Bombelli's *Algebra* (1572) and a new mathematical object

GIORGIO T. BAGNI

In a famous passage of his *Questions Concerning Certain Faculties Claimed for Man*, Peirce (1868/1991) suggests that it is impossible to "think without signs" (p. 49) We are symbolic creatures through and through, both creators of, and created by, signs. Peirce's semiotics, nevertheless, hardly explains the complexity of sign-based human thought processes and the manner in which those processes relate to their corresponding historical settings The historical dimension of cognition and its cultural subbasement have, however, been a recurrent theme in contemporary research (see, *e.g.*, Bradford & Brown, 2005; D'Ambrosio, 2006; Khan, 2008). This recurrent theme reappears in particular in recent sociocultural perspectives where cognition is conceptualized as "a cultural and historically constituted form of reflection and action embedded in social praxes and mediated by language, interaction, signs and artifacts" (Radford, 2008, p. 11). Sociocultural perspectives lead to both new conceptions of cognition and new views about knowledge and the cognizing subject:

> knowledge is produced by cognizing subjects who are, in their productive endeavours, subsumed in historically constituted traditions of thinking. The cognizing subject of sociocultural theories is a subject that thinks within a cultural background and that, in so doing, goes beyond the necessities of mere ahistorical adaptive urges. (*ibid.*)

In this paper I deal with the historical cultural context and meaning of imaginary numbers, focusing in particular on the work of Rafael Bombelli.

Commerce, algebraic symbolism, and imaginary numbers

The cultural movement of the European Renaissance had a deep impact on mathematics. It produced a major shift in mathematical ideas and the way they were expressed. While in the Middle Ages mathematics was a rather speculative part of the *Quadrivium*, in the Renaissance the study of mathematics acquired a practical interest as a result of a tremendous expansion of trade and commerce. Since the fourteenth century a new group of individuals made their appearance on the social, economical, and educational scenes - the abacists, teachers specialized in the domain of calculations. These teachers - sometimes private entrepreneurs, sometimes individuals appointed by the city - had to teach merchants and their sons both the Hindu-Arabic decimal place-value system and the algorithms for using it. During the fourteenth and fifteenth centuries they extended those methods to new problems and introduced abbreviations and symbolisms, creating at the same time new methods for dealing with algebraic problems

Imaginary numbers arose precisely in the midst of calculations required to solve cubic equations. They appeared in the work of N. Fontana (Tartaglia, 1500-1557) and G. Cardan (1501-1576) (although, as often happens with new ideas, Cardan did not completely understand them). Rafael Bombelli (1526-1573), too, is one of those who participated in the elaboration of imaginary numbers. In his masterwork *Algebra*, Bombelli (1572/1966) became the first mathematician to give the explicit rules for multiplication of complex numbers. He showed that correct real solutions could be obtained from Cardan's method and used considerable new symbolism (*e.g.*, *R.q.* for square root, *R.c.* for cube root, and symbols for powers)

The question of the emergence of algebraic symbolism in the Renaissance is hence, I want to suggest, entangled with, and in fact a precondition of, the emergence of imaginary numbers.

Algebraic languages

Since Nesselmann (1842), many historical accounts on algebraic language draw a distinction in rhetorical, syncopated and symbolic algebra. Rhetorical algebra refers to the use of words and full sentences to express algebraic ideas and calculations. Babylonian Algebra is usually taken as an example of rhetoric algebra. In syncopated algebra some symbols are integrated into words and sentences. Syncopated algebra spans from Diophantus's *Arithmetica* to European algebra until the seventeenth century, and includes the algebraic language of Viète. Symbolic algebra rests on symbolic expression of calculations. This distinction, however, has not been without its criticisms. It is, indeed, quite superficial: it attends to modes of expression only and assumes a suspicious developmental account of modes of algebraic expression, ignoring important contextual and cultural elements. It has been argued that instead of representing a stage of conceptual development, syncopated algebra was "a mere technical strategy that the limitations of writing and the lack of printing in past times imposed on the diligent scribes that had to copy manuscripts by hand" (Radford, 1997, p. 27). As regards Diophantus, for instance, it is likely that all extant manuscript copies of *Arithmetica* derive from one Byzantine copy in Greek minuscule. The practice of using letters in sentences (such as the stylized "s," the last letter of the word *arithmos* with which the unknown appears in the copies of Diophantus' *Arithmetica*) may merely have been a practical move that proved useful to save time in the time-consuming task of copying manuscripts (see also Heeffer, forthcoming)

The roots of the algebraic symbolism of the Renaissance should hence be sought elsewhere. I will return to this point later. Now I turn to the work of Bombelli

Rafael Bombelli

In Bombelli's *Algebra* we do not find the use of the symbol i for $\sqrt{-1}$. Bombelli wrote *pdm* (*più di meno*, plus than minus) and *mdm* (*meno di meno*, minus than minus) for i and $-i$ (nevertheless he sometimes used *p. di m.* and *m. di m.*)

The equation $x^3 = 15x + 4$ (in modern symbols), solved by Cardan-Tartaglia method, leads to

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}}$$

$x^3 = 15x + 4$

$[x^3 = px + q]$

$(4/2)^2 - (15/3) = -121$

$[(q/2)^2 - (p/3)^3 = -121]$

$x = \sqrt[3]{2+11i} + \sqrt[3]{2-11i}$

$x = (2+i) + (2-i) = 4$

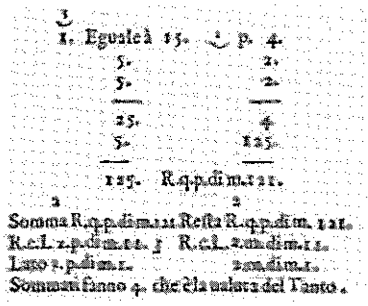
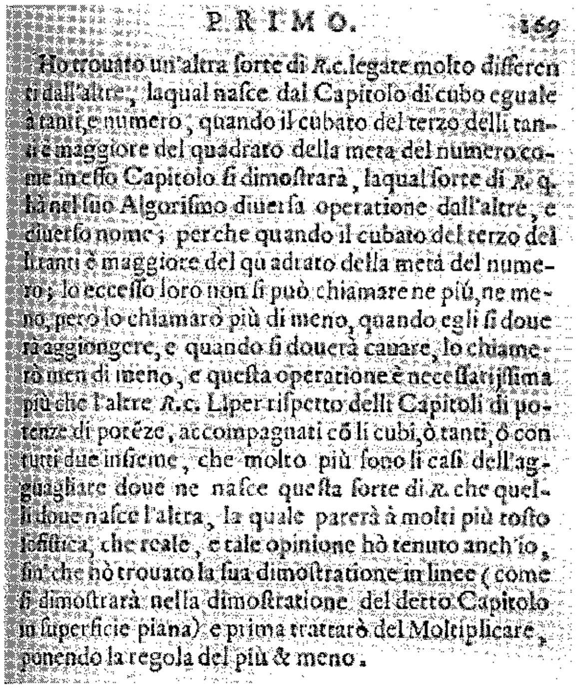


Figure 1 *Bombelli's use of imaginary numbers in the solution of a cubic equation*

so, in the considered case, to $x = \sqrt[3]{2+11i} + \sqrt[3]{2-11i}$ where $2 + 11i = (2 + i)^3$ and $2 - 11i = (2 - i)^3$. A solution is $x = (2 + i) + (2 - i) = 4$. In Figure 1 we can see the resolution on page 294 of Bombelli's *Algebra*. In Bombelli's symbolism, powers of the unknown are represented by a curved sign on top of which the exponent is written

The symbols *pdm* and *mdm* were hence used in order to make reference to "quantities" whose meaning was still unclear. In fact, Bombelli wrote (see fig 2): they "cannot be considered +1, -1, so I shall call them *più di meno* when it must be added and *meno di meno* when it must be subtracted, and these operations [*i.e.*, the consideration of these particular quantities in order to solve the examined equation] are very necessary in this case" (Bombelli, 1572/1966, p. 169; the translation is ours). And later, in the same figure, he says about these strange roots - the imaginary numbers - that they are *"più tosto sofistica che reale,"* that is, "more properly sophistic than real" (the translation is ours) where "sophistic" means "over-elaborated" and "excessively quibbling" in 16th-century Italian



P R I M O. 169

Ho trouato un'altra ſorte di R.c. legate molto differenti dall'altre, laqual naſce dal Capitolo di cubo eguale à tanti, e numero, quando il cubato del terzo delli tanti è maggiore del quadrato della metà del numero come in eſſo Capitolo ſi dimoſtrarà, laqual ſorte di R.q. hà nel ſuo Algoriſmo diuerſa operatione dall'altre, e diuerſo nome; perche quando il cubato del terzo delli tanti è maggiore del quadrato della metà del numero; lo eccesſo loro non ſi può chiamare ne più, ne meno, però lo chiamarò più di meno, quando egli ſi douerà aggiongere, e quando ſi douerà cauare, lo chiamerò men di meno, e queſta operatione è neceſſariſſima più che l'altre R.c. Li per riſpetto delli Capitoli di potenze di poteze, accompagnati cō li cubi, ò tanti, ò con tutti due inſieme, che molto più ſono li caſi dell'agguagliare doue ne naſce queſta ſorte di R. che quelli doue naſce l'altra, la quale parerà à molti più toſto ſofiſtica, che reale, e tale opinione hò tenuto anch'io, fin che hò trouato la ſua dimoſtratione in linee (come ſi dimoſtrarà nella dimoſtratione del detto Capitolo in ſuperficie piana) e prima trattarò del Moltiplicare, ponendo la regola del più & meno.

Figure 2 *Bombelli explains the still unclear meaning of complex numbers*

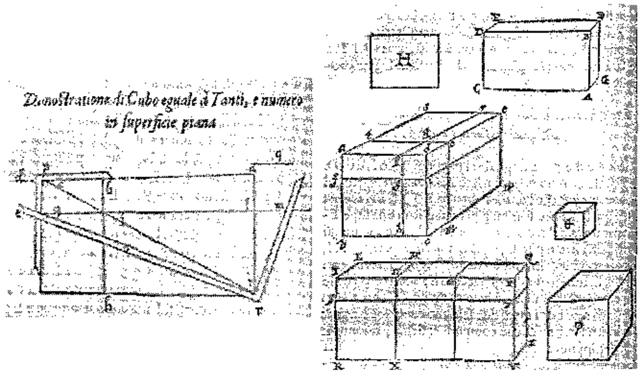


Figure 3 *Two- and three-dimensional constructions of the roots of a cubic equation*

Bombelli's geometric constructions

The mysteries surrounding the nature of these emergent bizarre square roots, Bombelli thought, could be elucidated through a geometric proof. He suggested two- and three-dimensional constructions (see fig. 3). Bombelli was drawing here on the Arabic tradition articulated by Al-Khwarizmi.

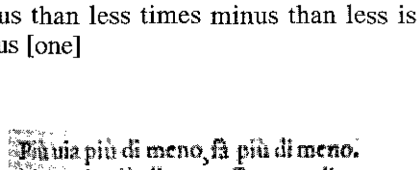
As regards the two-dimensional construction, it is worth noting that it is a general proof of existence of the roots of the cubic equations (see details in Bombelli, 1572/1966, pp. 228-229). In fact, geometric proofs have an important epistemic role: if the root of an equation can be constructed geometrically, then *surely* (according to the Arabian algebraic tradition) it is correct. Thus, if we can construct geometrically the root of a particular cubic equation that can be found by using imaginary numbers (*pdm* and *mdm*, that is, *i* and *-i*), then imaginary numbers become somehow legitimated.

Symbols, operations and emergent objects

Figure 4 shows Bombelli's original "rules" to calculate on the bizarre emergent objects

The first three rules and the last one read as follows:

> plus [one] times plus than less is [equal to] plus than less;
>
> minus [one] times plus than less is [equal to] minus than less;
>
> plus [one] times minus than less is [equal to] minus than less;
>
> minus than less times minus than less is [equal to] minus [one]



Più uia più di meno, fà più di meno.
Meno uia più di meno, fà meno di meno.
Più uia meno di meno, fà meno di meno.
Meno uia meno di meno, fà più di meno.
Più di meno uia più di meno, fà meno.
Più di meno uia men di meno, fà più.
Meno di meno uia più di meno, fà più.
Meno di meno uia men di meno fà meno.

Figure 4 *Rules of calculations of imaginary numbers (Bombelli, 1572/1966, p. 169)*

In modern terms, the three first formulas can be expressed as: $(+1) \cdot (+i) = +i$; $(-1) \cdot (+i) = -i$; $(+1) \cdot (-i) = -i$. The last formula corresponds to $(-i) \cdot (-i) = -1$.

Although there is still no name with which to refer to these strange roots that we now call imaginary numbers, in the previous rules Bombelli tells us how they behave. These rules do not define imaginary numbers in an axiomatic way. Rather, the only meaning they can borrow comes from calculations. It is worth noting (Mazur, 2003) that the symbolic expression on which the previous rules are based can be interpreted as a (possible) algorithm for the production of instances of the concept it represents. So the possible operations on the object become embedded with the representation (Heeffer, forthcoming).

But what is it that makes thinkable these new mathematical creatures? Klein has suggested that, in opposition to ancient mathematics and its emphasis on objects (circles, triangles, etc.) and the investigation of their properties, sixteenth and seventeenth century mathematics "turns its attention first and last to *method as such*. It determines its *objects by reflecting on the way in which these objects become accessible through a general method*" (Klein, 1968, p. 123). And method here is based on the idea of calculation and the symbols in which calculations and objects are expressed. Such a shift in mathematics, particularly in the new algebra of Cardan, Bombelli and other Renaissance mathematicians should be linked, Radford (2003) argues, to the rise and spreading of manufacture and a new systematic dimension in human actions. More specifically such a shift was encompassed by the appearance of

> a world in which machines and new forms of labour transformed human experience, introducing a systemic dimension that acquired the form of a metaphor of efficiency, not only in the mathematical and technical domains, but also in aesthetics and other spheres of life. (Radford, 2006, p. 512)

Final reflections

In this paper I offered some comments on the historical context and meaning of imaginary numbers. By way of concluding, I want to point out that, from an educational viewpoint, Bombelli's resolution may be a useful tool. As mentioned previously, the introduction of imaginary numbers did not take place in the context of *quadratic* equations, as in $x^2 = -1$. It took place in the course of solving cubic equations. Sometimes, the resolution of *cubic* equations does not take place entirely in the set of real numbers, even if their solutions are real numbers. A substitution of $x = 4$ in the equation above ($4^3 = 15 \cdot 4 + 4$) is possible in the set of real numbers. Would it be advantageous to introduce imaginary numbers in such a way? Without endorsing a recapitulation view of knowledge, an empirical research showed that imaginary numbers *in the calculation* of the resolution of an equation, à la Bombelli, are frequently accepted by pupils (Bagni, 2000). Of course, a more detailed investigation is required. At any rate, the progressive emergence of imaginary numbers, as well as the awareness of their properties and their ontological nature, is something that was not accomplished suddenly by the mathematicians of the past. Nor should we expect our students to achieve it from one day to the next. But history allows us to see how, bit by bit, these strange creatures that we now call complex numbers came into life. History allows us also to gain an awareness of the cultural context that made imaginary numbers possible. Last but not least, history may also encourage us to envision new ways in which to talk about these numbers in our classrooms.

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