A Neuropsychodynamical Theory of Mathematics Learning*

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* An address given at the 6th PME conference, Antwerp, 1982

1. Introduction; mental achievements as networks

Recent findings in the neurosciences throw some light on the working of the human brain as it stores and brings forth mathematical knowledge. We develop here an epistemological-psychological picture of mathematics consistent with these findings, adding a number of (falsifiable) conjectures derived from mathematics as a *creative* and *discovering* activity.

Every conscious mental activity in animal and man, aimed at achieving a certain immediate or distant goal – as these findings support – integrates a number of, dynamically stored, former achievements into a new one, *viz.* the mental achievement corresponding to that goal. Thus, every achievement is itself a *network* linking other achievements. On this view, brain development, starting at the moment of conception, is a confluence of networks of different sorts. Some come into being as "primitives", *i.e.* they are preprogrammed as innate senses or instincts, but in man almost all other achievements are learned. There are those that, once learned, may be changed or copied, as is the case when we have stored many mental pictures of a triangle, much in the same way as we all use the number "2" many times over in different contexts of achievement.

Usually, however, we copy achievements only partially when integrating them into new ones, since we tend to "throw out" inessential concomitants of the former, making it unnecessary to store every individual achievement.

For instance, not only instinctual achievements but also learned ones are often stored on a long term basis as moulds or *templates for routine use*, so that it becomes quite unnecessary to store every individual cast of the template. Hence, *skills* of all sorts are achievements, that, even in pure mathematics, involve the senses, be it only because even here pencils are indispensable. All mathematical discovery integrates former skills into new networks of reasoning. These networks may be discarded, changed and learned by others, eventually producing new skills. We ask: "How did you arrive at this or that result?", so that we may be able to re-achieve it, but in a different set of brains. In pure mathematics we find the strongest bifurcation of all between template and cast, in that the former is made part of a formal language, whereas the latter as an example or as a model of the theory leads a life relatively independent of the template.

It is noteworthy that the individual examples may be very rich in content since they may involve a lot of concrete imagery drawn from experience. A "model" for a theory is a set-theoretically organised structure of examples. There is a neurophysiological basis for the duality template-cast in mathematics, as we shall see.

There is also the question of "memory". Several kinds of memory are being discerned today : cognitive- and motor-memories, short- and long-term memories, etc. Emerging physiological insights into these kinds of memory are precious since they start to relate memory to neural networks as well as to biochemical processes. They are conducive to understanding the connections between the two different levels at which mental achievements and operations can be studied, *viz.* (i) the *micro-scale level* of (brain) physiology, and (ii) the *macro-scale level* of behaviour and functional experience, psychology, epistemology, etc., including pedagogy.

In the light of the present state of knowledge in brain physiology as it relates to behavioural function, we may describe the central task of *neuropsychology* (and neuro-epistemology) as follows:

> *Find those two-way mappings between the macro- and micro- level of the functioning of the human brain which map*
>
> mental achievements *to* neural networks
> *and*
> mental activities *to* flows in neural networks

It should be noted that we do not *identify* mental achievements with neural events, thus avoiding the trap of philosophical materialism. We believe that progress in the neurosciences is often blocked by over-simplifying philosophical choices. We shall say that neural networks are the *substrate* of mental achievements, indicating thereby that neural networks exist, develop and extend hand in hand with mental networks; whence the mathematical term "mapping". These mappings are not to be construed as in some sense one-to-one, since one and the same mental achievement may be mappable to many different networks. For instance, following the necrosis of certain brain tissues, other tissue may take over the former's function. The enormously high degree of connectivity within the cerebral cortex grants it a flexibility and rerouting capability, conflicting with any one-to-one speculation. (Karl Pribram uses the metaphor of the hologram – which is essentially not one-to-one – in connection with the brain's processing of sensory stimuli. See, for instance, *Cognition and Brain Theory*, 4,2: 105-122 (1981).)

For the Learning of Mathematics 3, 1 (July 1982)
FLM Publishing Association, Montreal, Quebec, Canada

2. On the difference between "wiring" and "circuitry"

The word "network" refers of course to the "wiring" of the brain, *i.e.* to the manifold nerve fibres that connect the nerve cells or neurones. There are mainly three types of wiring in the brain, *viz.* (i) the *projection fibres,* including those connecting the older and lower centres with the "newer" part or *neo*cortex, (ii) the *commissural fibres,* which run mainly between "homologous" areas of the neocortex of the two cerebral hemispheres, and (iii) the *association fibres,* including the longer ones connecting distant areas in one and the same hemisphere as well as the short ones between adjacent areas [cf. B. Kolb and I.Q. Whishaw, 1980, ch. 20]. The *wiring* of the brain, in the first instance complete at birth, undergoes, in the second instance, a further myelinization and a further connecting-up with neurones in later life. Through the wires, at any point during life, run the seemingly random chemo-electric pulses that are controlled by the "pacemakers" of the thalamic nuclei in the midbrain. The well-known E.E.G. only picks up an averaged-out picture of these pulses as they define electrical changes in the neocortex.

A *circuit,* on the contrary, is a subnetwork of the wiring that has been "potentiated", *i.e.* for a shorter or longer time it has been made to support special, repeatable, non-random reverberating firing patterns, in such a manner that it maps to a learned ability (or achievement). The change from mere wiring to circuitry is being studied intensively nowadays [cf. T.J. Teyler, 1978]. The words "axonal growth" and "synaptic potentiation" refer to changes of this kind, but much has still to be discovered. It is generally believed though that well-defined circuits map to achievements, thus defining the usage of the (neo)cortex as it supports instinctual and habituational tasks, learned activities, skills and achievements. The dynamic integration of networks of circuitry is thus believed to form the substrate of all human action and understanding.

A non-random special pulse firing is not necessarily seen on the E.E.G., though the onset of it usually is (desynchronization). It need neither be the result of consciously willed behaviour, nor necessarily of such a nature that it maps to voluntary behaviour. There are obviously many mental achievements arrived at un- or subconsciously.

If no special firings take place then circuits are in a holding pattern from which they may be recalled; they are dynamically stored. Long term memory (l.t.m.) is thought to be encoded in such circuits; when the circuit is being fired then information encoded in it becomes mappable to the mental level. However, circuits or parts of them may become "depotentiated", so that long term memories become hard to get at. It is *circuitry* that is preserved on a long term basis rather than isolated bits of information.

The role of the frontal lobes in higher mental function should be stressed. These lobes are of primary importance for the *egocentric spatial relations* and the planning of movement by the individual. More generally, the coordination and integration of body motion, the stability-in-time of emotional and social behaviour, as well as long term planning, turn out to be disruptable by lesions in the frontal areas.

By contrast, the *allocentric spatial relations* and body tasks turn out to be disruptable primarily by lesions in the parietal and temporal lobes [for a survey picture of the lobes see Kolb and Whishaw, 1980, ch. 1]. However, frontal lesions do not significantly affect l.t.m., once these have formed. Short term memories (s.t.m.) however, are affected by these lesions, notably in an asymmetric manner, as follows. For example, left frontal lesions may affect verbal recency – which involves wirings running between the left frontal, parietal and temporal areas –, and right frontal damage affects non-verbal (pictorial and musical) recency.

Taking into account existing studies on cognitive learning and long term memory, we *conjecture* that all l.t.m. is based on what we call "long term ideation" (l.t.i.). L.t.i. is that particular function of the mind which – at its strongest during the initial stages of one's life – maps to (flows into) large trees of thalomo-cortical and cortico-cortical fibres, *initializing* them and turning them into trees of circuitry in the process.

> Initializations of this kind may take place via the same feed-forward and feedback (l.t.m. and l.t.s.) loops as described in Eccles [1980, p. 175 ff.] on the basis of known anatomical connections and physiological evidence [Kornhuber, Bliss, Lϕmo etc., loc cit.]. They are known to involve the "emotional brains" or the limbic system with a principal role accorded to the hippocampus and the amygdala.

The most important example of l.t.i. is the discovery of one's own personal gifts and talents. It is a selfrecognition process, often followed by more or less clear plans to develop the gifts. Thus, on this theory, the ongoing growth of an individual's l.t.m. during life, is the superstrate of the dynamical extension and connection of l.t.i. circuitries into denser and denser networks until all the available wiring and firing patterns are exhausted.

As a corollary to the conjecture, s.t.m.'s are circuits not connected to l.t.i.'s. They may or may not be written into l.t.m. (e.g. in a manner as indicated by Eccles). On the other hand, l.t.m. of bits of information (names, facts, etc.) are to be viewed as the accessibility of small subcircuits within greater l.t.i. networks; according as a subcircuit forms part of a greater number of other subcircuits having the property that they can be independently and/or randomly accessed, the remembering is more facile and fast. The access need not run via the frontal lobes, as we have seen (hippocampectomy). Since pronounced mathematical talent often shows itself at a young age, future neuro-epistemological research on mathematical thought should start with youngsters in the lower grades so that their development can be followed over an extended period of time (although the study of historical prodigies of mathematics may be of help, cf. §5).

3. Neuropsychology and the status of the number and space concepts

Since mathematics is termed an "objective science" with fairly simple basic concepts, the neural mechanisms to which these concepts map must admit of being tracked down. Whereas among mathematicians there is no disagreement on the spontaneous *use* of number and space concepts, philosophical considerations still tend to pit some form of realism against some form of idealism. There is the well-known chicken-and-the-egg-question: do mathematical concepts or templates arise from (the experience of) some given order in the world that is independent of the knowing subject, or is the order that we experience a result of *a priori* mathematical concepts being applied to the world? Let us tackle this question with our dynamical theory of circuits. We look at natural number and space concepts.

Number has an *ordinal aspect* produced by such sequential neural firing patterns and processing modes that map to sequences of spoken or written discrete signs or symbols, etc., or to sequences of purely mental operations on the same (e.g. silent mental counting). Indeed, it has become known to what areas of the neocortex silent, digital counting maps: namely, to the topmost part of the superior frontal cortex, just before the premotor area, in the *left* hemisphere [cf. Roland, Larsen, Lassen and Skinhφj, 1980, fig. 5 and 7]. One cannot, however, say that *all* counting maps to the left brain, since for example *counting musical tones* maps to areas high up in the *right* temporal lobe [Ronald, Skinhφj and Lassen, 1982])*.

It is to be noted that counting, calculating, and concomitant picturing, is *not* as such a mathematical activity presupposing a number concept. But mental discernment of lawlike features involving *all* numbers (as, for example, expressed in the Peano axioms) and reasoning with them, certainly *is* a mathematical activity.

Number also has a *cardinal aspect*, showing up when one forms mentally-created collections or *sets* of things, leading to "magnitudes" and the cardinal of a set. This is a *holistic* processing mode, binding the things into a pictorial conceptual unity. If this binding together into a set takes place in extrapersonal space (usually involving the senses), then it is known that neural mechanisms in the *right* parietal areas are involved. If the act is purely mental, without language in parallel, and with the ears plugged and the eyes closed, then the involvement of these areas is still conjectural (but extremely probable). However, there are indications that in that case there is a greater blood flow in a distinct part of the right superior frontal cortex [Roland and Larsen, 1976].

Modifying slightly our former *complementarist* view [cf. Kuyk, 1977], we summarize as follows.

Since natural numbers are at once cardinal and ordinal numbers, there is only one irreducible concept of natural number, involving the whole brain, and mapping to the two principal complementary processing modes of the cerebral hemisphere.

Number theory is that mental activity of the brain that aims at the discovery of lawlike features governing the natural numbers and at expressing or encoding them in a logical-linguistic framework.

> Comments. Our theory unifies creative as well as discovery features. In a very real sense, radical Formalism and Intuitionism tend to reduce mathematics to operations mapping to circuitry in the left hemisphere ("idealism"). Cantorism claimed the reduction to imagery and sets ("realism"). Our number concept is falsifiable. For instance, on our theory, splitbrains, who before callostomy did not have a number concept to begin with cannot be supposed to form one after the operation. They cannot do any creative mathematics; they may at the most repeat and extend the formal language part of their previous knowledge. They may be able to improve their counting and their picturing ability, but the connection between these is seriously disturbed. They may do the machine-like parts of mathematics rather well. Note that no computing or thinking machine is in the offing which one expects to form a number concept all by itself. Hilbert in his *Grundlagen der Geometrie* in a sense proposed the systematic development of the two processing modes of the hemisphere separately by keeping them in opposition, thereby according the right hemisphere an appositive role. He discovered topological concepts in the process.
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> Further, Cantor's continuum hypothesis is an expression of the two different processing modes. Finally, Gödel's well-known theorems say, in effect, that formal language operations (i.e. abilities mapping to processing modes in the left hemisphere) are significantly poorer than the language of spontaneous number theory; a fact, we submit, that is due to deleting right processing modes from the spontaneous theory, which integrated the two at a higher level of operation.

Mathematical *space concepts* also have aspects mapping to the two cerebral hemispheres. Firstly, they all have a *pictorial aspect*, even if the dimension is high, and the space as a whole cannot be represented within the neural networks that are primarily reserved in the right parietal-temporal lobes for sensori-visual patterns and memories. It is known from the investigation of splitbrains that for the formation and preservation of sensori-visual and sensori-auditory "pictures" the right hemisphere does not need the left one. On the other hand, it is not known to what extent splitbrains and normals use the same lobes for imaginative-creative imageries. At any rate, producing pictures or evoking them is not as such mathematics. There has to be a complementary *analytical-logical aspect* in order that one may speak of mathematics. That analytical aspect is supported by networks in the left hemisphere, which is the primary "seat" of logical reasoning and linguistics as well. Mental operations are manifest in the multiple ways in which the left brain acts on the right one, forming sets of spatial pictures (points, low-dimensional figures, curves, shapes, spatial motions, etc.) that become

* Note that we are talking about people with normal brains. Recently, a lot has been learned about functions lateralized in one of the two cerebral hemispheres when they function *in separation* (callosotomy, or if one hemisphere is drugged, etc.). Normally, any excitation of an area in one hemisphere also excites the "homologue area". The question is: which of the two areas "has the lead", i.e. is more excited than the other?

part of the highly personalized imagery of a mathematician, while giving holistic *meaning* to his abstract language. It is clear that in mathematics the left brain is "in charge", even if features abstracted from sensorial and imaginative pictures are often conducive to the construction of a theory. As has been suggested before [Kuyk, 1977, ch. 3], mathematical language and imaginative "models" mutually define each other in a dialectical process.

The reader may add comments to this as we did after dicussing the number concept. It is important to note that spatial pictures in mathematics may range from Gestalt image (as in classical geometries) and loosely connected point sets to tactile/kinaesthetic high dimensional imagery and continuously deforming topological entities (as in differential equations and dynamic theory). Also the auditory images, making a mathematician "hear" his theory, should not be underestimated. Presumably, these images map to the whole range of association areas between the (primarily) right frontal, parietal and temporal lobes. Recent stereognostic investigations [Roland and Larsen, 1976], and those on voluntary movements in extrapersonal space cited already, are consistent with the complementarist view here exposued. Future research in this area hopefully will be done with a view to finding areas to which very elemental intentional mathematical acts map.

4. Catastrophe models and the dynamics of some brain functions

"Flows and networks", mathematically-physically speaking, evokes a picture of reverberations, interference, feedforward and feedback mechanisms, excitation and inhibition, smooth transitions and catastrophic jumps, input-output systems, control parameters, etc. In the case of the highly complicated network of the brain one may expect all these features to occur, even if the mind partakes in the control. We assume from the outset that the maps of 1 are like "diffeomorphisms", i.e. they map the smooth parts of processes to smooth images, but they map discontinuous jumps to discontinuous jumps.

In mathematical learning jump features are prominent: the sudden recognition of a pattern in problem solving, but also the discovery that certain features fit into a comprehensive formal framework, are examples. In a former paper [Kuyk, 1981] we have noted that switches between language and imagery occur at all levels of mathematical thought, ranging from those where the cognitive depth is relatively little (as in problem solving, exercises, etc.) to those where an extensive imagery already exists (as when a model for a whole theory is involved). These transitional jumps were attributed to symmetries in subnetworks of the wiring of the brain, which, in fact, facilitate switching.

Thus the idea arose to model the higher functions of the human neocortex by means of a cuspcatastrophe (Figure 1). We suppose the reader to be familiar with the general meaning of this picture. [cf. Zeeman, 1977] The lower sheet of the surface M is the "language sheet"; any mental ability mapping primarily to the left hemisphere may be thought to lie on it. The normal control parameter is "analysis" as opposed to "synthesis". The latter gives the direction of holistic modelling on the upper sheet, which contains the primary functions mapping to the right hemisphere. The area of M above the cusp is the area of tense thought; here the concentration cursor may jump to the other sheet following a Formalization/Axiomatization catastrophe (the arrow F/A) or a Creative Imagination catastrophe (the arrow CI). On the lower sheet, the farther the cursor moves away from the cusp the more machine-like the mental operations become. Likewise, the more the cursor moves away from the cusp on the upper sheet, the less amenable to formalization the imageries become.

By the way, once a theory has been established then jumping has come to an end and one can "walk" from language to model via "normal". What once took place at a great "cognitive depth" for someone has now become "trivial".

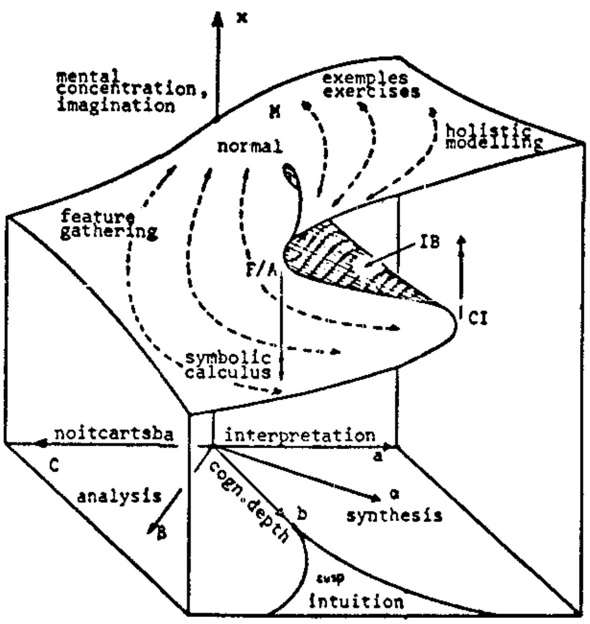

Figure 1
Cuspmodel of mathematical concentration and discovery. It does *not* model the motivational drives that make the thought cursor move on the convoluted surface M, but not on the Inaccessible Behaviour part.

We verify the five characteristic features of a cuspcatastrophe, *viz.* sudden jumps, divergence, inaccessibility, hysteresis and bimodality [Zeeman, 1977, p. 18]. Sudden jumps are part and parcel of discovery [cf. Hadamard, 1945, passim]

Divergence takes many forms. A very interesting one is "proof by handwaving" as opposed to "formal proof". Handwaving is an holistic language (body language) which neuroscientists know the right cerebral hemisphere is well-equipped to produce and understand. A handwaving, muscleflexing researcher proves that he is working from, and acting on, an egocentric model. His task is to translate the features carried by the handwaving into a more formal, and hence more communicable language, so that he can be checked by others. Every mathematics teacher should use, at the right time and place, handwaving to communicate images, etc., since purely formal proof is only part of the whole mathematical story

Inaccessibility refers to the IB-part of the sheet M, where the cursor cannot run. An example is the gap between the discrete integers and the geometrical continuum. Let us cite Fraenkel [1958, p. 198] for this:

> To understand the nature of the problem one should stress the fundamental difference between the *discrete*, qualitative, *individual* nature of number in the "combinatorial" domain of *counting* (arithmetic) and the *continuous*, quantitative, *homogeneous* nature of the points of *space* in the "analytical" domain of *measuring* (geometry). Bridging the abyss between these two heterogeneous domains is not only the central, but also the oldest problem in foundations of mathematics and in the related philosophical fields (two underlinings deleted by the author).

Hysteresis is the often long delay between the first glimmers of insight into how one can view things in terms of a processing mode of the other sheet, and the total breakthrough when the cursor makes the final jump to the other sheet

Bimodality obtains everywhere a formula or statement admits of a direct interpretation in terms of imagined or real models. Every theory also has language elements that do *not* admit of direct interpretations. The loose connection between theory and model has often led to the discovery of new models for an existing formal theory.

Anyone familiar with the theory of cuspcatastrophes as models for dynamical systems will recognize that the above five characteristics *prove* something if we could also show the *stability* of the brain dynamic; i.e. if we could show that our cusp is "locally equivalent" to a generic cusp in the set of all 2-parameter families of potentials. In fact, we believe this is *not* the case, although it is certainly true that for most mental functions a small change ("perturbation") of the excitation in the area of the neocortex to which it maps does not destabilize the function (this is widely attributed to the great redundancy of fibres and neurones in the excited area)

However, since every sheet of M maps to many many different functions (modes or parameters), and since there exists a high degree of connectivity between most parts of the neocortex, we rather view the present model as a "blow-down" of a very high-dimentional catastrophe model. Thus, Figure 1 represents one of the principal symmetries (if not *the* primcipal symmetry) of that high-dimensional model. It may be viewed as being obtained by squeezing two groups of its many control parameters into the two principal ones of Figure 1, *viz.* the "analysis" and the "synthesis" parameters as normal parameters (or alternatively, the "abstraction" and "cognitive depth" parameters as splitting parameters). This answers a question posed by Peter Hilton: whether the usual stability and/or genericity conditions for cuspcatastrophes are fulfilled in our model

The neocortex must be regarded as harbouring a huge number of attractors (and repellers) instead of only two. One is tempted to let the so-called "focal areas" seen in the active brain (and that surface everywhere in neurophysiology) correspond to the attractors of the assumed very high-dimentional model.

5. Odds and ends

In the light of a "dynamical theory of circuitry" the usual textbook terminologies stating that mental functions are "lateralized" or "localized" or "more left- than right-lateralized", etc., are wrong. What is meant is that a certain fixed area of the surface of the neocortex is physiologically "active" when that function obtains. Or that *some* area (in the left or right cortex) is active, as in tachistoscopical and dichotic listening tests. Our network theory implies that the cerebrum remains *one* even after complete callosotomy (without the bisection of the massa intermedia). One is simply left with a mutilated network. However, in view of our infinitesimal knowledge of the cerebral network, the textbook terminology may be used with these provisions in mind. As a matter of fact, *any* mental ability maps to the two hemispheres, either because the mind may act directly on the "liaison areas" of the neocortex [Eccles, 1982] or just because any excitation of an area *ipso facto* also excites the homologous area (by way of the callosum), thus calling on the (or an) other processing mode of the same ability.

> Sherrington's and Eccles' theory on the liaison brain, in view of the recent results by Roland, Larsen and Skinhφj [1980], raises the extremely important question, whether the same experiment by the latter four authors, but applied to split-brains, would also show increased blood flow in *both* the supplementary motor areas. Their theory implies that, in split-brains, not only "cross cueing" may supposedly take place along the limbic areas, but also via the "mind". There is no conclusive evidence for this as yet. It seems that the mind is "helpless" without the brain

The question is whether the latter excitation is strong enough to map back in a way so as to *enrich* the mental function. In any case, there seems to be a triangular pattern of interaction between the mind and the two hemispheres, whereby the mind usually leads the action by helping to *unfold* the potentialities of the brain

In this light it is understandable that, *with respect to a given ability* people often show a so-called *preferred mode*. Such a preferred mode may have been brought about by educational (environmental) influences, or it may be a natural disposition and innate. (One should be very careful in calling it the one or the other in any given case.) Poincaré's well-known *psychological typology* [Poincaré, 1913, 1914] of (the ability of) mathematics students ("logical", orderly "types" *vs* "geometrical", intuitive "types") really cuts ice against the background of the cuspmodel [Kuyk, 1981, p. 370]. He had a keen understanding of the two main mathematics processing modes, although his use of the word "intuition" still smells very Kantian There are mainly *two* kinds of intuition in mathematics, since one can break out of the cusp in two directions, making different kinds of progress in the process. In view of the many different tasks that map to any one hemisphere, the two corresponding preferred modes appear in many guises, such as : logical *vs.* geometrical (Poincaré), algebraic *vs.* topological [Kuyk, 1977], discrete *vs.* continuous [*loc cit.*] analytical *vs.* holistic, digital *vs.* analogue, verbal *vs.* nonverbal, quantitative *vs.* qualitative [Fraenkel, *loc. cit.*], syntactical *vs* semantical, formal system *vs.* interpretation, pure *vs.* applied, template *vs.* cast, analytic mathematics *vs* non-analytic mathematics [Davis and Anderson, 1979]. A call to a return to non-analytic mathematics usually involves a call to return to applied aspects and the senses [Davis and Anderson, *loc. cit.*, Goodman, 1979; and Hersh, 1979].

Grattan-Guinness [1980], in a study of the prodigies of mathematical physics in the first three decades of the 19th century, discerns *three* preferred modes of thinking, *viz.* the algebraic one (e.g Poisson, Laplace, Ampère, Carnot, Lagrange, de Prony), the geometric one (e.g Euler, Fresnel, Fourier, Hachette, Monge, Lamé, Navier), and the analytical one (e.g. Cauchy, Duhamel, Liouville, Sturm). The list shows that "preferred modes" have nothing to do with the choice of discipline of mathematicians One may be an algebraist with a geometrical way of thinking, etc. On Grattan-Guiness' criteria (the use of pictures, the way these mathematicians prefer to introduce the integral, etc.), Gauss does not fit into any category.

We suggest that his analytic category, which is the smallest, contains the "bimodals". They show no clear preferred mode since the projection of their "thought cursor" in the control place C just as easily breaks out of the cusp on either side Presumably, their abilities are not "lateralized" in the fashion of the 96% of all right-handed people (on whom the design of Figure 1 is based) This raises a lot of questions

For instance, psychological investigations on handedness and sex [Springer/Deutsch, 1981 and Craig, 1980] suggest that the remaining 4% of right-handed people and non-familial lefthanders (and perhaps most women) might be *bimodals.* There are indications that these people have more language ability mapped to the right hemisphere and more spatial ability to the left. (It is an unresolved problem whether greater lateralization also means greater ability.) As an example, a bimodal, when presented with a mathematical function, might be better able to plumb (and synthesize) simultaneously the nature of that function as an algebraic entity and as a graph (in space, for example), than a unimodal. So Grattan-Guinness' "analytic" category should contain relatively many from the remaining category of 4% righthanders. It is interesting anyway to know which mathematicians mentioned in this paragraph are natural lefthanders. As a kind of hypothesis we propose that bimodals have not one but *two cursors* running on M, and hence have an ability of *parallel processing.* This hypothesis is supported by investigations on left-handed people who were confronted with (dichotic) rhythm tasks. Out came the surprising result that left-handed subjects differed significantly from right-handed subjects in performance on a task that involved the analysis, synthesis, and response to dichotic rhythm patters.[1] "Possibly, the left-handed subjects are better able to analyse simultaneous stimuli and integrate differential information from them into a coherent structure. A bi-dominant theory of left-handedness would support such an interpretation of the data" [Craig, 1980, p. 618].[2]

The relation between Grattan-Guinness' two other categories and the cuspmodel is clear. Let us add to his groupings, partly as a kind of speculation, some other names To the "algebraic category" belong: de Fermat, Newton, Poncelet, Kummer, Weierstrass, Möbius, Kronecker, Dedekind, G D. Birkhoff, Polya. To the "geometric category" : Descartes, Leibniz, Dirichlet, Galois, Steiner, Klein, Carton, Hamilton, Hermite, Hadamard, E. Noether, Weyl, Einstein, Brouwer, Thom. Further bimodals could be : Gauss, Jacobi, Riemann, Poincaré, Hilbert, Weil. Until we have clearer scientific ways of discerning the exceptional right-handed people and the non-familial left-handed subjects, and know more about their peculiarities, all our categorizing is a shot in the dark, just as most present-day knowledge on the relation between sex and mathematical ability is

6. Epilogue on mathematics education

What, if any, conclusions can be drawn for teachers and mathematics education?

This paper should *not*, in the first place, encourage teachers or anyone else to categorize learners into "algebraic" and "geometric" types; since it appears foremostly that mathematics learning – and especially mathematical creativity – thrives on making hierarchies of *switches between the two principal modes.* Categorization often leads to harmful stigmatization. Besides, there are many modes between which switches are being made (e.g. between automatic calculation and carefully crafted new mathematical language forms; both modes map dominantly to the same hemisphere). It looks as if teachers should cater to both kinds of students, challenging those minds that tend to stick to one mode into overcoming the blocks that may prevent them from switching to the other. It is not altogether impossible that blocks of this kind have a physiological cause, and a training of the weaker function may prevail on the tremendous plasticity and adaptability of the student's brain to influence his physiology in a favourable direction. Some teachers themselves may even become aware of their own one-track minds and correct the necessary restrictions it imposes on the minds of the

1) Right-handed people have a right brain dominance for music appreciation and a left brain dominance for dichotic rhythm tasks [cf. Bever and Chiarello, 1974]. Professional right-handed musicians show left brain dominance for music, however, probably since they listen "analytically" to music more than non-professionals.

2) A cursor may be regarded as the "center of gravity', of an "active" subnetwork of the brain to which a conscious mental activity maps at any given moment Thus the dichotic tests suggest, that for some time at least, a task is being done in each hemisphere independently, until they join, leading to a single cursor again.

students – but presumably not on the best and most gifted students.

Secondly, of equal importance is the fact that mathematics education as a whole, proceeding from grade school to university, is a *process*, in which, by the way, the division between the different schools is only artificial. Hardly any mathematics educator will doubt this, but there is a considerable difference of opinion regarding which concepts or subjects should be taught first and what later. For instance, what should the role of mechanical and mental calculation and arithmetic be, and to what extent should Euclidean geometry still be taught and in what form? These questions are cognate to what in neurophysiology are called "the committed and the uncommitted areas" of the neocortex. At birth certain subnetworks of the brain wiring appear, upon potentiation, predestined to map to specific macro-scale functions. We call these subnetworks *committed networks*. At present we are still far from knowing all the committed wiring, although the discovery is in high gear. (We do not speak here of pathological situations where congenital brain damage forces other wiring to "take over" the function of damaged ones). According as the function is "lower" (e.g. sensorial, movement, movement planning, somaesthesia, etc.) the committed wiring mapping to it shows up better on the surface of the neocortex than when the function is "higher" (e.g. counting, spatial orientation, etc.), with the language functions somewhere in between. Apparently, the higher functions, by way of the deeper projection, association and commissural fibres, seem to organize the lower elements into a more complex unity. In fact, the radiotracer techniques used up until the present time are not very suited to looking into the (networks of the) deeper layers, so that even the networks of the lower functions must be said to be only partially known. One may expect that the Positron Emission Tomography technique which is in the offing will provide better insights into the paths of the lower and higher functions as well.

As for the *uncommitted networks*, they are not just subnetworks of the brain that can be separated from the committed ones, since they assume them, in the sense that they interweave them. They reflect the great extent of *freedom* that the mind exercises when unfolding the lower functions to make new wholes. Perhaps the best way to look at them is as follows. Regard for a moment every distinct (predestined) firing pattern of any committed network as one unit, by identifying all the connected neurones of that network into one unit. We call this unit a "theoretical node". Then the total network of the brain simplifies into a new one as follows. In the new network a *node* is either a neuron not uniquely involved in any predestined firing pattern, or it is a theoretical node. In it a *vertex* is a nerve fibre that, under at least two different unfoldings of the brain, may take part in firing patterns that map to (at least two) different functions. This simplified network conveys an image of the total "uncommitted wiring" of the brain.

The present state of brainphysiology does not permit a guess as to how many theoretical nodes there are, nor give information on the degree of connectivity of the uncommitted network. Certainly, the l.t.i. networks (cf. §2), along which the l.t.m.'s are being laid down, belong to it; but also those wirings that are used for combining phonemes (committed!) into wiring that maps to different word sounds.

The flexibility with which uncommitted wiring (u.w.) may be used reflects in the magnificent variety of human creativity. There are thousands of more cultural achievements possible than we know, and every culture tries to perpetuate itself by an educational *system*, making for a particular way of using the u.w.

The perennial philosophical question as to the difference in nature between "the sciences", and "the humanities" has the difference between committed and uncommitted wiring as its cerebral substrate. This becomes clear if one realizes that the sciences look for structures that are invariant over human cultures and volition, even if the structures are man-made and materially encoded so that they can be kept constant over a long period of time (e.g. machines, computers, economic systems, nation states). On the other hand, it is in the social sciences, medicine and psychology that the (methods of the) natural sciences vie with human will and freedom. Human creativity is clearly restricted, meaning that natural laws and the committed networks exercise constraints on the u.w. The extent to which this is the case can presumably be studied in the different human cultures that are known and in their history. All cultures share in the brain development that has led mankind to survive over eons of time. It is therefore reasonable to assume that there are higher organizations of u.w. especially geared to typical human functions such as love, social behaviour, aesthetics, ethics, justice and belief(s), much in the same way as fine motor systems and the cerebral language centres, etc., served the same survival.

What we see everywhere in human history is that forced commitments of u.w. – e.g. attempts to dominate over healthy people's minds by imprisoning them in rigid social structures – sooner or later lead to a reclamation of freedom of commitment on the part of the imprisoned subjects (whether in the name of justice or on ethical grounds). These facts undergird our claim that typically human functions are gifts that map to higher organizations of networks. It is remarkable that these functions can be changed or even killed by lesions in the brain or by drugs – although not selectively as yet.

We conjecture the existence of a *scale of committedness* for cerebral networks, ranging from the lowest level of total committedness (e.g. instincts) to those that admit of a greater variety – but still restricted number – of unfoldings. This scale would have the "sliding scale of exactness of reasoning", ranging from the exact sciences via psychology and the social sciences to even ethics and theologies, as its superstrate. Elsewhere we will substantiate our claim that in every teaching and learning situation all the typically human functions of the brain should somehow get their due if the teaching is to be rewarding and not frustrating to teacher and learner alike.

As for mathematics, it shares in all the traits of general culture. It is neither entirely free, or otherwise it could not be checked for correctness. Nor are its achievements fully predestined, or otherwise it could be equated with an instinct. It is experiential, or otherwise it could not discover anything; it is creative, or otherwise there would be no results beyond the level of counting. It is bound up with all the other faculties of the brain, or otherwise it would not be so effective in almost all areas. Insofar as it concerns itself with some concrete representation it is in danger of begetting abstractness; and as soon as it becomes abstract there are no things safe from its application. It may prosper by separating features from an observed wholeness and it may flourish by attaching features to what is observed as well. Finally, it is undefinable since it itself searches for the definable.

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The object of the pure Physic is the unfolding of the laws of the intelligible world; the object of the pure Mathematic that of unfolding the laws of human intelligence.

J.J. Sylvester

Infinity is nothing but a peculiar twist given to generality.

C.S. Pierce
