numbers divisible by three, etc. When students start converting vulgar fractions to their decimal expansions, strange things start to happen and sometimes the division doesn't come to an end. One could go on for ever and ever; another experience with infinite sequences. But these sequences are very special: they are periodic. Unending calculations return when we consider the question of the place of irrational numbers on the number line. The square root of 2 turns out to be "squeezed" between two infinite sequences of rational approximations. Calculation of the areas of figures even as simple as rectangles can give rise to questions leading to infinite sequences. If the sides of the rectangle are commensurable, then the formula "area=length×breadth" is easily explained in the frame of the conception of multiplication as repeated addition. But what if the sides are incommensurable? This demands a reconceptualization of the notion of multiplication. Iterating functions, producing sequences of numbers and sequences of geometrical transformations (possibly with the use of a computer), approximating solutions to equations, maxima and minima, tangents, areas, velocities, are further domains of experience with infinite sequences. (Interesting methods of working within these domains in the mathematics classroom can be found in Hauchart [1987]

This experience can bring about the first act of understanding the notion of CNS: the identification of a new object worthy of study:

U(lim) 1: IDENTIFICATION OF INFINITE SEQUENCES AS OBJECTS WORTHY OF STUDY

2. The predicate: *approach*

The verb is the most important part of the sentence: where there is no verb, there is no sentence. That is why, having entered the world of infinite sequences, the most important thing is to distinguish among them those that "approach" something or "tend to" something or "converge" to something. Sequences that "approach" or "tend" or "converge" must, at some moment, become "the figure" in our picture and all the other infinite sequences the "background". Students who experience this act of identification can be heard exclaiming: "Oh! it comes closer and closer (approaches, tends, converges) but will never become equal (or reach)".

U(lim) 2: IDENTIFICATION OF SEQUENCES THAT APPROACH SOMETHING

This act normally results in the development of a certain sensitivity to convergent sequences. However, this development is not possible without a number of shifts of attention as far as certain aspects of the notions of number and sequence are concerned. In particular, focusing on the form of the numbers, or on the stabilization of decimal digits, or on the rule for generating terms of a sequence, or on the set of terms of the sequence, can all function as obstacles to the identification of sequences that approach something. Below we shall make some comments on these obstacles.

The form we shall use to name an obstacle is analogous to that which Lakoff and Johnson [1980] have used to name metaphors. Metaphors, according to these authors, are symptoms of conceptions and, since obstacles are also based on conceptions, the use of an analogous form of coding is not misplaced, I hope.

Number: NUMBER IS A WRITTEN FORM

This obstacle consists in focusing on the form of a number and not on its value; the length of the writing, the digits used in it, are taken to be more important than the place of the number on the number line or in the sequence.

Having to classify (according to a rule of their choice) the following set of series:

A: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$;	B: $1+2+3+4+\ldots$;
C: $1-1+1-1+\ldots$;	D: $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots$;
E: $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots$;	F: $1+1+1+\ldots$

some students put the series B and E into the same class not because they knew they were both divergent but because in both of them the terms are composed of consecutive natural numbers. A and D were put together in another class, again because the same digits appeared in consecutive terms; the third class comprised C and F because both series are composed of repetitions of the number 1.

The act of overcoming this obstacle amounts to:

U(lim) 3: DISCRIMINATION BETWEEN NUMBER AND FORM OF NUMBER

Another obstacle is:

Convergence: CONVERGENCE IS THE STABILIZATION OF DECIMAL DIGITS

The above conception easily develops if sequences are introduced through excessive use of calculators or computers and the production of decimal approximations to terms. Students may be brought to make unjustified inductive jumps and to believe that if they observe the stabilization of digits after a hundred terms, then this means that the sequence belongs to the class of convergent sequences. Kuntzmann warned against this obstacle a long time ago [1976] when pocket calculators started to be openly used in schools. In my experiments one student displayed an interesting combination of the above obstacle and the previous one (NUMBER IS A WRITTEN FORM). Observing the numerical approximations of terms of a sequence converging to 4 from below (the last terms on the screen were: 3.999998, 3.999999) he would say: "Here, it tends to *nines*". Another sequence was "tending to sixes" (0.666666), and still another to "zeros" (3.000002, 3.000001). In fact, he focused his attention on the form of the approximations, and not on the values of the numbers they were approximations of

U(lim) 4. DISCRIMINATION BETWEEN CONVERGENCE AND STABILIZATION OF DECIMAL DIGITS

Let us now consider the obstacle:

Sequence: SEQUENCE IS A RULE FOR PRODUCING NUMBERS

A numerical sequence is a function defined on the natural numbers with values in the real numbers. This function is normally defined in some way; let us call this definition "rule". A sequence is then a synthesis of its arguments, values, and rule. If our attention focuses on only one of these elements it will create an obstacle. In the obstacle presently discussed the focus is on the RULE. Sequences may then be classified by their rules and not by the mutual relations between the values of their terms. In classifying a set of sequences students are able to put identical sequences into two different categories just because they have different rules [Sierpinska, 1989].

U(lim) 5: DISCRIMINATION BETWEEN SEQUENCE AND RULE FOR PRODUCING NUMBERS

Another obstacle to overcome is:

Sequence: SEQUENCE IS A SET

Here the focus is on the terms of the sequence (the values of the sequence seen as a function). These terms are conceived of as forming a set; their order is not important. It is inessential how the values of the terms change, whether they approach something or not. At best, attention may be attracted to the bounds of the set of terms. For example, in the sequence 1, 1.9, 1.99, 1.999, etc., neither of the numbers 1 and 2 will be more distinguished than the other [cf. Sierpinska, 1987]. This obstacle is, in fact, inherent in the notion of sequence: the denotation of the concept of sequence (as function) is sometimes described by "situations in which the set of values of the function is more important than the function [Maurin, 1977, p. 17]". To overcome this obstacle, one has to make the...

U(lim) 6: IDENTIFICATION OF ORDER OF TERMS AS AN IMPORTANT FEATURE OF SEQUENCES

3. The subject revisited; identification of *terms of sequence* as the subject of *approach*

Novices often treat convergence as a kind of phenomenon which does not call for naming the responsible agent. They may say "it converges" as they would "it rains" or "it snows" with an impersonal "it" [cf. Sierpinska, 1989_2]. Such an attitude towards convergent sequences leaves no room for the question of *what* is converging, in fact

Convergence: CONVERGENCE IS A NATURAL PHENOMENON

Some students, especially in situations where they must calculate a certain number of terms of the sequence, identify the subject of APPROACH with the person who calculates, i.e. with themselves: "*We* are approaching something" as *we* calculate more and more terms. The sequence becomes a sequence of calculations or operations. This raises the question of the infinity of the sequence, puts physical limitations on the arbitrary choice of the epsilon, and further leads to questions concerning the nature of mathematics (is it constructed? discovered?). Of course, one can deliberately choose the constructivist philosophy of mathematics and assume that...

Sequence: SEQUENCE IS A SEQUENCE OF CALCULATIONS

but if this conception is unconscious it functions as an obstacle.

4. Attribute of *sequence: infinite*

A focus of attention on the large number of terms of the sequence (i.e. of the arguments of the sequence seen as a function) is another obstacle:

Sequence: SEQUENCE IS A VERY LONG LIST OF NUMBERS

"Very long" may mean many different things to students [Sierpinska, 1987_2].

It is exactly this focus on the length of the sequence and not on the values of its terms that is the basis on which the paradoxes in "Achilles and the tortoise" and "Dichotomy" are built. The stories are told in such a way that the listener's attention is caught by the infinite number of steps to be made; the diminishing lengths of the consecutive steps are left in the shadow. The number of steps being infinite, it is "obvious" that Achilles will never catch up with the tortoise, and that one can never get out of the room one is in.

Conversely, if the number of steps is not perceived as infinite there is no paradox. This happened in an experiment by B. Caľa [1989] who inquired into the spontaneous explanations of Zeno's paradoxes by 16 year old students of electronics and their reactions to the usual explanations of these paradoxes in terms of summing numerical series. The students were interviewed before and after the introduction of the formal notion of limit in the mathematics class. Neither before nor after this introduction did the students see any paradoxes in the stories about Achilles and getting out of the room. Some students said that after some time the distance between Achilles and the tortoise is so small that it can be neglected: the number of calculations is finite. Other students said that the reasoning is wrong because it neglects the huge difference in the velocities of Achilles and the tortoise: the reasoning is obviously wrong, so no wonder the conclusion is absurd. There was no feeling of paradox.

The feeling of paradox appeared only when the students were shown the explanations in terms of summing numerical series. In the mathematization of "Achilles" the following numbers were taken: $v_A = 20$ km/h; $v_T = 0.2$ km/h. Achilles starts running 9.9h after the tortoise. The time Achilles needs to catch up with the tortoise is then the sum of the series

$$9.9+0.099+0.00099+0.0000099+\ldots$$

and amounts to 10h. Now *this* was the really paradoxical result for the students. They would say that this sum is 9.9999999... and that this is not 10: the two numbers are not equal. And since they got 10h by solving the problem using kinematic formulas they said that the above explanation explains nothing. On the contrary, it proves that Achilles will never catch up with the tortoise since 9.999... just approaches 10 without ever reaching it.

Such an attitude results from, first, a focus of attention on the number of terms of the numerical sequence; second, the conception of sequence as a sequence of calculations; third, the conception of infinity as a certain potentiality to go on further and further in the number sequence.

The two paradoxes of Zeno cannot be solved by the concept of the sum of a series. The mathematization turns out to be equivalent to the paradoxes themselves. They cannot be solved by mathematics at all. The existence or non-existence of an actual infinity is, after all, not a mathematical but a philosophical question. And philosophy does not give definite answers to such questions, it can only discuss the possible consequences of different answers. The Weierstrassian definition of the limit of a sequence in terms of epsilon and N does not solve the problem of whether or not the sequence reaches its limit. Its static and symbolic formulation eliminates this problem from mathematics and makes it senseless to pose it [Sierpinska, 1985_1].

In order to understand Zeno's paradoxes and be able to appreciate the Weierstrass definition one must become aware of all this and consciously consider one's own and other people's conceptions of infinity, their advantages and limitations.

Obviously in understanding limits a particularly dangerous conception linked with infinity is the belief that what is infinite is necessarily unbounded. All convergent sequences, albeit infinite, are bounded. This belief may be linked with a focus on the very large number of terms. It is the shift of attention from arguments to values of the sequence that may lead to the discovery of a "bounded infinity" [Sierpinska, 1987_1; Thomas' story].

Overcoming obstacles related to infinity seems to be a necessary condition for a conscious synthesis of the concept of sequence. This is why we choose the following order of obstacles and acts of understanding:

Infinity: DIFFERENT CONCEPTS OF INFINITY

These conceptions function as obstacles if unconsciously admitted as absolute truths. Below we distinguish one of them as particularly undesirable from the point of view of understanding limits:

Infinity: WHAT IS INFINITE IS UNLIMITED

U(lim) 7: IDENTIFICATION OF DIFFERENT CONCEPTIONS OF INFINITY: POTENTIAL INFINITY, ACTUAL INFINITY, LARGE UNDETERMINED NUMBER, ARBITRARILY LARGE NUMBER, ...

U(lim) 8: IDENTIFICATION OF INFINITE AND BOUNDED SETS

Convergence: THE PROBLEM OF REACHING THE LIMIT IS A MATHEMATICAL PROBLEM AND THEREFORE THERE EXISTS A MATHEMATICAL SOLUTION TO IT

Philosophy of mathematics: DIFFERENT PHILOSOPHICAL ATTITUDES TOWARDS MATHEMATICS

Again, these attitudes function as obstacles if unconscious and dogmatic.

U(lim) 9: IDENTIFICATION OF DIFFERENT PHILOSOPHICAL ATTITUDES TOWARDS MATHEMATICS

U(lim) 10: IDENTIFICATION OF THE PROBLEM OF REACHING THE LIMIT AS A PHILOSOPHICAL PROBLEM CONCERNED WITH THE NATURES OF MATHEMATICS AND INFINITY

U(lim) 11. SYNTHESIS OF THE CONCEPT OF NUMERICAL SEQUENCE

It is only then that we can speak of a conscious...

U(lim) 12: IDENTIFICATION OF THE SUBJECT OF "APPROACH"

5. Attribute of *sequence: numerical*

Analysis of the ancient "method of limits" presented in the XIIth Book of Euclid's "Elements" [e.g. Wygotski, 1956] brings to our awareness the importance of the concept of real number in understanding the notion of limit. Today, the notion of real number seems so familiar, so omnipresent in mathematics, that we hardly pay any attention, in theorems concerning limits, to assumptions such as "$a_n \varepsilon R$".

Asked, in a problem, to provide the area of a figure, we produce a number; but have we really detached this number from the figure? Do we always take care to say "area of circle" instead of "circle" when we mean the number — the measure — the area?

Perhaps, unconsciously, we still make a distinction between numbers and continuous magnitudes which are wholes containing within themselves their qualitative as well as quantitative aspects? Perhaps the convergence of a numerical sequence is something other than the convergence of the sequence of 2^n-gons inscribed in a circle: in the former, the difference $a_n - L$ is a number; in the latter, this difference is a difference of shapes: the higher the value of n, the less difference there is between the shapes of the 2^n-gon and the circle [cf. Sierpinska, 1985_1, p. 51-52].

This is why as the next obstacle we include:

Convergence: THE MEANING OF THE TERM "APPROACH" DEPENDS UPON THE CONTEXT: DIFFERENT IN THE DOMAIN OF NUMBERS AND IN THE DOMAIN OF GEOMETRICAL OR PHYSICAL MAGNITUDES

U(lim) 13: SYNTHESIS OF THE CONCEPTS OF "APPROACH" AND NUMBER

This act should result in establishing the meaning of the term "approach" in terms of distances and not in differences of shapes, positions, etc

6 Object of *approach*: number called *limit*
Sometimes students do not conceive of "approaching" as "approaching something" but as "approaching one another": as n grows, the terms of the sequence come closer and closer to one another [Sierpinska, 1989_2]

> Convergence: CONVERGENCE CONSISTS IN A DECREASE OF DISTANCE BETWEEN THE TERMS OF THE SEQUENCE

This is an intuition concerning Cauchy sequences rather than convergent sequences. There is no difference between these in the real (in general, complete) domain, but.

> U(lim) 14: DISCRIMINATION BETWEEN CAUCHY AND CONVERGENT SEQUENCES

is an important step towards understanding not only the notion of limit but also the notion of real number itself, of the meaning of the Dedekind axiom in particular, and thus of the essentiality of the assumption that $a_n \in R$ in theorems concerning monotonic and bounded sequences

> U(lim) 15: IDENTIFICATION OF THE "GOAL" OF "APPROACHING" (i.e of the limit of the sequence)

7. Adverbial phrase of *approach* *as close as we please to*
Students' first conceptions of convergence may not differentiate between the approaching of a sequence like 0.8, 0.88, 0.888, ... to the number 0.9, and the approaching of a sequence like 0.9, 0.99, 0.999, ... to the number 1. To make this distinction one has to overcome the obstacle:

> Convergence: CONVERGENT SEQUENCES ARE SEQUENCES THAT APPROACH SOMETHING

> U(lim) 16: DISCRIMINATION BETWEEN "APPROACHING" AND "APPROACHING AS CLOSE AS WE PLEASE TO"
> (cf Sierpinska [1982_2] the case of Robert)

8. Attribute of *terms*: *almost all*
Another distinction that must be made is between sequences like: 1, ½, ⅓, ¼, ⅕, ... and 1, ½, 2, ¼, 3, ⅛, ... In both sequences infinitely many terms tend towards zero, but only in the first one do almost all terms tend to zero In the second sequence an infinite number of terms run away from zero [cf. Sierpinska, 1989_2, p. 33-34].

> Convergence: CONVERGENCE IS WHEN AN INFINITE NUMBER OF TERMS OF THE SEQUENCE APPROACH SOMETHING

> U(lim) 17: DISCRIMINATION BETWEEN "INFINITELY MANY TERMS APPROACH THE LIMIT" AND "ALMOST ALL TERMS APPROACH THE LIMIT"

9. Reference of the sentence as a whole: what is the relation between the subject (TERMS OF INFINITE NUMERICAL SEQUENCES) and the object (LIMIT)?
Let us call this relation "passing to the limit".
In solving equations by approximate methods, calculating areas, tangents, rate of change of variable magnitudes, etc., one can easily come to the conclusion that

> Passing to the limit: PASSING TO THE LIMIT IS A HEURISTIC METHOD USEFUL IN SOLVING CERTAIN KINDS OF PROBLEMS

On the other hand, formal teaching of the notion of limit based on introducing the formal e – N definition and then proving (by definition) for concrete a_n and L that $\lim_{n \to \infty} a_n = L$ may lead the students to develop the following obstacle:

> Passing to the limit: PASSING TO THE LIMIT IS A RIGOROUS METHOD OF PROOF OF RELATIONS BETWEEN SEQUENCES AND NUMBERS CALLED THEIR LIMITS
> [cf Sierpinska, 1987_1]

Hence there is a need to make a.

> U(lim) 18: SYNTHESIS OF PASSING TO THE LIMIT AS A MATHEMATICAL OPERATION DEFINED ON CONVERGENT SEQUENCES AND WITH VALUES IN R

A mathematical operation should be well defined: a sequence should not have more than one limit. This brings forth another condition:

> U(lim] 19: SYNTHESIS OF THE NOTION OF PASSING TO THE LIMIT AS A MATHEMATICAL OPERATION AND THE PROPERTY OF MATHEMATICAL OPERATIONS OF BEING WELL DEFINED
> (awareness of the uniqueness of limit).

Cauchy, who introduced the symbol "lim" did not demand that it denote a single object. However he proposed marking sequences having "many limits" with double brackets: e g "lim((sin 1/x)) has infinitely many values contained between –1 and 1 (Cauchy, A. *Cours d'analyse*)"

The above synthesis (19) may lead to some degeneration in the students' conceptions of limit, especially if they tend to reduce the new operation to some well known one and to apply the same methods. One such degeneration, very common among students (and having its counterpart in Fermat's method of "omitting certain terms"), is:

> Limit: LIMIT OF SEQUENCE IS VALUE OF THE SEQUENCE AT INFINITY

Students are led to consider writings like 1/∞, ∞, 1/0 as numbers, and thus to conceive of numbers like Leibniz and Cauchy rather than like Weierstrass [cf. Sierpinska, 1985_1, p. 46] In students' exercise books one can find writings like:

$$\lim_{n \to \infty} \frac{1}{2^n} = \frac{1}{2^\infty}$$

$$\frac{0.777...}{n} = \frac{\overbrace{777...7}^{n}}{10^n}$$

In the first of the above writings, ∞, and, in the second, n stand rather for "large undetermined number" than for "infinity" or "arbitrary natural number" [cf Sierpinska, 1987_2, 1989_1]

U(lim) 20: DISCRIMINATION BETWEEN NUMBERS AND CONCEPTS SUCH AS INFINITELY SMALL QUANTITY AND INFINITELY LARGE QUANTITY

U(lim) 21: DISCRIMINATION BETWEEN THE CONCEPTS OF LIMIT AND VALUE

10. Understanding a formal definition of the limit of a numerical sequence

The sentence "Almost all terms of the infinite numerical sequence approach as close as we please to a number called limit of the sequence" can be used as the definiens of an informal definition of a convergent sequence. Let us consider now the conditions for understanding the formalization of such a definition

In the formalization all nouns will have to be translated by letters, all predicates by symbols of mathematical relations. In traditional algebra, and in physics, letters may denote either variable or constant magnitudes. But it is not this concept of variable we have in mind when formalizing the definition of LNS. A letter — a variable — must be conceived as just a name for an element of a class (or set). The important thing is to know which variables in the definition are bounded and which are free, and not which letters denote "variables" and which "constants". This conception of "variable" is very far from what is usually meant by it in algebra or in physics [cf. Freudenthal, 1983]

Symbolic language: LETTERS STAND FOR VARIABLES OR CONSTANT MAGNITUDES

U(lim) 22: DISCRIMINATION BETWEEN THE USE OF LETTERS IN ALGEBRA AND LOGIC

Suppose we have chosen S for "infinite numerical sequence" (i e. S is a name for a representative of the class of infinite numerical sequences), t_n for "term of the infinite numerical sequence" and L for "a number called limit of the sequence".

The property defining convergent sequences is that almost all t_n approach as close as we please to a certain number L. This number is not arbitrary, it depends upon S. L is a function of S since S may have only one limit. We might mark this in our formalization and write L_S instead of just L. This however would make the formalizm very heavy, so let us just write L.

The first condition is therefore that there exists such a number L: $\exists L \varepsilon R$.

Now the question arises what does it mean, in mathematical terms, that the numbers t_n "approach" L? Approaching is linked with decrease of distance. Can this distance be measured somehow? Modulus is a notion invented for just that purpose: $|t_n - L|$ is the measure of distance between t_n and L. The numbers $|t_n - L|$ should be getting smaller and smaller

Having overcome the obstacles related to the concept of sequence and being aware that the terms of the sequence change not over time but with the growth of n, we can say that as n grows to infinity, the distances $|t_n - L|$ get smaller and smaller.

Now, in comparing how the numbers 0.8, 0.88, 0.888, ... approach 0.9, with the way in which the numbers 0.9, 0.99, 0.999, ... approach 1, we can see that in the first case the distances $|t_n - L|$ can never become smaller than 1/90, whereas in the second case, for any distance (say e), however small, we can find terms t_n distant from L less than this distance, i e such that $|t_n - L| < e$.

What does it mean to say "*we* can find"? This expression has to be de-personalized.

In formalizing the definition of CNS one must become aware that what points to a term in a sequence is its index: the index determines the place of the term in the row of terms and shows the particular term we are concerned with

U(lim) 23: IDENTIFICATION OF THE ROLE OF ARGUMENTS IN A SEQUENCE AS INDICES OF THE VALUE-TERMS

This act of understanding allows for a de-personalization of the notion of approaching, as well as for reducing some conditions on terms to conditions on their indices. This leaves the variable "term of sequence" (t_n) unbounded, because, instead of having, e.g. " $\exists t_n : |t_n - L| < e$ ", we can put: " $\exists n : |t_n - L| < e$ ". This way the experience with sequences that "do not approach very close to their limits" leads to a tentative formalization in the form:

$$\exists L \varepsilon R \forall e > 0 \ \exists n : |t_n - L| < e.$$

However, when investigating the logical consequences of this definition we fall upon monsters which do not look convergent at all: only some or, at most, an infinite number of terms approach L, not "almost all" of them. "Almost all" means that all but at most a finite number of terms approach L. Let this number be k. This number depends on the choice of the distance e. The number of terms from the beginning of the sequence is visible in the index. So we may say that terms with an index greater than k are distant from L by less than e. Such an index k must exist, whatever the value of e. As k depends upon e and not vice versa, we put:

$$\exists L \varepsilon R \forall e > 0 \quad \exists k \varepsilon N \quad \forall n > k \ |t_n - L| < e.$$

This way, all elements of the structure of the definiens have been taken into account. So, finally, we can formulate the whole definition:

A sequence S: $N \to R$, $n \to t_n$ is called convergent $\leftrightarrow$

$$\exists L \varepsilon R \forall e > 0 \quad \exists k \varepsilon N \quad \forall n > k \ |t_n - L| < e.$$

There exists a formalization where the condition $\exists k \varepsilon N$ is replaced by the condition $\exists k \varepsilon R$. I have seen a teacher parachuting this definition on 17 year-old humanities students. The students were completely lost; they could not see the meaning of this condition. Of course, logically, the two definitions are equivalent, but the meaning of $\exists k \varepsilon N$ is different from the meaning of $\exists k \varepsilon R$ and therefore they are not equivalent psychologically. The condition $\exists k \varepsilon N$ points to an index and thus to a term of the sequence after which all the terms of the sequence are distant from L by less than e. This allows one to understand the definition even without having completely de-personalized the choice of terms that are in the interval (L−e, L+e). But if we write $\exists k \varepsilon R$ then we must conceive the indices as numbers and remember that these numbers are embedded in the field of real numbers. Now, when thinking of sequences we do not normally regard indices (arguments) as having the same status or as belonging to the same category as the terms (values). Indices are not numbers; not *nombres* but rather *numéros* (like those you get in a cloakroom). And this point of view has to be overcome if the formalization with $\exists k \varepsilon R$ is to be understood and accepted

Sequences: INDICES OF TERMS OF A SEQUENCE ARE NOT NUMBERS

U(lim) 24: IDENTIFICATION OF INDICES OF TERMS OF A SEQUENCE AS NUMBERS

11 Generalization

Further extensions of the understanding of the notion of CNS are concerned with the generalization and synthesis of this notion

First, the activity of formalizing the definition of CNS may lead to a more conscious overcoming of the belief that the "problem of reaching the limit" is a mathematical problem. This act of understanding already has the status of a synthesis; it is, in fact, an evaluation of a mathematical result, a perception of its relevance:

U(lim) 25: SYNTHESIS OF DISCUSSIONS AROUND THE PROBLEM OF REACHING THE LIMIT IN THE LIGHT OF THE FORMAL DEFINITION OF LIMIT: AWARENESS THAT THE FORMAL DEFINITION AVOIDS RAISING THIS PROBLEM AND IS ACCEPTABLE WITHIN MANY DIFFERENT CONCEPTIONS OF INFINITY.

Looking for new domains of application of the concept of limit leads to questions concerning the relevance of the different conditions in the definition of the CNS. What can we change without losing the general idea of approaching and preserving the passing to the limit as a mathematical operation? It may be tempting to define the concept of limit of a sequence in any topological space. Preserving passing to the limit as an operation demands however that we restrict ourselves to Hausdorff spaces Unawareness of this is another obstacle which has to be overcome.

And so on. There is probably no end to the possible generalizations and syntheses.

12. Summary

Twenty-five acts of understanding have been identified in the paper. Only six of them were strictly related to the definition of convergent numerical sequence (U(lim) 12-17). Others were either of a more general nature and concerned the identification of infinite sequences among other mathematical objects (1), global grasping of the idea of convergent sequence (2, 4, 10, 18, 19, 21, 25), or the natures of infinity and mathematics (7, 8, 9), or were concerned with basic notions such as number (3, 20), sequence (5, 6, 11, 13), or symbolic language (22).

Strictly related to understanding the definition were such acts as: identification of the terms of the sequence as that which "approaches" (12); connection between the vague idea of approaching and the mathematical concepts of distance and number (13), awareness of the fact that the terms of a convergent sequence have to approach something or that "approaching" must have a goal (which act leads to discriminating between the Cauchy and the convergent sequences, and therefore between complete domains such as R and other domains, like Q, where a Cauchy sequence may not have a limit) (14, 15); grasping the meaning of "approaching as close as we please" as something different from "approaching" *tout court* (16), as well as the meaning of the condition that almost all and not only an infinite number of terms so approach the number called limit (17).

These acts of understanding are by no means the first ones we would expect to occur to a student Rather, they are linked with the verbalisation, clarification and formalisation of intuitive ideas that should have developed through dealing with and discussing infinite processes (not always convergent) in many different contexts. The identification of infinite sequences as an interesting object of study and a useful tool for describing certain phenomena or relationships, or else for calculating certain quantities, is a prerequisite for any act of understanding related to understanding convergent sequences (1).

Limit of a sequence is often the first context in which a student encounters infinity, and although thinking or talking about infinity in a Calculus course can be avoided by providing students with a bunch of tricks or powerful theorems, this does not seem to be the way towards deeper understanding of the topic. Identification of one's own and other possible conceptions of infinity (7) as well as of views on the nature of mathematics (9); awareness of the fact that the problem of "whether the sequence reaches its limit or not" is mainly a philosophical problem that cannot be solved in a "yes or no" manner (10, 25) are, it seems, important acts of understanding the concept of convergent sequences.

In addition to, naturally, the identification of sequences that approach something among infinite sequences in general, the global understanding of convergent numerical sequences comprises such acts as: grasping the passing to the limit as a mathematical operation which assigns a well defined number to a convergent numerical sequence (18, 19), awareness of differences between convergence and stabilization of decimal digits (4) as well as between the concepts of limit and the value of a function (21).

Final remarks

The methodology of "levels of understanding" in epistemological analyses of mathematical concepts [Herscovics & Bergeron, $1989_{1,2}$] seems to focus on the evaluation of knowledge in students. The methodology of "acts of understanding" is concerned mainly with the process of constructing the meaning of concepts. However, the partial ordering of acts of understanding would probably allow for the definition of something like "depth" of understanding. Depth of understanding might be measured by the number and quality of acts of understanding one has experienced, or by the number of epistemological obstacles one has overcome Of course, there is the problem of devising methods which provide the evidence that, in a particular person, such and such an act of understanding has taken place. These methods will probably have to be elaborated separately for each act of understanding. There is also the practical educational problem of how to provoke these acts in students within the classroom and how to check, without making detailed interviews, that they have occurred

The methodology of "acts of understanding" is not very precise. Perhaps it cannot be made more precise without loss of generality. For example, the choice of sentence(s) the sense and reference of which are analysed is not well defined. In our example of the concept of convergence, this sentence was a part of an informal definition. Has it always to be a definition? This might work with the notion of function (e g. "changes in the magnitude of y are related to changes in the magnitude of x in a well-defined way"), but one can hardly do the same with the concept of number, or even such an apparently simple thing as the concept of area of a rectangle [cf. Sierpinska, 1990]

However, in spite of all these difficulties, and whatever the methodology, the usefulness of epistemological analyses of the mathematics taught at different levels seems indisputable, whether for the practice of teaching or writing textbooks or as a reference for all kinds of research in mathematics education

References

BACHELARD G. [1938] *La formation de l'esprit scientifique* Paris: PUF

BYERS, V and ERLWANGER, S [1985] Memory in mathematical understanding *Educational Studies in Mathematics* 16, 259-281

CAŁA, B [1989] *Spontaniczne wyjaśnienia paradoksów "Achilles i żółw" i "Dychotomii" przez uczniów oraz akceptacja przez nich wyjaśnień opartych na sumowaniu szeregów*, Master's degree thesis, Warsaw University Department of Mathematics, Computer Science & Mechanics

CARR, W and KEMMIS, S. [1986] *Becoming critical. education, knowledge and action research.* London: Falmer Press

DAVIS, R.B [1980] The postulation of certain specific, commonly-shared frames *The Journal of Mathematical Behavior* 3 (1), 167-201

DEWEY, J [1988] *Jak myślimy?* Warszawa: PWN (Polish translation of "How we think?" first published in USA, 1910)

DILTHEY, W [1970] *Der Aufbau der geschichtlischen Welt in den Geisteswissenschaften* Frankfurt a.M., p. 99

DUVAL, R., GAGATSIS, A. PLUVINAGE, F [1984] Evaluation multidimensionnelle de l'activité de lecture, Université Louis Pasteur. IREM de Strasbourg

FILOZOFIA A NAUKA, Zarys Encyklopedyczny [1987] Warszawa: Ossolineum, p. 788

FREGE, G [1884] *Die Grundlagen der Arithmetic. Eine logisch mathematische Untersuchung über den Begriff der Zahl.* Breslau

FREGE, G. [1967] Sens i nominat (Polish translation of "Über Sinn und Bedeutung"), in: Pelc, J. *Logika i język Studia z semiotyki logicznej* Warszawa: PWN, 225-252

FREUDENTHAL, H [1984] The implicit philosophy of mathematics history and education, Proc of ICM-82, Warsaw 1983 Vol 2, 1695-1709 (cf also *Didactical phenomenology of mathematical structures*)

GAGATSIS, A [1984] Préalables à une mesure de la compréhension *Recherches en Didactique des Mathématiques* 5.1, 43-80

GAGATSIS, A. [1985] Questions soulevées par le test de closure *Revue Française de Pédagogie*, No 70, 41-50

HADAMARD, J. [1964] *Odkrycie matematyczne* (Polish translation of "An Essay on the Psychology of Invention in the Mathematical Field") Warszawa: PWN

HAUCHART, C. and ROUCHE, N. [1987] *Apprivoiser l infini, un apprentissage des débuts d'analyse* GEM, Proposition 14, CIACO éditeur Belgique

HEIDEGGER, M. [1979] *Sein und Zeit.* Tübingen

HEISENBERG, W [1969] *Der Teil und das Ganze* München: R Piper & Co Verlag

HEISENBERG, W. [1989] *Physics and philosophy*. Penguin Books

HEJNÝ, M [1988] Knowledge without understanding *Proc Int'l Symposium on Research and Development in Mathematics Education* Bratislava: Aug. 3-7, 1988, 13-74

HERSCOVICS, N. [1980] Constructing meaning for linear equations *Recherches en Didactique des Mathématiques* 1 3, 351-386

HERSCOVICS, N and BERGERON, J -C [1989_1] Analyse épistémologique des débuts de l addition *Actes de la 41e Rencontre CIEAEM* Bruxelles: 23-29 juillet 1989

HERSCOVICS, N and BERGERON, J.-C [1989_2] A model to describe the construction of mathematical concepts from an epistemological perspective, *Proc. of the 13th Meeting of the CMESG* St Catharine's, Ontario: May 26-31, 1989

HERSCOVICS, N and BERGERON, J -C [1989_3] The kindergartners' construction of natural numbers: an international study, *Proc. of the 13th Meeting of the CMESG.* St Catharine's, Ontario: May 26-31, 1989

HOYLES, C [1986] Scaling a mountain: a study of the use, discrimination and generalisation of some mathematical concepts in LOGO environment. *European Journal of Psychology of Education* Vol 1, no 2, 111-126

HOYLES, C and NOSS, R [1987] Children working in a structured LOGO environment: from doing to understanding, *Recherches en Didactique des Mathématiques* 8/1 2, 131-174

HUME, D. [1985] An enquiry concerning human understanding, in: *Modern philosophical thought in Great Britain*, Part I, Gogut-Subczyńska I., ed Wydawnictwa Uniwersytetu Warszawskiego, 55-65

KRASNODĘBSKI, Z [1986] *Rozumienie ludzkiego zachowania* Warszawa: PIW

KRYGOWSKA, Z. [1969-70] Le texte mathématique dans l enseignement *Educational Studies in Mathematics*, Vol 2, 360-370

KUNTZMANN J [1976] Peut-on employer des calculatrices pour donner une initiation à l'analyse? in: *Quelques apports de l'informatique à l'enseignement des mathématiques*, Publication de l'APMEP. No 20, 124-127

LAKATOS, I [1984] *Preuves et réfutations: essai sur la logique de la découverte mathématique* Paris: Hermann

LAKOFF, G and JOHNSON M [1980] *Metaphors we live by* University of Chicago Press

LINDSAY, P.M and NORMAN, D A [1984] *Procesy przetwarzania informacji u człowieka, Wprowadzenie do psychologii* (Polish translation of "Human information processing An introduction to psychology") Warszawa: PWN

LOCKE, J [1985] An essay concerning human understanding, in: *Modern philosophical thought in Great Britain*, Part I, Gogut-Subczynska I., ed., Wydawnictwa Uniwersytetu Warszawskiego, 32-39

MAŁA ENCYKLOPEDIA LOGIKI, Second Edition Wrocław: Ossolineum [1988] p 256

MATZ, M [1980] Towards a computational theory of algebraic competence *Journal of Mathematical Behavior* 3(1), 93-166

MAURIN, K [1977] *Analiza, Część I Elementy* Warszawa: PWN

MINSKY, M. [1975] A framework for representing knowledge, in: P Winston (ed) *The psychology of computer vision* New York: McGraw-Hill

OTTE, M. [1990] Arithmetic and geometry: some remarks on the concept of complementarity. *Studies in Philosophy of Education* 10, 37-62

RICOEUR, P [1989] *Język, tekst. interpretacja* (Polish translation of: 1 Interpretation Theory: Discourse and the Surplus of Meaning; 2 La tâche de l'hermeneutique, 3. La fonction hermeneutique de la distanciation; 4. La métaphore et le problème central de l'hermeneutique; 5 L'appropriation) Warszawa: PIW

ROSNICK, P and CLEMENT, J [1980] Learning without understanding: the effect of tutoring strategies on algebra misconceptions *Journal of Mathematical Behavior* 3(1), 3-28

SIERPINSKA, A. [1985_1] Obstacles épistémologiques relatifs à la notion de limite. *Recherches en Didactique des Mathématiques* 6.1, 5-68

SIERPINSKA, A [1985_2] La notion d'obstacle épistémologique dans l'enseignement des mathématiques *Actes de la 37e Rencontre CIEAEM* Leiden: 4-10 août 1985, 73-95

SIERPINSKA, A [1987_1] Humanities students and epistemological obstacles related to limits. *Educational Studies in Mathematics*, Vol 18, 371-397

SIERPINSKA, A. [1987_2] Sur la relativité des erreurs *Actes de la 39e Rencontre CIEAEM.* Sherbrooke, Canada: 27 juillet - 1 août 1987, 70-87

SIERPINSKA, A [1989_1] *Propozycja pewnej sytuacji dydaktycznej w zakresie nauczania początków analizy matematycznej.* Preprint B18 of the Institute of Mathematics of the Polish Academy of Sciences, p 64 (to appear in Dydaktyka Matematyki 12)

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