On the Learning of Mathematics Through Conversation[1]

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I. Introduction

Since the National Council of Teachers of Mathematics published new standards for mathematics instruction [1989, 1991], reform in the teaching of mathematics has proceeded with energy The present paper contributes to the discussion of instructional strategies that may prove useful in reforming classroom mathematics.

To introduce the analysis, we need first to clarify the aims of mathematics instruction—the goals to which it aspires. For the evaluation of instructional approaches depends upon their relation to desired (and undesired) outcomes. Among other things, the NCTM Standards [1989] state that students should: (1) learn to value mathematics; (2) become confident in their ability to do mathematics; (3) become mathematical problem solvers (p. 5). Similarly, Schoenfeld [1994] maintains that the ability to reason sensibly, particularly about mathematical material, is a goal of school mathematics instruction, and that this goal has implications for classroom practice He writes:

> If we believe that doing mathematics is an act of sense-making; if we believe that mathematics is often a hands-on empirical activity; if we believe that mathematical communication is important; if we believe that the mathematical community grapples with serious mathematical problems collaboratively, making tentative explanations of these phenomena, and then cycling back through those explanations (including definitions and postulates); if we believe that learning mathematics is empowering and that there is a mathematical way of thinking that has value and power, then our classroom practices must reflect those beliefs [Schoenfeld, 1994, pp 60-61]

When Schoenfeld suggests that mathematics is a kind of sense-making, he seems to envision the discipline as one grounded in what we, with John Dewey [1933/1971], call "reflective" thought. This is thinking which, as Schoenfeld puts it, "grapples with serious problems"—the genuine problems which one wishes to resolve yet cannot. Reflective thinking, then, is what the NCTM standards seem to mean by "problem solving " Furthermore, it is thinking which may take place in a community, as people work together to solve the problems. The thinking that occurs seems to involve offering "tentative explanations" and then "cycling back" or re-examining these explanations to determine their strengths and limitations—very much the vision of reflective thinking that Dewey presents [1933/1971, pp. 106-118]. There is, then, continuous reflection upon the ideas and explanations in order to determine their usefulness in a given situation. Finally, Schoenfeld appears to urge that mathematics classes engage students in the kind of conversation he describes

If one accepts the goals of mathematics instruction as defined by the NCTM standards and eloquently elaborated by Schoenfeld, one may ask: How might teachers prepare themselves and actually interact with students so as to foster a caring for mathematics, the ability to think reflectively about its problems, and confidence in that ability? It is to this question that we now turn and in so doing, offer an experiment that we conducted in the Fall Quarter, 1992, at Northwestern University. The experiment was carried out in a graduate course entitled "Conversation in Mathematics and Music: A Study of Teaching." The purpose of the experiment was to discover whether a conversational approach called "interpretive discussion," which had been used with success in high school English classes by the first author [Haroutunian-Gordon, 1991], could contribute to the teaching of mathematics. So, given the results of our experiment, we reflect upon whether interpretive discussion seems to facilitate reflective thinking, particularly about mathematical content, and whether it helps students to value mathematics and become confident in their ability to approach and resolve mathematical problems. We argue that the approach seems to work toward these goals and conclude by discussing the preparation of teachers to engage students in interpretive discussion

II. The experiment

The experimental class consisted of twelve registered students, eleven graduate and one undergraduate. In addition, there were three regular auditors and occasional visitors who generally contributed to the conversations. Of the twelve registered students, two were in doctoral programs in mathematics, two were in master of music programs, and the balance were preparing to teach in the elementary or secondary school in fields not limited to mathematics. With the exception of the two doctoral students, the others had no unusual preparation in mathematics. But all of the students were very much interested in the process of learning, the role of the teacher in the classroom, and how novel approaches, such as teaching through interpretive discussion, might be used in teaching mathematics and other subjects

As will be described below, the course focused upon mathematical issues Topics covered in depth during the

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course included Euclid's proof of the infinity of prime numbers, the uncountability of the real numbers and countability of the rationals, the Cantor set, and relations of volume to surface area in solids with applications to packaging problems. The topics were selected because they were felt to be ones to which all the students would have access, regardless of previous study of mathematics. No attempt was made to "cover" whole areas of mathematics, although many aspects of the areas considered were explored in depth, as the excerpts below suggest

The approach we followed, interpretive discussion, required that the students and instructors work together to develop questions about the meaning of the texts—questions they wished to resolve but could not, at least not definitively. The questions were, then, "genuine questions," as the NCTM Standards [1989] defines these: "Situation(s) in which, for the individual or group concerned, one or more appropriate solutions have yet to be developed" [p. 10] The "texts" for the course were mathematical, philosophical, and musical.[2] The class consisted of conversations about the meaning of the various texts and about the questions themselves so that both texts and questions became clarified While there were occasional spontaneous lectures—moments when someone stood at the front of the room and explained a point which was to serve as a basis for further discussion—nearly all the class time was spent raising, clarifying, and trying to resolve questions to which the resolution was unclear, even to the instructors. Hence, the class was unlike many described in the literature on mathematics education, in which the teacher, but not the students, knows the answers to the questions on the floor [3]

The reader may wonder why the course included musical and philosophical texts as well as mathematical ones. Part of the answer is that if one begins with questions to which the answer is unclear, then one must discover not only the resolution but, implicitly at least, the criteria that a satisfactory resolution must meet. Indeed, in the excerpt from the class discussion which follows, one sees the students debating whether "indirect" and "direct" proofs of the infinity of prime numbers are equally acceptable, given that the latter is "messier" and "doesn't give you all of the primes," as students put it. Music and philosophy, it turned out, helped us to explore issues that arose in discussing the criteria that solutions to mathematical problems must meet.

For example, G. H. Hardy [1967], in his book, *A mathematician's apology*, states that:

> The mathematician's patterns, like the painter's or the poet's, must be beautiful; the ideas like the colors or the words, must fit together in a harmonious way Beauty is the first test: there is no permanent place in the world for ugly mathematics [p. 85].

But what does Hardy mean by saying that mathematics must be "beautiful?" He speaks of it being "serious," and "significant," and one has the sense that these characteristics are related to mathematical beauty, which he says is "hard to define" [pp. 85, 89, 90]. To think about the nature of mathematical beauty, we found it helpful to think about beauty as it appears in other contexts, particularly music. We selected music because we are both musicians, and during the eleven-week course, we played four violin and piano sonatas by Mozart for the students.[4] Talking about beauty in music seemed to give us ideas about what a "beautiful" mathematical proof might look like.

Both students and instructors prepared for class by writing interpretive questions about the meaning of the texts—genuine questions which related passages to one another in order to develop interpretations and address the problems Although we, the instructors, provided written comments on the students' questions so as to help clarify points of doubt, the class discussion, which strongly encouraged interactive conversation, emphasized the contributions of all equally, as may be seen in the excerpts below

Over the course of the quarter, we observed evidence to support the following claim—a claim intuited by Fawcett many years earlier [Fawcett, 1938, p. 21 and passim]: Once the expectations for the conversation are established, the discussion of any particular segment of the material follows an entirely natural path Now, what do we mean by "natural"? The question may be addressed by looking briefly at the experiments conducted by Fawcett [1938/1995] and another daring voyager in the territory of mathematics education, Benezet [1991].

Fawcett's experiment on the teaching of geometry lasted for two years and involved fifty high school students divided into two groups that met for forty minutes per day [p. 20]. The procedures followed in the study for the experimental group of twenty-five (Group A) assumed that the students had experience with problems of reasoning, that they should be allowed to reason about geometry in their own ways, that the development of the work in the course should be based upon the "logical processes ... of the pupil and not those of the teacher" [p. 21]. There was no definite sequence of theorems arranged in advance [p 24] and no emphasis on the amount of material covered [p. 25] Instead of memorizing definitions, students worked to develop their own of both geometric and non-geometric terms [e.g., pp. 46-65]. Instead of building proofs upon unquestioned assumptions, students worked to detect and criticize the latter [pp. 75-86]. The study of mathematics was natural in the sense that it proceeded according to student ideas and understanding, as that understanding directed the sequence of activities in the classroom.

Benezet was superintendent of schools in Manchester, NH, in the late 1920s and 1930s. In the Fall of 1929, he chose certain of his schools at random and in those schools entirely eliminated the standard teaching of arithmetic from grades 1 through 5. Instead, the students learned what little mathematics they needed for daily life (telling time, making change, e.g.) but spent the time reading and discussing books, talking about current events, and doing interesting projects that involved their minds and bodies [p. 3]. In general, they engaged in activities requiring their judgment and opinions: their individuality and thoughts were not only permitted expression but actively encouraged. Here, the study of mathematics was natural in the sense that it was integrated into children's experience as needed for various activities; it was not imposed as an additional discipline to be studied for arbitrary or abstract reasons.

The approach we present seems to create a classroom in which the students, like Fawcett's and Benezet's, create the path of conversation that was followed. It is natural in the sense that conversation in these other situations is natural: the discussion of mathematics flows out of the ideas, beliefs, and concerns expressed by the students, and the topics pursued are the ones they needed to pursue in order to further their understanding. Hence, the topics were not imposed upon the group with the aim of covering certain matters in advance. While we, the teachers, sometimes posed questions, these were, for the most part, natural questions for us, in that they were ones to which we were seeking answers, not ones to which we had answers in mind. Many of the questions we posed arose as the conversation progressed. We, like the students, came to the class seeking understanding, and the questions we raised grew out of our experiences—in life, with each other, the students, and with the texts.

Is mathematics learned more effectively if conversation about it is allowed to take a natural course? Fawcett reports that after one year of the study, the two groups involved in the experiment plus two additional groups in another school that had followed the traditional geometry curriculum were given a post-test. All four groups had achieved about the same scores on the pre-test. However, Group A, which had studied geometry as described above, scored significantly higher than the other groups (24.2 as opposed to 12-14 by the other groups) on the post-test. The difference existed despite the fact that Group A had covered only one half of the geometry curriculum [pp. 103-4]. Other measures led Fawcett to conclude that: 1) Studying a small number of theorems using the approach described "Yields control of the subject matter of geometry at least equal to that obtained from the usual formal course"; 2) Following these procedures improves reflective thinking ability; 3) The usual formal course in demonstrative geometry does not improve the reflective thinking of pupils [p. 119].

Likewise, Benezet reports that in the beginning of the sixth grade, students in the experimental classes were exposed to arithmetic, which they learned quickly. And by the middle of the sixth grade, they had surpassed the students who had been studying arithmetic diligently for five and one-half years [pp. 11-13]. Benezet reports that beyond test scores, the students' intuitions about which answers were reasonable, and what questions made sense—their "common sense"—was on a sounder footing than that of the students who had followed the traditional curriculum [p. 14].

Because interpretive discussion allows students to explore issues of concern to them naturally, i.e., as their own beliefs, concerns, and questions direct, it was deemed appropriate for helping them to reason reflectively and sensibly about mathematics. The value of the approach may be seen by reviewing transcripts of the classes. So, we turn now to excerpts from the second meeting of the class, for which the students had prepared by reading four descriptions of Euclid's proof of the infinity of prime numbers. These included Euclid's own proof and versions of it given by Hardy, Courant and Robbins, and Wittgenstein [Courant & Robbins, 1969, pp. 22-23; Hardy, 1967, pp. 93-94; Wittgenstein, 1991, pp. 307-308]. The accounts differ in difficulty, and in degree of abstraction. We give them here with their salient points so that the reader may appreciate the classroom conversation that took place.

1) The "indirect" or "non-constructive" proof which proceeds by *reductio ad absurdum* argument (given in slightly different forms by Hardy and by Courant and Robbins): suppose there are only finitely many prime numbers,[5] $P_1, P_2, \ldots P_n$; then form $Q = 1 + P_1 \times \ldots \times P_n$. Either Q is prime (a new prime number, for it is larger than any of the original ones) or it is not. If it is not a new prime number, take its prime factorization—the prime numbers which if multiplied together would give us Q. These factors must all be new, since the original primes all left non-zero remainders (actually, remainders of 1) when divided into Q.

2) The more constructive or "direct" proofs, which offer a procedure for finding new prime numbers in a limited period of time:

a) Euclid's version: Suppose some primes, say A, B, and C, are given. Euclid showed that there must be another prime, different from those. He does this by taking the product of A, B and C and adding 1 and calling the result Q: $Q = A \times B \times C + 1$; as above, either Q is prime (and new) or is not and has a prime factor (different from A, B and C).

b) Wittgenstein's version. He shows that there will always exist a prime in a prescribed range. Given any number r he shows that there will exist a prime larger than r but no larger than $r! + 1$.[6] The proof is quite similar to Euclid's, though, for Wittgenstein, the constructed number is $Q = r! + 1$. Either this Q is prime and we are done, or it has a prime factor. But that prime factor cannot be 2, 3, 5, ..., $r - 1$ or r since these all leave a remainder of 1 when divided into Q. Hence it is a new prime number bigger than r, but of course less than Q itself. Again, we have the desired result, a prime larger than r but no greater than $r! + 1$.

To begin the discussion, we asked the following question calling for interpretation of the texts, a question to which we were unsure of the answer: If Hardy believes there could be another proof of the same result that is as beautiful as [his version of] Euclid's, what qualities would it need to have? The approach we followed to generate the question is described in Part III below. For now, we note only that we came to this question, in part, because the notion of "beauty" had been introduced in the previous discussion. After some initial remarks, the group soon turned to the issue of Q's beauty:

> **(Excerpt 1)**
> [George] "And to me, that's ... setting it up ... the relation between the Q and a total proof is where I would perhaps find that which I call beauty."

The beauty, for George, seems to lie in the way Q proves the infinity of prime numbers. But, how exactly does Q do its job in the two types of proof, the direct, constructive approach as opposed to the indirect, non-constructive route? The group seemed to grow more and more unclear about this issue:

(Excerpt 2)
[Mort] "It's the same idea [i.e , the direct and indirect proofs], right? It's the same idea, it's just you say it in a different way and you don't . . there's something very . kind of ... in contradicting proofs there's something very symmetrical about it. You start and assume that the statement is not true and you come up with the contradiction for it, you tie it up, close it and boom, you're done. As long as you accept the *reductio ad absurdum* and contradiction proofs, there's no more you have go through. But there's . if you want to be a little more messy you can do constructive proofs. Messy being not as nice and simple, as open and closed, but a nice constructive proof is [a proof] just as well. And the idea's still there and the idea is still very strong."

[George] "I don't want to get in dialogue here, but you already said that the constructive proof would be .. it could get a little more messy. In that sense , the constructive proof would be a bit less beautiful than ."

[Mort] "I already regretted saying that as soon as I said it, so ."

[George] "Oh, no, I would . in seeing the two proofs I have this sense that the indirect proof is more beautiful. The result may be the same, but for some reason this proof seems more beautiful."

[SHG] "Why do you think that?"

[George] "I think because it seems to somewhat relate to my previous statement that it seems to fit together."

Here, then, the issue of Q's beauty gives way to the issue of the relative beauty of direct and indirect proofs. As this topic is pursued, the features of each type of proof are explored, and a new question arises:

(Excerpt 3)
[Mike] "I think It's important to mention, which took me a while to remember, that numbers sitting next to each other actually mean multiplication. Like P_1 times P_2 times, times, times, all the way to P_n."

[DT] "And this whole thing, this is the number Q."

[Mike] "Right."

[DT] "Okay , so this is a big number. It's huge."

[SHG] "You mean, if you multiply those factors, P_1 times P_2, assuming that there's a lot of P's in there, then you would get a huge number."

[DT] "Okay, so there's this huge number. What does it say? Either it itself is prime, in which case, ... is that good? [Euclid] seems to be saying it either is or it isn't. If it turns out to be prime are you happy or unhappy?"

[Lucy] "Why would it make you happy or sad either way?"

[Sam] "I would think it made you happy if it came out prime .. it seems like that's the supposition, that's what you're hoping for is . . hoping to show that this thing, this big number will be prime."

[Mike] "I'm not sure I understood that last part. So by doing this method of doing all the primes [multiplying all the primes you have together] plus [adding] one, there are times when you're not going to find a prime number?"

[DT] "Right."

[Mike] "So you're still guaranteed to find the prime number bigger than P_n different from the ones you've already got?"

[Several voices at once]

[Sam] "You just have to find the right combination of prime numbers to get it. You plug into this Q formula so that you find that . that prime number that's supposedly out there, because if you plug in certain . I ... I think what I'm hearing is that if I plug in certain combinations of primes I will get a composite number."

[Sue] "That doesn't make sense if the proof says that if you multiply prime numbers by each other .. Do you have to do it in a series . . numerical series? Do you know what I'm saying?"

[DT] "Maybe. I'm not sure."

[Sue] "If it's just prime times prime times prime plus one then the proof said that's going to be prime."

[DT] "Does it?"

The discussion of whether Q must be a prime number or not intensifies the issue of the difference(s) between direct and indirect forms of proof:

(Excerpt 4)
[Ann] "Why don't you back up and start all over. I mean, the differences between a constructive proof and an indirect proof. I mean, this is something .... what is the difference between the two?"

Here, we seem to see the group, as Schoenfeld put it, "cycling back" through the explanations offered previously. Ann, evidently unclear about the differences between direct and indirect proofs, wants to "start all over" defining these. As the group addressed Ann's question, another question arose:

(Excerpt 5)
[Lisa] "Okay, so you guys are saying that the equation will not get a prime?"

[Sam] "No, it will *get* a prime. All I said is that it won't *give* you .. it doesn't give you all the primes. But it does allow you to keep computing primes as long as you want."

[Sue] " . you *will* end up with all the primes. If there's one you don't know it will eventually pop up. Because like he .. we .. "

[Sam] "Oh, I see, okay."

[Sue] "All you really need to start with is the definition. You try two and find out it's a prime number. You start like he did and it gives you three. You know? And then if you went with two and three it gives you seven. If you went with two, three and seven, … I mean, eventually you'd find out five is a prime number "

At this point in the conversation, the group turned to consider examples Perhaps it embarked upon this strategy because doing so was thought to help address the issues that had arisen, viz., Would Q always be a prime number? Would the procedure for generating a constructive proof give all the prime numbers in a sequence, or would it skip some?

(Excerpt 6)
(DI is at the chalkboard, working out an example. He is trying the product of primes 5×7 and then adding 1, which yields 36)

[Mort] "Well, thirty-six isn't a prime number, right?"

[several people] "No"

[Lisa] "So what are its factors?"

[Mike] "This proof doesn't seem quite so beautiful now that you (DI) are up there "

[laughter]

[Sam] "It's so much trial and error And like you were saying . I mean .. you have .. you have to presuppose, it seems to me, a little bit more. Maybe I'm wrong about that, but it seems you have to presuppose a little more with this proof [than the indirect proof, which is based upon the *reductio ad absurdum*] because you have to actually *get down with the numbers* "

Once the group started looking at examples, it seemed to naturally "cycle back" to a question raised previously, and the thinking about the issue seemed to shift:

(Excerpt 7)
[Sue] "I think also we're kind of hung up on finding all of them [all the prime numbers] and I don't think this was a big deal with this proof I don't even think anybody cared about finding all of them I think they cared about knowing there were an infinite number of them And in that respect if we use this [the constructive, direct proof] to say we want to find them all, well yeah, it's not going to be very pretty, cause that's not the way it was intended "

[Sam] "Except my concept of infinity is .. is damaged if I can't find all of them. I mean, I know that's not necessary [laughter] but if I leave out seven I have a problem "

[Sue] "And we know there's always one more, too "

[Sam] "I know, I know, I know My mind tells me it works, it's just . not as beautiful, that's all."

[Mike] "You'll spend the rest of your life computing all the rest of them."

[Sam] "Yeah right, hoping I'll get to seven "

[Mort] "I don't think that this proof is intended to give you all prime numbers [in a sequence]."

[Fay] "Yeah, I don't think so either."

[Linda] "I think this proof is nice because you have freedom of choice about what primes you start out with and everyone can start out with different ones and there's room for individual freedom "

[Sam] "The freedom of primes "

With this last statement, there seems to be a sigh of relief, even a release of tension, in the group. The realization that one can begin with any prime takes away the sense, all too prevalent in mathematics, that there is only one so-called "correct" way to proceed with the construction. We interpret the student comments to suggest that some in the group may have even entertained a new criterion of mathematical beauty, namely that of freedom And as each student's preferred method of construction was seen to be just as good as—no better and no worse than—any other, some people, at least, seemed to behave as a more confident and capable members of the class. Witness the approach to Wittgenstein:

(Excerpt 8)
[DI] " . [someone] mentioned . Wittgenstein I think he does it differently. I think he doesn't just take the product of prime numbers, he takes the biggest one and takes its factorial Takes the product of all the numbers smaller than or equal to it "

[SHG] "Where is that?"

[Nancy—reading Wittgenstein] "The Euclidean proof of the infinity of prime numbers might be so conducted that the investigation of the numbers between P and P factorial plus one was carried out on one or more examples, and in this way we learn a technique of investigation The force of the proof would of course in that case not reside in the fact that a prime number greater than P was found in *this* example And at first sight this is queer

It will now be said that the algebraic proof is stricter than the one by way of examples, because it is, so to speak, the extract of the effective principle of these examples But, after all, even the algebraic proof is not quite naked. Understanding, I might say, is needed for both!" [p 307]

(Excerpt 9)
[Lisa] "Can I ask a question? When he says P factorial does he mean prime numbers factorial or all numbers factorial?"

[DI] "I guess he doesn't say, does he? What do you think?"

[Lisa] "I was under the assumption it was prime numbers factorial—all the primes "

[DI] "Does it matter? I don't know "

[Mort] "Well, it seems like when he says P, that means you take a prime number, and you take its factorial, which would mean all numbers "

[George] "Which would include some numbers that weren't prime."

[Sue] "Right."

[Lisa] "Right "

There now follows a discussion in which another proof is developed: Given P, Q is defined as 1 plus the product of all numbers less than or equal to P The group then carries out an analysis of particular examples. So we ask: Given these excerpts from the classroom conversation about Euclid's proof of the infinity of prime numbers and what followed them, is there evidence interpretive discussion helps students to explore mathematical material, to think reflectively about it, and to otherwise move towards meeting the goals of mathematics instruction as defined at the outset of the present paper?

The questions, topics, and moods that arose in the conversation suggest that the answer to the question is: yes. To begin with, they seem to have arisen from reflection upon the subject matter of mathematics—Euclid's proof of the infinity of prime numbers—and emerge from an attempt to understand the subject. Insight into the features of Euclid's proof seems to be achieved as students discover that Q itself need not be a prime number, that the constructive form of proof generates new prime numbers, if not in sequence, and find patterns for reflecting upon different versions of the proof (including Wittgenstein's). There is evidence, then, that interpretive discussion helped these students to define and pursue resolution of mathematical problems, i.e., to think reflectively about mathematical issues.

Furthermore, the issues arising in the conversation seem to have a real and direct meaning to the students, that is, they seemed to care about resolving the issues that they raised. For example, when DT asks, "If it turns out prime, are you happy or unhappy," Sam responds:

> "I would think that it made you happy if it came out prime ... it seems like that's the supposition, that's what you're hoping for ... hoping to show that this big number will be prime." [Excerpt #3]

Sam seems to have a personal stake in settling the question of whether Q is a prime number or not. When Mort continues by asking whether the constructive method yields a prime number Q [Excerpt #5], he seems to feel genuine concern about the resolution, and declares that DT's claim that the constructive method might yield a composite number "doesn't quite make sense." Having a genuine question and caring about its resolution—as seems to be the case for Mort and Sam at this point—suggests that the conversation is bringing students to care about the resolution of mathematical issues. The caring seems to arise not out of concern for good grades or teacher approval but out of growing interest in the mathematical issues themselves—interest in identifying and resolving them.

We note also that members of the group seem to work together to try to resolve issues. Why? Perhaps it is because they are asking questions they wish to resolve but cannot. The issues are "shared": that is, they are issues that the members of the group have worked to define and in so doing, have developed a desire for resolving. The desire for resolution may explain why the talking to one another seems very direct, and why the comments made seem to be taken seriously. For example, Sue says:

> "I think also we are kind of hung up on finding all of them [the prime numbers] and I don't think this was a big deal with this proof. I don't even think anybody cared about finding all of them. I think they cared about knowing there were an infinite number of them. And in that respect ... it's [the constructive form of the proof] not going to be very pretty, 'cause that's not the way it was intended." [Excerpt #7]

Here, Sue directly addresses the issue of whether the constructive form of proof gives all the primes in a sequence. She seems to have a definite view on the matter, which she works to express. In so doing, she also returns to the issue of whether the constructive proof is "beautiful"—another topic which the group has brought up repeatedly. Sue's comments are made in response to Mort, Sam, and others, who, like she, are working to understand the features of the direct and indirect forms of proof of the infinity of prime numbers.

As a third point, we observe that as students work together to define and resolve mathematical issues of mutual concern, they seem to become more confident about their ability to pursue the issues. Indeed, the final question—How does Wittgenstein's version of the proof differ from the others?—might have been most frightening at the outset, but given what preceded it, was not. Students began by raising questions about the meaning of Wittgenstein's version of the proof. Some, at least, seemed ready to pursue the issue and had a clear and productive idea about how to do so. (Excerpt #9)

When Lisa poses the question of whether $P!$ refers to the prime numbers less than P or all the numbers less than P, she responds immediately to DT's probe, although she gives no reason for her answer. Mort offers a different resolution, and George draws the implication of Mort's suggestion for Lisa's question. Lisa seems easily persuaded that her initial suggestion was in error, and the conversation moves quickly to the analysis of Wittgenstein's proof. Lisa's willingness to question Wittgenstein's text in combination with the quickness of Mort, George, Sue, and again Lisa, to pursue and resolve the issue, suggests that at least these students are focused upon the question and confident about its resolution. They then proceed to explore the meaning of Wittgenstein's version of the proof using particular examples—again, with assurance.

Readers must judge for themselves, but in seems to us that students valued the conversation, cared about defining and resolving the issues on the floor, made progress with the issues, communicated more clearly with one another as the discussion progressed, and likewise, became more confident in pursuing the topics. In short, the goals of mathematics instruction, as defined above, seem to have been met by the class. Assuming that we are justified in our claim, we now ask: how did the conversation help to achieve the objectives?

Perhaps the first point to establish is that the course of discussion seemed to be a natural one in that topics were not forced upon the group by the instructors. While we opened the conversation with a question, the issues of concern to the group were allowed to evolve as the conversation progressed. Interestingly enough, the initial question was explored as other, more focused questions, which arose naturally and whose resolution had implication for its resolution, were pursued.

For example, the issue of what constitutes the difference between direct and indirect forms of proof seems to have been raised anew after much discussion because at least one person developed greater perplexity about the matter as the conversation progressed The repetition of the question seems perfectly reasonable in the context of what had been said, for it is asked at a point when the differences had not yet been clearly stated (Excerpt #4). Hence, there is evidence that the conversation, proceeding naturally as it did, allowed the questions to mature, so that the points of genuine doubt that concerned the group were clarified. Furthermore, the clarification of differences between direct and indirect forms of the proof had implication for identifying the qualities which, according to Hardy, a proof equal in beauty to Euclid's would need to have

Without the pressure upon us to "cover" a certain amount of material or set of topics, it seems to us that the discussion allowed a flow of questions and moods that permitted a kind of maturation to occur. The growth seems to start with an identification of the basic elements in Euclid's proof, which was followed by an exploration of the qualities that each element has, a return to the starting point (the differences between direct and indirect forms of proof) for clarity and reassurance, the emergence of strong hopes, the testing of these (sometimes unrealistic) hopes by means of specific examples, the realization that one "can't have it all," and finally ends with the recognition that the more constructive proof of the infinity of prime numbers has features which, if recognized, can bring a sense of joy and freedom. Once the maturation through conversation has occurred, as it did in many other classes as well, very hard, technical mathematical work could take place, which, in this instance, consisted of studying one of the most difficult of philosophers (Wittgenstein) and his version of the proof.[7]

III Concluding remarks

In summary, this experiment seems to us to suggest that mathematics, like so many other parts of life and domains of understanding, can be explored and learned in a natural way, much as Fawcett, Benezet and others like Papert [1980] have argued. The process need be no more forced or imposed or disagreeable than learning to walk or talk Furthermore, our evidence suggests that the shape of the discussion could never have been predicted in advance and must not be artificially confined if the path it takes is to fit the group. In conclusion, we address briefly the role and preparation of teachers to lead interpretive discussion.

In light of the excerpts above, how might one describe the role of the teacher in a class conducted through interpretive discussion? To some extent, it is analogous to that of the spotter in gymnastics. When gymnasts work out on a trampoline, there is the possibility that they will hit the metal edge or even land outside the trampoline altogether. Consequently, individuals—"spotters"—are stationed around the trampoline to nudge the gymnast back into the large central area where the work can continue. We, as teachers, functioned not by imposing topics but by raising questions that aimed to help discussants clarify their ideas and points of doubt so that the conversation could become increasingly focused upon issues that concerned them. Our questions, then, helped the discussants to identify what they did and did not understand, thereby, directing the discussion toward the creation and resolution of issues that were their own rather than ones we determined.

Despite the spontaneous character of the discussion, extensive teacher preparation went into each class, however, as it was necessary to read the texts at least twice, writing down questions about their meaning as they occurred during the readings. We each reviewed our own questions, excluding those which had been resolved or which could not be addressed through textual analysis. We exchanged the remaining questions, looking for relations between them that suggested deeper issues of concern to us—issues of which we may have been previously unaware. Then, we worked to formulate a question which clearly expressed a shared point of doubt. Finally, we modified the questions and prepared additional ones which allowed us to explore various aspects of the text whose interpretation might have implication for resolving the central issue of concern (what is called the Basic Question) The question cluster in Appendix A was the product of our efforts prior to the class excerpted above. Although we did not ask the follow-up questions in class, preparation of the question cluster was critical for providing enough familiarity with the texts to permit the close listening that is needed in order to guide discussion toward the cultivation of the students' shared issues.

Now, it may be argued that it is difficult to teach traditional mathematics curricula through conversation between students and teacher interacting on equal footing as we have been describing. Even the reformed mathematics curricula[8] would need careful review to determine how interpretive discussions might be successfully integrated into them We maintain that the approach needs extensive study before its broad applicability can be argued.

Our belief—in need of further investigation, but supported by the evidence presented above—is that if the aims of mathematics instruction are to develop the ability to reason reflectively and sensibly about mathematics, to develop affection for the subject and confidence in one's ability to pursue its problems, then interpretive discussion may contribute to meeting the goals. That it may do so seems to arise from the fact that it allows students, by conversing about the meaning of texts, to cultivate questions and pursue resolution of the issues. Because the issues are defined by the group, its members seem to care about finding resolution and working together towards that end. It seems to us that the opportunity for interpretive discussion may help to create what Schoenfeld [1993] calls a "local intellectual community," in which "growth" as defined by Dewey [1916/1966] takes place:

> Since growth is the characteristic of life, education is all one with growing; it has no end beyond itself. The criterion of the value of school education is the extent in which it creates a desire for continued growth and supplies means for making the desire effective in fact [p 53].[9]

Notes

[1] The authors wish to thank Allan Collins, Karen Fuson, Alan Schoenfeld, and David Wheeler for helpful comments on earlier drafts of this paper. We also thank Kristin Field for her technical assistance with the manuscript.

[2] The texts included: R. Courant and H. Robbins, [1947], *What is mathematics? An elementary approach to ideas and methods* Oxford: Oxford University Press; A. Einstein [1991], *Autobiographical notes*, trans. P A. Schilpp, La Salle, IL: Open Court Publishers; E. Hanslick [1986], *On the musically beautiful*, trans., G. Payzant, Indianapolis: Hackett Publishing Company; G. H Hardy [1992], *A mathematician's apology*, Cambridge: Cambridge University Press; T W. Surette [1957], *Music and life*, Boston: E C. Schirmer Music Company; L. Wittgenstein [1967], *Remarks on the foundations of mathematics*, G H von Wright et al (Eds.), Cambridge: MIT Press

[3] The research reported by Lampert [1990], Schoenfeld [e g , 1993, 1994], Fravillig *et al* [1995], indicates that at moments, there may be questions under discussion to which the teacher does not know the answers In general, however, such questions are exceptional rather than typical in the classroom conversations.

[4] E minor, K. 304; G major, K. 301; C major, K 296; B-flat major, K 378.

[5] A prime number is one that has no integer factors other than one and itself.

[6] $r!$ is read "r factorial" and is defined by $r! = r \times (r-1) \times (r-2) \times \times 3 \times 2 \times 1$

[7] Further analysis of excerpts presented above suggests that the conversation followed patterns found in fictional and even musical conversations See Haroutunian-Gordon, "How does thinking proceed: A study of thinking patterns in interpretive discussion" (forthcoming).

[8] Consider those being developed at the University of Wisconsin [Carpenter et al., 1995], University of Chicago [Diamond and Fuson, 1995], University of Pittsburgh [Forman *et al.*, 1995], to mention but three examples.

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Appendix A: Cluster of questions for Euclid's proof of the infinity of prime numbers

Basic question: *If Hardy believes there could be another proof of the same result that is as beautiful as Euclid's, what qualities would it need to have?*

(1) Would Hardy say that Euclid's proof is beautiful because "it corresponds to a paradigm that is already in us" [*Remarks on the foundations of mathematics* 171; 100]—that it answers a question that arises once we know how to count integers and identify prime numbers through calculations? If so, does its beauty depend upon conditions in us rather than the object itself?

(2) When H says that when math is beautiful, "the ideas fit together in harmonious ways," [*A mathematician's apology* p 85], does he mean that the rules that make up the proof must fit together this way, or that the proof must fit the question it addresses in this way? In either case, would H say that Euclid's proof is beautiful because it meets this criterion?

(3) When Hardy says, "the best mathematics is serious as well as beautiful," (*AMA*, p 89), does he mean that being serious is something in addition to and not part of being beautiful? If so, why does he say, "the beauty of a mathematical theorem depends a great deal on its seriousness" [p. 90]? Does this mean that H would say that E's proof is beautiful because it is serious?

(4) Would H say that E's proof is "serious" because it connects already "significant" ideas or because it creates significant ideas by linking mathematical ideas together in a way which "leads to important advances in mathematics itself" (*AMA*, p. 89)?

(5) Would H say that mathematics is like music in that "it finds its perfection in itself without relation to other objects"? [*Music and life*, p. 5] If so, would H say that its beauty is related to this characteristic?

(6) Would H say that E's proof is "beautiful" because it combines insight, feeling and imagination, and hence, is what Surette would call a kind of "wisdom" [*ML*, p. 15]? If so, would Hardy say these are elements of "seriousness"? Of "beautiful" mathematics?

(7) Do Courant and Robbins feel that the "indirect proof" they give initially is weaker than the direct proof

they discuss in small print [p. 23]? If so, would H agree, or would he say that Euclid's is as powerful?

(8) When Surette speaks of Beethoven's Eroica symphony as having "all the force of a mythological epic in which the heroes are pure spirit-types of humanity, of no age or time—gods, if you will, and above the human limitations" [p. 24], is he making the claim for music that Hardy makes for mathematics when he says, "A mathematician, on the other hand, has no material to work with but ideas, and so his patterns are likely to last longer, since ideas wear less with time than words" [p. 85]? Is the timeless, universal element part of what makes both mathematics and music beautiful?

(9) When Surette says that "music is the one perfect medium for this dream [that of Carlisle] of humanity. In its expression of human emotions it enjoys the inestimable advantage of entire irrelevance [p. 24]," is he contradicting the claim he makes elsewhere that emotion or feeling belong on the lowest rung of the ladder of qualities essential to great music? Would S and H say that great mathematics and music must express human emotion? DT/SHG 9/28/92

The unity of the human mind is such that, just as there is intellectual knowledge in all that a man does, or makes, there seldom is complete absence of art in what a man knows. Elegance is a quality highly prized in mathematical demonstrations. The same elegance is perceptible in the dialogues of Plato, so much so that some of them—for instance, his *Symposium*—constitute in themselves exceptionally perfect specimens of literary art. But this is not our question. Even if it is truly aesthetic in nature, mathematical elegance is entirely at the service of cognition; it aims to achieve an expression of truth highly satisfactory to the mind.

Etienne Gilson
