DIVISION IS PRETTY MUCH JUST MULTIPLICATION

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Are multiplication and division over the real numbers the same operation?

A reasonable reader's response may be immediate, 'no'. Multiplication is defined for a zero factor; division is not defined for a zero denominator. Multiplication is commutative; division is not. However, an equally reasonable reader might respond 'yes', as this abstract algebra student stated while considering whether division is a binary operation:

> Yes, because [division is] pretty much just multiplication.

Why might a student respond in this way? Are division and multiplication "pretty much" the same? The mathematics education literature offers some insight into how the two operations may be regarded as the same or different depending on context. For example, Taylan (2017) states that students in a third-grade classroom should learn to distinguish the operations:

> Some students in the class had difficulties in understanding the *difference* between multiplication and division (p. 268, emphasis added).

In contrast, by fourth and fifth grade, Olive (1999) documented a productive reorganization of students' views of the two operations, highlighting their sameness:

> The reorganization involved an integration of their whole-number knowledge with their fractional schemes whereby whole-number division was regarded as the *same* as multiplication by the reciprocal fraction (p. 279, emphasis added).

In both of these cases, *same* and *different* are used without articulating *what* makes two operations the same. If, by the elementary grades, two operations can be treated—without conflict—as implicitly both same and not the same, we may need to attend more closely to the question of *same in what way?*

Motivation for exploring sameness

In mathematics, determining sameness (of numbers, figures, objects, properties, definitions) is a common activity. For example, one can ask:

- Are 5×3 and 3×5 the same?
- Are two figures in the plane similar? Are they congruent? Do they have the same area?
- Are 2, $^2/_1$, $^4/_2$ and 1.999… the same?
- Are $f(x) = x - 2$ and $g(x) = \sqrt{((x - 2)^2)}$ the same?
- Are substitution and addition algorithms for solving a 2×2 system the same?
- Are Z_4 and the rotational symmetries of the square the same?

Such ideas of sameness are frequently leveraged when solving mathematical tasks. For example, in algebra when two expressions are the same, one can be substituted for the other when solving an equation or graphing. Whether a student makes the substitution is predicated on how a student interprets *expression sameness*. Another example comes from abstract algebra. A common task is to determine if a given subset forms a subgroup under the *same* operation as the parent group. Correctly unpacking the definition of a subgroup in order to apply the subgroup test is predicated on how a student interprets *operation sameness.*

In mathematics, there are myriad ways to make precise the ideas about sameness (*e.g.,* equality, congruence, isomorphism) and the appropriate one is context-dependent. Our purpose is not to posit new mathematical definitions for 'sameness' nor to offer an over-arching definition. Rather, our goal is to make sense of unexpected claims students make about sameness in mathematics. Treating sameness as-if-well-defined in pedagogical contexts obscures the students' thinking. For example, what is it about students' experiences with multiplication and division that would lead them to claim the two are the same operation? If they see multiplication and division as the same operation, what might we infer about the psychological or linguistic notions of *sameness* or non-normative mathematical properties they might be considering when making their claims? If we take the position that students' responses reflect sensible-to-them reasoning, then the onus is on us to resolve our puzzlement about their conclusions by making sense of *sameness* in a way that accounts for how students could see two objects as the same.

We begin with a look at the philosophical roots of sameness, historical problems with sameness, and posit an operationalization of *sameness* we think is fruitful for unpacking students' thinking. We then leverage this operationalization to make sense of students' ways of understanding sameness from both the literature and our own exploration.

Sameness is multidimensional

Our main task in this reflection is to reach an operationalization of 'same' that supports conjecture about why students may respond affirmatively (or negatively) about the sameness

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of two mathematical objects. We begin by exploring, through a concrete example, why questions of 'sameness' are ambiguous in the first place. Piaget's (1965) decanting experiments offer an opportunity to illustrate this point. In one experiment, a child is shown two identical containers A and B holding the same volume of two different liquids. The child is asked whether A and B are the same. The liquid in container A is poured into a differently sized container, A*, and the child is asked whether A* and B are the same. In the second experiment, the liquids are replaced by beads, and the procedure from the first experiment is repeated. In the third experiment, the child is presented with two rows of equally-spaced objects with row A equinumerous to row B. Then, row A* is formed by increasing the distance between the objects in row A. The child is again asked whether A* and B are the same. The main finding of these experiments was that younger children would often answer 'no' to the question of whether A* and B were the same and older children tended to answer 'yes'. A common inference is that recognizing that A* and B are the 'same' is a matter of cognitive development. Yet, the question itself is ambiguous—it depends "on what kind of same-ness the respondent brings to bear on the situation" (Cable, 2014, p. 228). A* and B have the same volume (numerosity), but are not in the same configuration. We argue that the different answers across age groups could reflect a deficit among younger students, but alternatively may reflect differing criteria for sameness where some object's properties are foregrounded while others are backgrounded.

Historical attempts to define sameness and the philosophical problems that follow

There are parallels between how students resolve questions of sameness and the philosophical roots of defining sameness. We can build on these to sensibly deconstruct students' claims. The classical approach is to lay out criteria according to which two objects can be said to be identical. Identity can be formalized into classical first-order logic by adopting the following postulates:

$x = x$ (Principle of Reflexive Identity [PRI])

$x = y \longleftrightarrow [\Phi(x) \longrightarrow \Phi(y)]$ (Leibniz's Law [LL]) [1]

The postulates capture the idea that two identical things are the same in every respect, *i.e.,* for all properties, Φ. In this way, objects can be partitioned into sets whose members are 'the same'. While the characterization seems intuitive, philosophical and practical problems arise applying it. Specifically, LL requires two things must be the same in *every* respect. Under LL, two things are the same if and only if they are identical. When considering all potential properties an object may have (including temporally limited properties or position), the equivalence classes become singleton sets.

One could imagine a student grappling with this issue during a geometry lesson. She is presented two congruent triangles that do not have the same orientation. Applying LL would prevent her from concluding that the two triangles are the same. The triangles are different because their orientations differ. Where a teacher may refer to two congruent triangles as 'the same', foregrounding properties such as congruent pairs of sides and angles, her student may not have an intellectual reason to foreground that (or any) subset of properties. Two triangles in different orientations are not the same across all properties. It is therefore reasonable to claim that the two triangles are not the same. The children in Piaget's experiments may have faced essentially this problem; one of two aspects (the material identity) is the same while the other (the form) differs.

These examples illustrate that LL is too strong a criterion for any practical inference of *sameness* in mathematics. Under LL, for any given object X, only X would be certain to fall into the same equivalence class as X. Thus, *sameness* is not even possible. Adopting LL prohibits one from concluding two objects are the same, even in mathematically normative ways (such as through congruence or isomorphism). So, although we may use 'same' as if it needs no further clarification, we do not typically operate under the LL paradigm. Thus, contextualizing claims about mathematical objects' *sameness* is essential.

Weakening Leibniz's Law

LL fails because when claiming two mathematical objects (or any objects) are the same, we often mean that the two objects are the same *in some (meaningful) respect* but are not the same *in every respect.* When considering identity in mathematics, we are often not attending to absolute identity, but rather to objects' *qualitative* or *relative* identity:

x and y are the same F, but x and y are different G's (Relative Identity [RI]) [1]

The idea is that we need not consider all properties of an object, only the salient ones. That is, an object x and an object y might be elements of the set of objects F all of whom possess property Φ_f, but only x is an element of the set G whose members possess property Φ_g. In this way, we could understand x and y as the same (with respect to Φ_f).

Relative identity would allow us to resolve the problems raised by LL above. We may claim that the two congruent triangles are the same in terms of side lengths and angles, but different in terms of orientation. That is, triangle y is a member of the set of triangles congruent to triangle x. However, triangle y is not a member of the set of triangles with triangle x's orientation. With RI, we can say two triangles are the same shape and size without being in the same position or orientation. In the decanting experiment, we can understand, as Piaget intended, that A* is the same as B because the liquids have the same volume (the beads are equinumerous), but the contents are not in the same configuration. Two objects may share some properties but not others.

Operationalizing a new paradigm for 'the same'

Violations of LL effectively discriminate between two non-identical objects, but it does not help to determine whether two objects can be called 'the same' nor does it help an educator determine why a child might assert that two objects are the same. From a relative identity perspective, it is possible to claim that two objects are the same as long as there is some property (or set of properties) that the two objects share. The paradigm a student applies depends upon the set of properties they foreground, which may or may not align with the (normative) properties foregrounded by the mathematical community. To make sense of students' claims about sameness in terms of properties, we operationalize Relative Identity as follows:

$$\exists\Phi \text{ such that } \Phi(x) \wedge \Phi(y) \therefore x = y \text{ (Shared Property [SP])}$$

$$\exists\Phi \text{ such that } \Phi(x) \wedge \neg\Phi(y) \therefore x \neq y \text{ (Unshared Property [UP])}$$

That is, if Φ is observed in x and Φ is observed in y, then x is the same as y. Interpreting student sense-making becomes uncovering what Φ could be foregrounded by the student that would lead her to claim that x and y are (not) the same, in tandem with which of the two sameness criteria she is operating under. It is as much a pattern of reasoning as it is attending to a specific Φ. For example, in Piaget's experiments, we can easily envision the interviewer attending to one quantity, such as volume (or numerosity) while the child attends to a different quantity, such as length or height, or to a different attribute entirely, like color. We can infer the interviewer applies SP with a particular Φ_a in mind, while the child applies UP with a different Φ_b in mind.

The mathematics education literature using the SP-Sameness Paradigm

In this section, we illustrate the feasibility of using the SP-Sameness Paradigm to interpret some well-known examples from the mathematics education literature.

Is *f(x)* the same as *g(x)*?

Consider two functions $f(x) = 1/x$ and $g(x) = (x - 2)/(x^2 - 2x)$. Are they the same? Solares and Kieran (2013) raise just this question in the context of a middle-grades algebra classroom. They argue that not only is the question 'Are f and g the same?' ambiguous, it is duplicitous. Both 'yes' and 'no' are correct answers, even from a normative textbook-correct perspective. To answer the question, they offered two different properties that can be used to determine functional equivalence based on a numeric or a syntactic view. From the numeric view, two expressions are equivalent if for each input value they yield the same output value. From the syntactic view, two expressions F and G are equivalent when they have a common algebraic rewriting. The algebraic rewriting may transform F into G through commutative, associative, identity properties and also factoring, canceling, long division, *etc.* The algebraic expressions for f and g are the same syntactically but not numerically. Solares and Kieran argue that for some encounters in algebra, a syntactic perspective is required whereas for others a numeric perspective is needed, and still for others, students must use both. Thus, whether the child receives positive or negative feedback for her answer will depend on whether she uses the same property as her teacher to arrive at her answer. Again, relative identity is useful for constructing an explanation for how it can simultaneously be the case that f and g "denote the same rational fraction, [but] they do not denote the same rational function" (p. 122).

Is 0.999... the same as 1 ?

Choi & Do (2005, p. 14) report that gifted middle grades students typically do not believe that 0.999... is the same as 1. The mathematical reasons included:

- 1 = 0.999…+.000…01, so 0.999… is not equal to 1.
- Theoretically, 0.999… can be equal to 1, but there exists a very fine tolerance between the two numbers.
- Just as 0.9, 0.99, 0.999 and so on are all less than 1, so 0.999… is less than 1.

What might it mean for these students who disagree that these two numbers are the same? It appears the students are operationalizing 'same' as 'numerically equal'. Because all three student disagreement types rely on the decimal expansion of 0.999… we can infer that these students have identified a property, *a number's decimal expansion,* and concluded the two numbers are UP-different because they do not have the same decimal expansion.

Table 1. Focal tasks comparing operations

<table>
<tr><td>Directions</td><td colspan="2">Determine if the following binary operations are the same. Explain why or why not.</td></tr>
<tr><td>Task 1</td><td colspan="2">Op 1: Division on ℝ,
Op 2: Multiplication on ℝ</td></tr>
<tr><td>Task 2</td><td colspan="2">Op 1: Addition mod 4 on the set {0, 1, 2, 3},
Op 2: Composition on the set of rotations on a square (0° rotation, 90° rotation, 180° rotation, 270° rotation)</td></tr>
<tr><td>Task 3</td><td>Op 1:
<table>
<tr><th>◇</th><th>0</th><th>1</th><th>2</th></tr>
<tr><td>0</td><td>0</td><td>1</td><td>2</td></tr>
<tr><td>1</td><td>1</td><td>2</td><td>0</td></tr>
<tr><td>2</td><td>2</td><td>0</td><td>1</td></tr>
</table></td><td>Op 2:
<table>
<tr><th>*</th><th>0</th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th></tr>
<tr><td>0</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td></tr>
<tr><td>1</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>0</td></tr>
<tr><td>2</td><td>2</td><td>3</td><td>4</td><td>5</td><td>0</td><td>1</td></tr>
<tr><td>3</td><td>3</td><td>4</td><td>5</td><td>0</td><td>1</td><td>2</td></tr>
<tr><td>4</td><td>4</td><td>5</td><td>0</td><td>1</td><td>2</td><td>3</td></tr>
<tr><td>5</td><td>5</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td></tr>
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Table 2. Overview of properties students attended to.

Φ	Description
Φ_1	⊛ can be manipulated into ⊗
Φ_2	⊛ possesses the same properties as ⊗
Φ_3	⊛ and ⊗ give identical outputs for identical inputs
Φ_4	⊛ and ⊗ afford an isomorphism between two structures
Φ_5	⊛ is the same type of operation as ⊗

The normatively accepted view appeals to limits, arguing that

$$0.999\ldots = \sum_{i=1}^{\infty} \frac{9}{10^i} = \lim \sum_{i=1}^{n} \frac{9}{10^i} = \lim(1 - 10^{-n}) = 1$$

The relevant property here is that the two numerals represent the same object if their series expansions have the same limit. However, even after seeing the computation, many students still do not shift their *sameness* criteria away from finite decimal expansions, likely because the left- and right-hand sides still are not 'the same' using that criterion. A promising pedagogical approach for helping students foreground the relevant property would be to appeal to a number line representation for the real numbers. Using SP (rather than UP), a teacher could suggest that two numbers are the same if they *occupy the same position* on the real number line. Attending to a shared property could lead to productive discussion as to why the distance between 0.999… and 1 is arbitrarily small and why that should mean the two are the same.

Using the SP-sameness paradigm to understand students' claims

We surveyed undergraduates at two institutions who were completing an introductory abstract algebra course about binary operations [2]. The students were given pairs of operations and asked to determine whether the binary operations were 'the same'. See Table 1 for focal tasks.

Even at such an advanced level of mathematical study, students' answers and justifications were unexpected. Using the distinctions we made above, we can conjecture *why* they might claim that two operations are (not) the same. Under the SP-Sameness Paradigm for determining sameness, this amounts to asking: What (set of) properties do students foreground when determining whether two operations, ⊛ and ⊗, are the same? (See Table 2 for an overview.)

Φ1: ⊛ can be manipulated into ⊗

Some students claimed that two operations are the same if one operation can be manipulated into a second operation. For example, Student 1 claimed:

> Yes, they are the same [operation]—division can be turned into multiplication (*i.e.,* divide by two or multiply by ½).

This response was initially surprising since abstract algebra tends to emphasize *structural sameness.* However, the Shared Property Φ_1 may be a productive understanding for students. As Solares and Kieran (2013) illustrated, two functions can be thought of as the same if one can be obtained through algebraic manipulations of the other. Here, division can be obtained through arithmetic manipulations of an expression. Even in the case of multiplication and division, it is propitious for students to see "whole-number division was regarded as the *same* as multiplication by the reciprocal fraction" (Olive, 1999, p. 279, emphasis added) because it can conceptually ground understanding of fractions through connecting the operations.

Φ_2: ⊛ possesses the same properties as ⊗

The Shared Property Φ_2 captures the notion that operations which are the same should share a common set of emergent properties when combining the elements they take as inputs. For example, Student 2 stated that multiplication and division are:

> Not the same, because they don't have the same properties. Multiplication is commutative, division is not.

In abstract algebra, it is common to focus on the properties of a binary operation that impose structure on a set, such as commutativity and associativity. Indeed, one can classify groups according to properties conferred by their binary operations. Student 2 picked out a property possessed by multiplication, but not possessed by division and applied UP. Along the dimension she focuses on, the two operations are not the same. A property-driven approach to evaluating operations is privileged by the abstract algebra curriculum and researchers. For example, students may determine whether two groups are isomorphic by systematically checking properties shared (or not) by the two groups (Leron, Hazzan & Zazkis, 1995). For the students in Leron, Hazzan & Zazkis's study, property checks were fruitful for establishing differences. However, because checking that a finite list of properties are possessed by two groups can mask important structural differences, the strategy was not fruitful for establishing whether two groups were the same, in the sense of isomorphism.

Φ_3: ⊛ and ⊗ give identical outputs for identical inputs

Shared Property Φ_3 appeals to the results of combining elements through the binary operations. It requires identical inputs to produce identical outputs. It explains the following two student responses:

Student 3	Multiplication and division are not the same because $2 \times 3 = 6$ but $2 \div 3 = 2/3$
Student 4	mod 3 and mod 6 addition are different because $(2 + 2) \bmod 3 \neq (2 + 2) \bmod 6$

As a strategy, checking that identical inputs yield identical outputs relies on a functional view of binary operation. In this way, it is consistent with the *numeric equivalence* of two functions suggested by Solares & Kieran (2013). Shared Property 3 can be quite powerful for determining differences at a group element level. It is precisely this view of *sameness* that is normatively used when asserting that a subgroup has the same operation as its parent group. Understanding that two operations are (not) numerically equivalent has consequences for determining, for example, whether {1, 2, 3} is a subgroup of Z_6 (Dubinsky, Dautermann, Leron & Zazkis, 1994).

Φ_4: $\circledast$ and and $\otimes$ afford an isomorphism between two structures

The Shared Property Φ_4 arises from considering algebraic structure imposed on a set by an operation. In this view, two operations are the same if an isomorphic map can be created between the structure $\circledast$ induces on set A and the structure $\otimes$ induces on set B. This view accounts for Student 5's claim that addition mod 4 (Op 1) and function composition on rotations of the square (Op 2) are the same operation:

> Yes. Because Op 1 and Op 2 are isometric *[sic]* to each other where Φ: $Z_4 \longrightarrow C_4$, $\Phi(x) = R^x$, $x \in Z_4$. This means there *[sic]* homomorphic meaning applying this operation *[to]* either set will result in the equivalent in the other set.

Attending to Shared Property Φ_4 also explains Student 6's claim that addition mod 3 and addition mod 6 are not the same:

> No, because it is impossible to create a bijective mapping between the two groups of elements.

Other students (with varying strength of justification) argued that multiplication and division are the same because an isomorphism could be set up between them.

We see Shared Property Φ_4 used in both the positive sense (same) and the negative sense (not the same). Attending to Φ_4 is appropriate because it is mathematically normative; it is also productive because it captures structural equivalence. Weber and Alcock (2004) reported that experts explicitly described isomorphism in terms of sameness (p. 218):

- Essentially the *same*
- Exactly the *same structure* and exactly the *same form*
- The *same group* except the elements and operation may have different names

The graduate student experts qualified their responses, appealing to relative identity to claim *sameness* rather than absolute identity. Thus, even with the precision afforded by the highly stringent mathematical definition of isomorphism there may be many ways in which the two groups are not the same including the group operations themselves. That is, two operations *can act the same* on their sets (producing isomorphic groups) while being *fundamentally different* operations (different inputs and outputs, Φ_3).

Φ_5: $\circledast$ is the same type of operation as $\otimes$

The final Shared Property we identified is, at first glance, the most superficial. Property Φ_5 requires only that two operations are the same kind. Student 7's response (Figure 1) illustrates using Φ_5 in the positive sense (SP claim) and Student 8's response uses Φ_5 in the negative sense (UP claim):

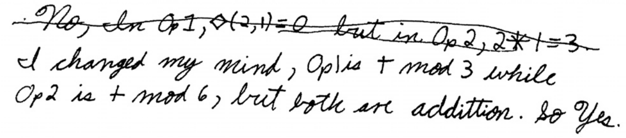

Figure 1. Student 7 claims that + mod 3 and + mod 6 are the same.

> Division is inherently different than multiplication

We conjecture that Student 7 would claim that all kinds of addition (any modular arithmetic) are the same. The *way* that elements are combined under addition is not only procedurally the same but the result can be interpreted in the same way, regardless of mod. We further conjecture that reasoning in the manner of Student 8 may preclude a student from recognizing Φ_4 as a viable Shared Property. The conjecture accounts for Student 9's response:

> Addition mod 4 composes two elements. Different than the rotations. They [the groups] are isomorphic, but the binary ops are different.

For Student 9, it is not enough that the two operations induce isomorphic structures on their respective sets. They are fundamentally *different* operations because of how they act procedurally on the necessarily different elements of their respective sets.

Student 9's lens for sameness is reasonable when reflecting on the curricular treatment of binary operations. Multiplication and division are introduced as separate (but related) operations that model different procedures: totaling equinumerous sets, or partitioning/measuring a set, respectively. Foregrounding Φ_5 would support claims that these are inherently different operations. In contrast, modular addition is often defined symbolically $(a + b)$ mod n and then exemplified procedurally: add a and b and then subtract multiples of n until the result is between 0 and n. The general formula introduces all modular addition, capturing a family of addition operations that work the same way procedurally. From a perspective considering Shared Property Φ_5, there is no inconsistency in believing both 'subgroups have the same operation as their parent group' and '{0, 1, 2} forms a subgroup of Z_6'.

Implications and further musings about dimensions of sameness

Mathematically normative claims about sameness depend on foregrounding some (interesting, relevant) attributes while backgrounding others. As an educational community, as teachers and researchers, we typically approach our mathematical interactions with students with specific attribute(s) in mind that *ought* to motivate claims of sameness between two mathematical objects (procedures, structures). The misalignment between the properties attended to by an educator and those held in mind by a student often lead to inferences about the student's capacity for mathematical reasoning. For example, based on children's assertions that A* and B are not the same, Piaget concluded that knowing the normative response was a matter of cognitive development. Likewise, it is common to interpret student claims such as '{0, 1, 2} forms a subgroup of Z_6' as indicative of lack of attention to the role of operation in subgroup rather than using a non-normative SP to claim that the operation is the same (*e.g.,* Dubinsky *et al.,* 1994). Our work and examples suggest that these claims may be too strong because they are based on an implicit criterion for sameness, which privileges the teacher/researcher's set of properties while excluding the student's.

We offer a more conservative interpretation of the student responses compatible with a relative view of identity: individuals attend to sets of shared properties among objects that are salient-to-them. A pair of mathematical objects might be Φ_a-same while being Φ_b-different. Furthermore, we argue it can be productive to explicitly address these properties. For example, addition mod 3 and addition mod 6 are Φ_5-same, but Φ_3-different. Classroom discussions centering on the possible properties permissible to use (or not) can draw attention to issues like why Φ_5-sameness and Φ_3-difference are important mathematically, such as for identifying families of structures and applying the subgroup test, respectively. Discussing some properties may serve to support students in examining mathematical objects at different levels of abstraction. For example, Φ_1 and Φ_5 reflect (or indicate) lower levels of abstraction since using them requires procedural manipulations or actions on individual elements, respectively. In contrast, Φ_2 and Φ_5 reflect (or indicate) holistic analysis of an operation, with Φ_5 further indicating consideration of the structures induced by an operation. Reasoning in each of these ways is mathematically valuable, depending on the problem at hand. Further, our interpretation of student work based on SP-sameness paradigm has actionable pedagogical implications whose efficacy are empirically testable. We conjecture that acknowledging, explicitly, the existence of viable dimensions of sameness and properties to attend to—and why—can support students in explicating their ideas of sameness and developing an understanding of what sameness properties are appropriate in different mathematical contexts.

The literature on *defining* reflects promise in such an approach to instruction. Defining is a foundational mathematical activity and depends on identifying what is the *same* (shared properties) among all mathematical objects of a particular type. For example, a triangle definition requires attending to the shared properties of closure and three segments. However, young children might claim that three unconnected line segments are a triangle because they attend to only the latter shared property (Satlow & Newcome, 1998). Thus, from an early age, students attend to properties shared among objects even though they may not be the complete set of normatively accepted properties. In our view, students' attention could be directed to additional properties the figures do (not) share through discussion about Φ's that others (including the teacher) notice. Zandieh and Rasmussen (2010) illustrated the potential of such an instructional sequence where undergraduates explicitly identified properties, tested the sameness of objects with respect to those properties, and adjusted the properties accordingly. These examples increase our confidence that the SP-Sameness Paradigm can serve as an empirically-grounded basis for interpreting student reasoning and provide pedagogically sound means for progressing student understanding.

We close by reiterating that when mathematical attributes and their properties remain implicit, students do not have access to the lens we use to make decisions regarding the sameness of objects. As educators, we can better serve students by reflecting on the Φ_a foregrounded by normatively correct mathematical claims in a given mathematical setting, the reasons such a property is valued by the mathematical community in that setting, and attending to the Φ_b a student may foreground. Through explicit discussion, common ground can be found to reconcile Φ_a with Φ_b, supporting pedagogical moves that center on student cognition rather than on cognitive development.

Notes

[1] See the relevant articles in *The Stanford Encyclopedia of Philosophy.* Online at https://plato.stanford.edu/archives/sum2018/entries/identity/ https://plato.stanford.edu/archives/fall2018/entries/identity-relative/; https://plato.stanford.edu/archives/win2016/entries/identity-indiscernible/;

[2] Additional analysis can be found in Melhuish, K., Ellis, B. & Hicks, M. D. (2020) Group theory students' perceptions of binary operation. *Educational Studies in Mathematics* **103**(1), 63–81.

References

Cable, J. (2014) La meme chose: how mathematics can explain the thinking of children and the thinking of children can illuminate mathematical philosophy. *Science & Education* **23**(1), 223–240.

Choi, Y. & Do, J. (2005) Equality Involved in 0.999... and (−8) $^{1}/_{3}$. *For the Learning of Mathematics* **25**(3), 13–36.

Dubinsky, E., Dautermann, J., Leron, U. & Zazkis, R. (1994) On learning fundamental concepts of group theory. *Educational Studies in Mathematics* **27**(3), 267–305.

Leron, U., Hazzan, O. & Zazkis, R. (1995) Learning group isomorphism: a crossroads of many concepts. *Educational Studies in Mathematics* **29**(2), 153–174.

Olive, J. (1999) From fractions to rational numbers of arithmetic: a reorganization hypothesis. *Mathematical Thinking and Learning* **1**(4), 279–314.

Piaget, J. (1965) *The Child's Conception of Number.* Oxford: W. W. Norton & Co.

Satlow, E. & Newcombe, N. (1998) When is a triangle not a triangle? Young children's developing concepts of geometric shape. *Cognitive Development* **13**(4), 547–559.

Solares, A. & Kieran, C. (2013) Articulating syntactic and numeric perspectives on equivalence: the case of rational expressions. *Educational Studies in Mathematics* **84**(1), 115–148.

Taylan, R.D. (2017) Characterizing a highly accomplished teacher's noticing of third-grade students' mathematical thinking. *Journal of Mathematics Teacher Education* **20**(3), 259–280.

Weber, K. & Alcock, L. (2004) Semantic and syntactic proof productions. *Educational Studies in Mathematics* **56**(2–3), 209–234.

Zandieh, M. & Rasmussen, C. (2010) Defining as a mathematical activity: a framework for characterizing progress from informal to more formal ways of reasoning. *The Journal of Mathematical Behavior* **29**(2), 57–75.