The Co-ordination of Meanings for Randomness

DAVE PRATT

1. The stochastic: a pedagogical challenge [1]

The domain of probability points particularly sharply to a fundamental difficulty in mathematical pedagogy. On the one hand, it is deceptively close to everyday intuitions and experience, even language: chance encounters, random behaviours, likely occurrences. One might easily suppose that these culturally embedded meanings could facilitate the transition to a mathematical way of thinking. Yet this is not the case. Probability is a notoriously difficult topic and it might appear that the only way for students to achieve satisfactory grades is to ignore the relationship of probability to everyday notions of chance altogether.

Perhaps more than in any branch of mathematics, the gap between potential everyday applicability and formal understanding is at its greatest in the domain of stochastics and probability. There are multiple opportunities to apply such knowledge as people go about their everyday lives; the playing of games with explicit random number generation (do I take the finesse of the queen of hearts?), in sport (should I adopt a long ball strategy?) and in parenting (should I let my daughter walk to school?) Yet such opportunities do not seem to have led to widespread construction of meaning for stochastic concepts. On the contrary, the research reported later in this opening section suggests that adults' understanding of such ideas is often impoverished, even misconceived.

In fact, mathematical discourse is simply *different* from everyday discourse, and the mathematical notion of probability is a scientific, rigorous concept in contrast to the fuzzy idea of chance which pervades everyday settings. But this simple statement masks the complexity of finding a pedagogical solution. I would argue that this complexity underpins a fundamental challenge of mathematical pedagogy: to construct situations which are rich in meanings for the learner, yet which point towards specifically *mathematical* meanings. As individuals make their way around their social and physical world, the intellectual tools at their disposal for mathematisation and abstraction are fairly impoverished. There is no need for them: everyday, pragmatic activity is adequately served by the fuzzy linguistic tools and artefacts that have emerged in the culture over the centuries. Thinking mathematically demands more. It presupposes that the learner has a more or less rich pool of intellectual tools at their disposal: algebraic notation, symbolism, and so on. These are precisely the intellectual tools available to a mathematician - and precisely those lacking in the naïve learner.

Hence, there exists a kind of pedagogical loop: as mathematics educationalists, we would like people to gain power over these tools so that they can make mathematical abstractions. But, in order to make mathematical abstractions, it seems that they need access to precisely these tools.

It is not clear that the pedagogical challenge is answerable. Piaget and Inhelder (1951) suggest that, in recognising the notion of a random mixture, the child comes to realise that the actions which conspired to determine that mixture cannot be identified. For them, the child in such a situation is in a position of conflict, since all operations previously constructed have been both composable and reversible. Yet here is a situation which is apparently neither. At this stage, therefore, the child may recognise random mixtures as those whose structure cannot be explained by operational understanding, a state of affairs which is not resolved until formal thought allows the concept of proportionality. Then, and only then, is the child able to construct a formal system of probability resolving the conflict. The key breakthrough is that the construction of probability enables the child to treat the random as if it were operational. A possible implication of Piaget and Inhelder's work might be that, prior to such an accommodation, children will rely upon unreliable intuitions, raising questions about the feasibility of a solution to the pedagogical challenge.

Borovnik and Peard (1996) suggest that the lack of consistent feedback from stochastic phenomena contributes to the underdevelopment of operational thinking. This line of argument suggests that concrete operations are missing, impeding the teaching/learning process and allowing the domination of unreliable intuitive thought:

> With probability, it has been noted, the concrete operations that form the basis of this process of concept acquisition are missing, thus hindering the reflection phase. The individual formation of concepts cannot be provoked by a hierarchical sequence of actions and reflections. This increases difficulties in any interaction between teachers and learners. Furthermore, the subjective domain of experience is more or less dominated by idiosyncratic and uncontrollable intuitive thought, as there is no direct feedback and control to real-world experience (p. 247)

A considerable body of literature, assembled over the 1970s and the 1980s, might appear to lend support to this view. During this period, attention was drawn to the heuristics that people (mostly adults) use when making judgements of chance. The main contributors, Daniel Kahneman and Amos Tversky, reported a catalogue

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of such heuristics, placing some emphasis on the systematic bias inherent in their application. I will focus on one such heuristic to illustrate their approach.

When using the representativeness heuristic, people predict the outcome which appears to be the most representative of the evidence (Kahneman and Tversky, 1973) In many situations, such a heuristic leads to valid conclusions, experiences which confirm the heuristic as an appropriate strategy. However, Kahneman and Tversky showed that, at least under certain experimental conditions, people's use of the representativeness heuristic can lead to bias

For example, they report on how this heuristic can lead to situations in which the conjunction of two events is seen as more likely than either of the constituent events They gave a group of undergraduates this thumbnail sketch (Tversky and Kahneman, 1983):

> Bill is 34 years old. He is intelligent, but unimaginative, compulsive, and generally lifeless. In school, he was strong in mathematics but weak in social studies and humanities. (p. 297)

The subjects were then asked to rank eight statements according to the extent to which they resemble the typical member of that class and secondly according to the probability that the statement was true. The subjects rated conjunctions such as "Bill is an accountant who plays jazz for a hobby" as both more representative and more probable than the constituent events "Bill is an accountant" and "Bill plays jazz for a hobby".

While it is acceptable that the conjunction should be regarded as of greater descriptive value and so more similar to the thumbnail outline of Bill, it is surely unacceptable that a conjunction of two events can be regarded as more likely than either of the events which make up that conjunction, in contradiction of one of the basic laws of probability, and indeed common sense.

Other researchers have reported on people's irrationality when making judgements of chance For instance, Marie-Paule Lecoutre (1992) found a tendency for people to assume that the different outcomes occupying the possibility space were equally likely, even when the individuals involved were grounded in probability theory. Hence, there would be a tendency to assume that, on shaking two dice, a total of 9 was just as likely as a total of 7, each total being *just a matter of chance*. Apparently, such notions of equiprobability are resistant to modification

Cliff Konold (1989) has found a tendency for people to interpret the use of 'probabilities' in certain situations as measures of causal dependence, rather than as measures of chance. He has called this way of thinking the "outcome approach". When inclined to the outcome approach, people use probabilities as modifiers, with 50% meaning that no sensible prediction can be made. In common with bias due to the representativeness heuristic, the outcome approach tends to disregard frequency information.

In the face of the above heuristics and predispositions, which betray the *naïveté* of stochastic intuitions even (especially?) among adults, teachers and students often engage in an avoidance strategy Advice to students often goes something like this:

> Ignore reality. Probability is just counter-intuitive. Always work with the definitions and standard methods. ... If a trial may result in any one of *n* exhaustive, mutually exclusive and equally likely cases, and *m* of these are favourable to an event A, then the probability that A will happen as the result of the trial is measured by the quotient.

There is a tendency to erect a coherent symbolic edifice which a few will understand but most will not

There is, though, a need to be cautious in accepting the validity of a conclusion which asserts that chance is counter-intuitive, since an apparent implication might be that the pedagogical challenge is insurmountable. Two implications could be drawn from the heuristics research, each reflecting a different interpretation of the nature of those heuristics. People make these errors in making judgements of chance either because:

(i) minds are somehow hard-wired to obey the intuitions that give rise, for example, to representative thinking;

or:

(ii) intuitions are insufficiently developed to deal with the tasks in an any more sophisticated way than that observed

The first of these two implications is the stronger, but either view itself has important implications for the teaching and learning of probability The strong implication would suggest that 'successful' teaching would present probability as a formal subject, disconnected from any everyday intuitions. The weaker implication leaves open the question as to how a more effective learning environment might be designed.

Though Kahneman and Tversky assert that the collection of heuristics constitutes a theoretical framework in itself, such a framework does not include the setting as a significant influence on judgements of chance Much of the research in this field has removed, or at least diminished, the influence of setting. In fact, one critical structuring force within an effective pedagogy might be precisely the tools and resources that we are able to provide, and which were not available to the subjects in most of this research

Some evidence has been found to suggest that three factors within a setting do indeed support the use of rational statistical reasoning. According to Nisbett *et al.* (1983), when there is a transparent clarity in the sample space and the sampling process, people are less likely to use the representativeness heuristic. Devices such as dice and playing cards benefit from this clarity as well as from the second factor, the ease with which the operation of random factors can be identified Such factors may also sometimes be identifiable in the playing of familiar sports. The third factor relates to the influence of culture. The increasing use of statistical-type language in everyday life (for example, in sports and weather forecasting) has, according to Nisbett *et al.*, a trickle-down effect on the accessibility of such ideas.

The view that pedagogical changes can enhance children's intuitions of the stochastic is supported by the work of Efraim Fischbein (1975). Fischbein was interested in the nature and role of intuitions as a vital form of knowledge in the process of working with ideas. He reported how, when subjects were required to predict the outcomes of a repetitive series of stochastic trials, they were, even from an early age, able to tune the proportions of their predictions to the relative frequencies of the outcomes It would seem, therefore, that even very young children have intuitions about relative frequencies. Elsewhere, Fischbein (1982) proposed how such intuitions could be actively developed if the child were to operate in contexts structured in supportive ways.

> For instance, in order to create new correct probabilistic intuitions the learner must be actively involved in a process of performing chan[c]e experiments, of guessing outcomes and evaluating chances, of confronting individual and mass results with *a priori* calculated predictions, etc. New correct and powerful probabilistic intuitions cannot be produced by merely practising probabilistic formulae. The same holds for geometry and for every branch of mathematics. (p. 12)

Fischbein would likely have claimed that the experiences which children enjoy in their games and pastimes are not structured in such a way as to make the underlying probabilistic ideas explicit. As a result, their intuitions do not, in fact, develop.

Even worse, he had earlier claimed that school, with its emphasis on causality and determinism, has a counter-productive effect on the development of such pre-operational intuitions.

> This is why the intuition of chance remains outside of intellectual development, and does not benefit sufficiently from the development of operational schemas of thought, which instead are harnessed solely to the services of deductive reasoning. (1975, p. 73)

The withering of intuitions about chance, through a lack of nourishment in school, perhaps provokes the construction of heuristics, subject to systematic bias, as a means of making sense of the stochastic.

Falk *et al* (1980) support this view, quoting evidence that children as young as six years of age were able to perform at levels better than random when asked to optimise the chance of choosing a pre-determined outcome by selecting the most appropriate possibility space Like Fischbein, they infer that more attention should be placed in schools on the stochastic.

> One of the aims of teaching about probability in the first grades should be to restore the balance in favour of indeterminism. (p. 202)

It seems then that there is a *prima facie* case that setting may shape intuitions of randomness, supporting the case for the weak implication from the heuristics research, and tending to refute the strong implication. The research reported has identified some of the factors which might contribute to a pedagogical solution, without providing the detailed observations necessary to mount a persuasive argument. In the next sections, I will first outline the approach of a study which aimed to observe young children as they constructed meanings for randomness in a computer-based setting Subsequently, I will provide a snapshot of two children working with the tools made available in that setting. This snapshot will be used to begin the formulation of a theoretical framework for the construction of meanings in one particular pedagogical setting – a stochastic domain of abstraction

2. The iterative study

As part of this study, I aimed to construct a setting in which individuals would meet the consequences of their beliefs My aim was to build a domain of abstraction in which the laws of probability mattered, in which it was possible to *work with* these formalisations, rather than to approach the formal as a separate domain grafted onto activity. My intention, then, was to put individual learners in situations where they could express their beliefs in symbolic (programming) form. Through the articulation of the beliefs that they hold, I hoped that learners would be able to reconstruct them in the light of their experiences. Those aspects of the domain relevant to the remainder of this article are described later

The computer setting enabled the detailed study of children's intuitions through the articulation of those intuitions during activity within the microworld using a conventional programming language. I was not, of course, able to observe the learner's thinking directly, but I was able to study and analyse the learner's actions as they were played out within an expressive medium on the computer. In this sense, the computer acted as a *window* (Noss and Hoyles, 1996) on the children's intuitions of the stochastic At the same time, for the learner, the computer can be regarded as a window through which the mathematics built into the system can be observed. [2]

Noss and Hoyles propose the term *web* to describe the union of those structures built into the software by the programmer and those mental structures *forged* and *re-forged* during the activity by the learner. In this sense, the study examined the webbing of stochastic intuitions with computer-based tools. The forging of connections can manifest itself as the construction of what they term *situated abstractions*, generalised heuristics which learners invent and apply within a relatively narrow or situated domain, but which empower them in their actions within that domain.

The development of the computer-based tools took place through several iterations during which the tools themselves were modified and at times built anew in the light of insights gained into the way that those tools shaped children's meanings for the stochastic. The methodology then was built around an iterative design in which the study of the children became increasingly systematic and focused as the tools converged on a design which appeared to be effective in enabling the observation both of initial meanings for the stochastic and the subtle changes in those meanings as new connections were forged during activity.

The final iteration involved 16 children, aged between ten and eleven years of age. [3] Each child was interviewed individually using a semi-structured schedule by means of which I tried to ascertain the sorts of meanings that these children held before working with the computer-based tools. The early interactions with the tools provided further insights into those initial meanings. For the sessions with the computer, the children worked in pairs for between 2 and 2.5 hours. The sessions were conducted as clinical interviews with myself acting as participant-observer. The actions of the children within the computer environment were videotaped directly and the discussions were simultaneously audio-taped.

I developed case accounts of the clinical interviews in which I merged the transcripts of the discussions and plain accounts of the children's actions. I interpreted these case accounts in the light of the pre-interviews to develop case analyses and identified issues as common to many of the cases or as atypical variations.

Later, in section 4, I shall present a synopsis of the work of Anne and Rebecca which illustrates the issue of how meanings for randomness can be co-ordinated through interaction with computer-based tools. First, in order that the subsequent analysis is meaningful to the reader, I will describe a small portion of the domain of stochastic abstraction which emerged through the iterative design process and became known as the Chance-Maker microworld.

A glimpse of the Chance-Maker microworld

The software tools were provided within a Boxer environment. [4] Boxer has evolved out of Logo and provides a powerful environment in which children are able to express their mathematical ideas in computational terms. The central objects of the microworld are a series of 'gadgets', computational tools which behave in many identifiable respects like their everyday counterparts. For example, in the DICE gadget (see Figure 1), the learner can 'throw' a dice with varying strengths, as well as inspect and change the 'workings' which produce the behaviour of the dice.

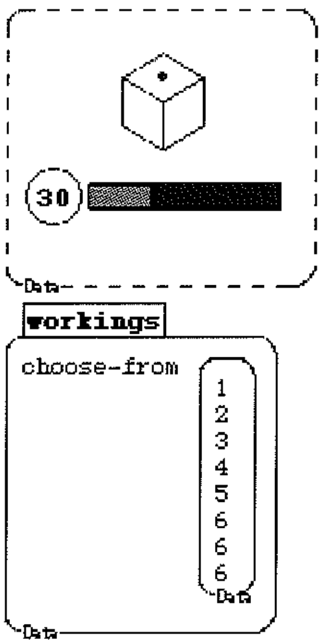

Figure 1

The DICE gadget is activated using the strength bar, depicted in Figure 1 as a solid black bar with a circular switch at one end. Imagine the child controlling the strength by allowing a tube (the black bar) to fill with a red fluid until the switch is clicked. The strength of the throw, 30% in this case, is represented by the amount of red fluid. Clicking directly on the dice itself causes the dice to be thrown with the same strength as last time. When the dice is clicked, it rolls 'dice fashion' and indicates the outcome on its top surface. The outcome is controlled by the workings box, which can be edited by the child. In its default form (as shown), the dice is biased towards sixes; the probability distribution in this default form defines Pr(scoring a 6) = 3/8.

Other gadgets include, *inter alia*, a coin and a spinner (default forms shown in Figures 2 and 3 respectively).

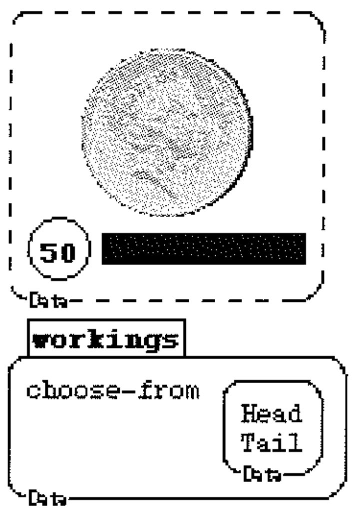

Figure 2

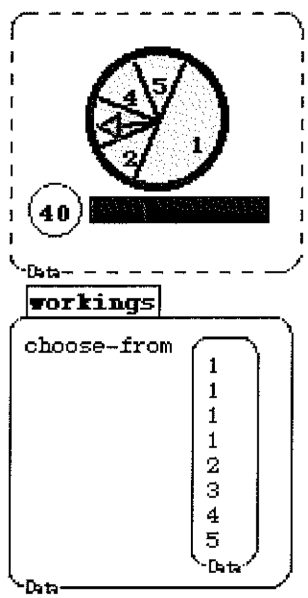

Figure 3

A gadget therefore embodies quite explicitly and accessibly a mathematical representation of how it works. This representation is both *instructive*, in the sense that the children need to make sense of it, and *constructive*, in the sense that it can be modified and used as a building block for extended ideas (see O'Reilly, Pratt and Winbourne, 1997, for further discussion of these notions).

The children were challenged to identify which gadgets they thought were working properly and to use the tools provided to mend those which they thought were not working properly. The design of the task was itself an important structuring resource, since it provided a sense of purpose for activity with the gadgets. The children were encouraged to explore and to use tools such as the workings box (as above), the 'Repeat' primitive (Logo-style), the graphing tools (see Figures 4 and 5 below) and the results box (a list of results).

3. Local meanings

Through successive iterations, children articulated meanings for randomness which appeared to be abstracted directly from everyday experience with artefacts such as dice, spinners and playing cards, often associated with the playing of board games. These meanings tended to focus on attributes discriminated from the immediate appearance of the random generating device or else though relatively short-term usage of those devices. I refer to these meanings as *local.* Though extensive data is available as to exactly how the children expressed these meanings, in the interests of brevity I choose instead to list the local meanings with only a brief explanation. I think most readers will find the list non-controversial and I do not wish to distract from the main argument in this article which relates to how these local meanings interact with the computer-based resources to enable the construction of new meanings.

Four meanings were expressed in the prior interviews and all four were recognisable in the children's early work within the Chance-Maker microworld. A fifth local meaning is more specific to computational environments. The paragraphs describing children's actions and discussions, as presented throughout the remainder of this article, are extracted directly from the case accounts.

(i) Unpredictability

A common type of activity throughout the eight case studies was for children to consider whether it seemed possible to predict the next outcome of any particular gadget. Attempting to anticipate the result of the next trial frequently took place as part of a game-playing type of activity invented by the children.

Gurdev and Neil, for example, engaged in predicting because they saw this as a fun activity in itself.

> Gurdev confidently predicts heads next. Neil says, "It will be tails again". It turns out to be tails. Gurdev then predicts heads whereas Neil predicts tails. It is a tail again. Gurdev now predicts tails, Neil predicts heads and it is heads. I ask again if they can predict. Neil: "Yes." Gurdev: "Now I think that it's heads next, and if not tails, it's going to be heads after that, then tails."

Equally often, predicting the next outcome seemed to be a natural strategy for testing whether a gadget was working properly. Generally, success in predicting was associated with non-random behaviour, whereas a lack of success was seen as indicative of random behaviour. The absence of a predictive capacity in a gadget was often seen as synonymous with its property of randomness.

(ii) Unsteerability

Children would often focus on the exertion of control through the degree of strength with which the gadget was activated. I term this type of control *steerability*. When a gadget could not be steered, children would tend to regard the gadget as working properly. For example, when Steve and Richard were asked whether they could predict the spinner's next result, Richard replied by referring to the strength control:

> Richard continues, "If you change strengths, I think you can, because if it starts from there, if you just do a few numbers, like" Richard tries strength 10 and got a 1. Then the same strength gave 4. Steve says, "Definitely random". Richard says, "Yes"

The fundamentalness of unsteerability is also seen in the way it is often used to 'explain' other meanings for randomness. Very frequently, children associated unpredictability with unsteerability. On such occasions, unpredictability was usually seen as the outcome of uncontrolled input.

(iii) Irregularity of results

A fairly common activity in the eight case studies was for the children to try to identify regular patterns in sequences of results. This behaviour was often linked closely to predicting, so that patterns were conjectured on the basis of past results and then used to make predictions, which were tested by further trials. For example, Neil and Gurdev, who seemed especially interested in pattern-spotting activity, were playing with the coin at the top level when this typical incident occurred.

> I ask if they can control the coin. Neil: "Sometimes, it's the same. It's a sequence. Sometimes." Gurdev: "Just say it was 60%, well, I've noticed that it was like just say heads first, then it goes heads again, tails, tails, heads, heads, tails, tails." I ask if this is just when it is 60%, to which Gurdev says, "Generally". I clarify, "So you think there is a pattern to what it does?" Gurdev: "Yes." Neil: "Yes, I think there is as well."

Subsequent testing led Neil and Gurdev to conclude that conjectured patterns were not sustained after all.

(iv) Fairness

The pre-interviews showed that fairness is, for many of the children in the eight case studies, a defining characteristic of randomness. This meaning of fairness was also articulated in the Chance-Maker microworld; the spinner was not random because of a lack of fairness in its physical appearance. Fairness was also articulated through the results themselves. Anne, whom the reader will meet again later, was concerned at a lack of fairness in the first few results of the coin gadget (before she had access to the workings box)

In fact it is a head. Then 100% strength gives a head, and then head again, and then head again. Anne says, "It's a bit unfair because it keeps going to the queen [i e heads]" They try 100% again and it gives a tail, and then head. Anne says, "Most of the time it goes to a queen, er heads".

The above local meanings were all evident in the pre-interviews as well as in the early interactions with the Chance-Maker microworld A fifth local meaning seemed to be closely related to the specific use of a computer-based environment.

(v) Computer-in-control
Some children did not, at least initially, regard the gadget as random because it had been programmed. Rebecca, for example, rejected the coin's randomness because there must be some sort of pattern in the way that the computer was programmed, even though she had not found such a pattern.

I ask if the coin is random. Rebecca: "Not really [.] it's probably been programmed to do it, in a loop." Anne says she doesn't know. I ask Rebecca what she meant by 'in a loop'. Rebecca: "Well, it's programmed to do heads, then maybe heads again and then tails." I clarify, "In some sort of pattern?" Rebecca: "Yes."

No pattern had been located, but the presumption that such a pattern must exist in a programmed environment was sufficient to reject randomness. [5] As the children became more familiar with the gadgets, they also expressed the four meanings above. The task of making the gadgets work properly was meaningful even when a computer-in-control meaning was articulated

Attributes of local meanings

• *Interchangeability of meanings*
There was considerable variation in which particular local meaning for random behaviour was cued, even for the same pair of children. On many occasions, different meanings of randomness were cued without presenting anything problematic Unsteerability was articulated alongside unpredictability; the fact that a situation which is not controlled is often not predictable allowed these meanings to be used interchangeably

• *Contradictory usage*
Different aspects of the setting would cue different local meanings and at times lead to an opposite conclusion without apparent discomfort. For example, in her pre-interview, Donna had been shown a non-uniform spinner; she had decided that the non-uniform spinner was not random because it was not fair, the sectors being unequal in size. In the early work with the computer's spinner, Donna began by conjecturing that the strength controlled the outcome but, after seeing some variation in the results, she changed her mind, presumably noticing the unpredictability. Only a few moments later, Donna accepted that the spinner was random. In the first case, Donna's attention had been on the physical appearance of the spinner (so cueing lack of fairness), whereas in the computer setting her attention had been drawn to the use of the strength control (and so unsteerability and unpredictability were cued) The change in the setting seemed to stimulate different meanings.

• *Weakly connected meanings*
The interchangeability of meanings suggests that local meanings are connected in the sense that there is an expectation that one meaning implies another. The connection, however, cannot be strongly held, since there were many examples where the gadget's behaviour might have cued contradictory meanings but did not. Thus, it seems that aspects of a setting may cue one local meaning, but the connection to another local meaning is not implicit and is not seen as highly reliable

• *Indiscriminate use*
Local meanings were used, at least initially, to construct meanings for long-term as well as short-term behaviour There was little attempt in the early interactions with the gadgets to diagnose their behaviour in terms of aggregated results. Even when relatively large numbers of trials were generated, the children searched for local regularities in results.

• *That which is **not** deterministic*
On many occasions, the short-term behaviour of the gadgets was explained by reference to cause and effect. A typical way of working was for the children to begin by conjecturing possible deterministic reasons for a gadget's initial behaviour. Further results would not support that conjecture and an alternative, deterministic conjecture would be found. When this failed too, perhaps yet another reason for the behaviour would be proposed though at some point, if sufficient testing were carried out, the search for such explanations would come to an end and the behaviour would be described as random.

There is a strong sense in which stochastic meanings are constructed to deal with those aspects of the world which cannot be deterministically explained. It is no coincidence that the children's actions usually involved searching for deterministic behaviour, since it is impossible to look for stochastic behaviour when it is characterised only in terms of the *absence* of various attributes: a *lack* of predictability, a *lack* of sequential patterns and a *lack* of control. A richer appreciation of the stochastic must involve the identification of positive features and these lie in long-term behaviour

4. The emergence of global meanings

The story of how new meanings emerged out of the interaction with the computer-based resources is most vividly captured by focusing on the activities of one pair of children, Anne and Rebecca. Their story illustrates important issues apparent across the cases in the final iteration of the study There appeared to be four phases in the emergence of new meanings, which focused on the aggregation of results in long-term behaviour. These phases did seem to have a certain progression to them, but this progression is not one in which new and better ideas replace previously established ones. On the contrary, new pieces of knowledge - situated abstractions - were constructed in such a way that their meanings were connected with, or distributed across, previous local meanings.

(i) Sense-making through well-established local meanings
Anne and Rebecca made sense of their early interactions with the Chance-Maker microworld through the use of local meanings. There were many references to unpredictability and unsteerability

Anne: "I don't think you can really estimate which one."

Rebecca: "You can't be too sure really."

Rebecca: "If you start on tails, it might land on tails again because it might not be a very good flick."

Anne: "The first time there were loads of tails, so I thought it was going to be tails again. But probably after a couple of goes, it will probably do tons of heads again."

At this point in time, Anne and Rebecca's only resources for making sense of long-term behaviour was through these sorts of meanings.

(ii) The number of trials controls the appearance of the pie chart

After 200 trials of the coin, the pictogram showed the two rows nearly equal in length (Figure 4).

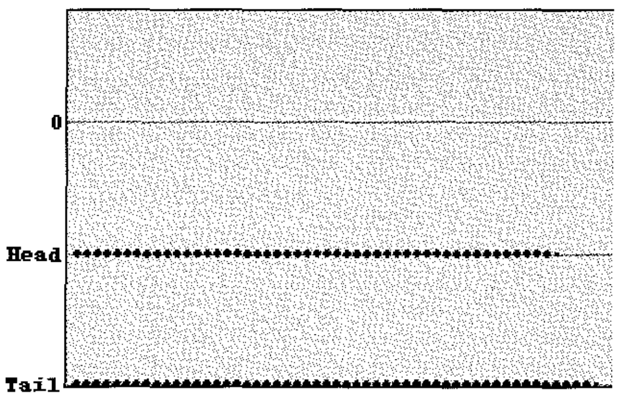

Figure 4

Anne thought there might be a reason for this even appearance of the pictogram, but she was unsure what that reason might be. Rebecca was unconvinced, suspecting that the evenness may have just been coincidence.

When they observed the evenness of the pie chart for 1000 trials (Figure 5), they abstracted the notion that the higher the number of trials, the more even was the pie chart

Anne: "I think it's the highest the number, the even more it gets."

Rebecca even advanced a corollary to this situated abstraction which suggested that lower numbers of trials would be not so even.

Rebecca: "Because the other time, when we did less numbers, it was half um even really."

It is clear in the detail of how Anne and Rebecca reached these conclusions that they have been constructed through interactions with structures which enabled the repetition of many trials (the 'Repeat' primitive) and provided images of aggregated results (the graphing tool, especially the pie chart)

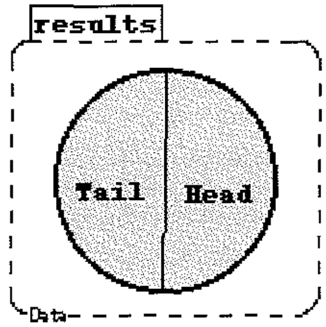

Figure 5

This knowledge, having been constituted through the webbing of internal resources and external tools, was likely to be deeply connected to those tools, and so one might regard the abstraction as situated in the domain of the coin gadget. Just how situated, and therefore constrained, this knowledge was apparent in the subsequent interactions with the spinner gadget

(iii) The workings box controls the evenness of the pie chart

In many ways, the coin gadget had not been problematic, since its workings were set to cause the gadget to behave exactly like an everyday coin. The girls had therefore not needed to engage with the workings box. Indeed, there had been no reference to the workings box until they began to interact with the spinner

Initial experiments with the spinner gadget had suggested to Anne and Rebecca that perhaps there were too many 1s appearing in the results. Consequently, a need arose for them to consider whether the workings box was causing a degree of unfairness. Anne and Rebecca were quickly able to edit the workings box so that each possible outcome appeared exactly once (Figure 6).

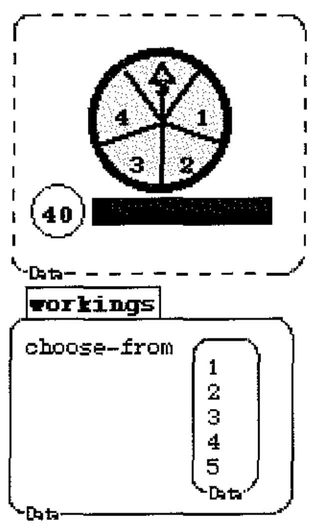

Figure 6

There now followed an extended period in which Anne and Rebecca carried out a sequence of experiments involving about 50 trials each time Each experiment generated a pie chart in which the possible outcomes were quite unequally represented. From the statistician's perspective, this is not surprising since they would not expect evenness in the pie chart for only 50 trials, even if the workings were fair

Anne and Rebecca, however, reacted by editing the workings to try to make the pie chart come out even First, they changed the workings to include two of each possible outcome, then, seeing too many 1s in the following pie chart, they removed one of the 1s from the workings box. When this failed to work they tried a workings box with one of each outcome represented - back, in fact, to where they had started.

It is clear from their actions that they expected there to be a direct relationship between the entries in the workings box and the appearance of the pie chart, even though they were only using 50 trials each time. The extraordinary thing about this episode is that they appeared to have 'forgotten' the situated abstraction from the coin that: 'the higher the number of trials, the more even is the pie chart'.

How might Anne and Rebecca's actions be explained? My interpretation is that the introduction of the workings box brought this very much to the forefront of their attention. A meaning that changing the workings box would change the pie chart was easily cued by its connection to well-established intuitions of deterministic behaviour The reliability of such intuitions was far in excess of recently acquired and therefore very tentative situated abstractions, which, in any case, were connected with the various surface features of the coin gadget, ones not apparent for the spinner

(iv) The number of trials *and* the workings box controls the evenness of the pie chart

It would be easy to dismiss Anne and Rebecca's tinkering with the workings box as misconceived. On the contrary, it seems to me that it is exactly these prior experiences which allow the co-ordination of a powerful situated abstraction which deals perfectly adequately, even from an expert perspective, with cases where the number of trials is low or high and the workings box is uniform or not.

Anne and Rebecca were challenged to make a dice gadget for a game in which there was a good chance of getting 1s, a fairly good chance of getting 2s, but a pretty low chance of getting anything else. Rebecca immediately edited the workings to an appropriate form and they tested the new gadget using 1000 trials While waiting for the pie chart, Rebecca predicted: "More 2s, more 1s and less of the others", indicating an expectation that 1000 trials would cause the pie chart to appear in proportion to the frequencies in the workings box. When asked what would happen if 50 trials were used to test the gadget, Anne explained, "A bit more uneven". There seemed to be evidence that Anne and Rebecca now saw control of the appearance of the pie chart as exerted jointly through the workings box and the number of trials.

5. Discussion

I would like to begin the discussion with some theory building The sketch in Figure 7 schematises the process by which new connections, in the form of situated abstractions, are forged out of juxtaposition of weakly connected, local meanings and external structures within the Chance-Maker microworld. Thus, meanings such as unsteerability are reserved for cases where the number of trials is low and new causal meanings for control emerge to make sense of long-term behaviour. Further co-ordination of causal influence of the number of trials and the workings box is also depicted in the emergence of a meaning in which they are combined.

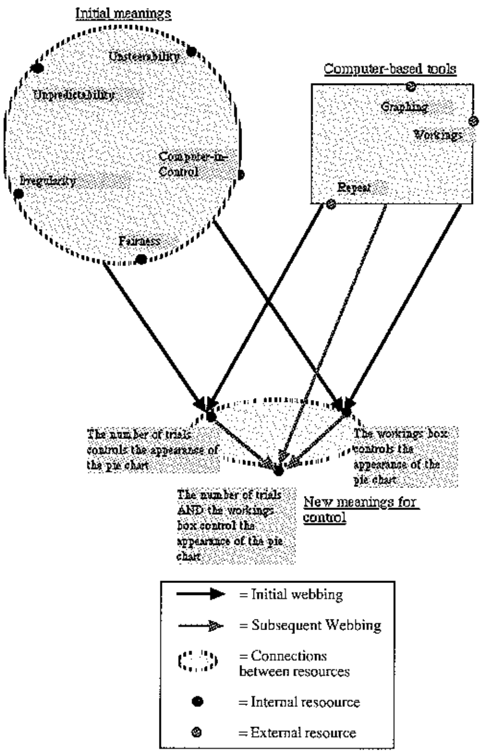

Figure 7

The five local meanings are depicted as contained within a closed set to illustrate the connections, such as interchangeability, among those meanings. As the children interacted with the Chance-Maker's gadgets, one or more of these local meanings were cued by features of that activity. Out of this connection emerged new meanings for control, in the form of the two situated abstractions: 'the number of trials controls the appearance of the pie chart' and 'the workings box controls the appearance of the pie chart'. Further activity with the computer-based tools allowed the co-ordination of these two situated abstractions into a third: a global meaning, which states 'the number of trials *and* the workings box control the appearance of the pie chart'. I see this type of connection between local meanings and the computer's tools and resources as a particularly clear example of webbing. Below, I will develop this idea in a way which I believe clarifies, and possibly extends, the notion as first presented by Noss and Hoyles.

It is important to recognise that the initial meanings continue alongside the newly formed situated abstractions, available for future sense-making activities In particular circumstances, such as when the number of trials is neither clearly large nor small, the meaning which is actually cued may appear quite arbitrary. Continued access to a wide range of possible meanings, which gives the meaning-making process its messiness, confounds attempts to present a smooth picture of how the children progress from one 'level' to another

The situated abstractions for the long-term behaviour of the gadgets take the form of causal relationships, where one specific factor, such as the number of trials or the workings box, causes a corresponding effect, such as the evenness of the pie chart. This is in stark contrast to the local meanings which describe attributes, linked to each other by association or inference: for example, the dice is unsteerable, so it is random, so it is unpredictable

The causal situated abstractions are not, however, disconnected from the local meanings. Local meanings become reliable as predictors of short-term behaviour, but are of reduced reliability for making sense of long-term aggregated behaviour New global meanings take on increasingly high reliability as they are found to explain long-term behaviour better than longer-established meanings and to be successful predictors of the behaviour of more and more gadgets [6] In this sense, the system of local meanings is restructured whilst new global meanings are created afresh. The co-ordination of local and global meanings can be seen as a condition, attached to, for example, the unsteerability meaning If indeed the number of trials is large, the condition activates the controllability global meaning A first attempt to schematise the co-ordination of local and global meanings for randomness is depicted in Figure 8

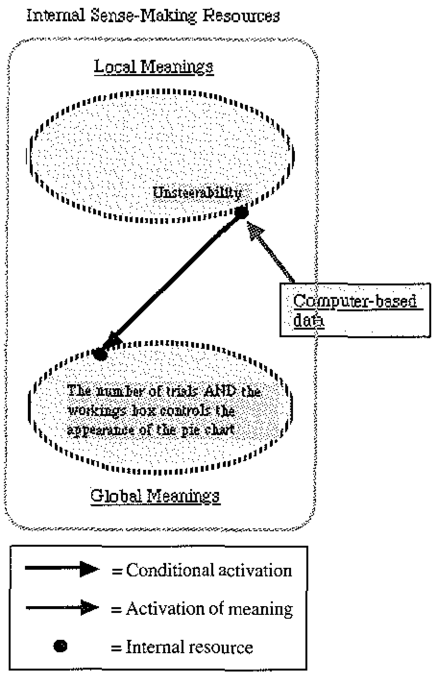

Figure 8

Start with the rectangular box to the right of Figure 8 Surface features of the activity from the behaviour of the computer-based phenomena, as viewed by the child, activate the unsteerability meaning in the set of local meanings, part of the overall sense-making resources The condition attached to the unsteerability meaning activates the global causal meaning This conditional connection can be schematised as:

if <number of trials> is large,
then cue <connected global situated abstraction>

The connection between the local and global meanings is depicted by a heavy black arrow.

This model can be extended in similar ways to incorporate unpredictability, fairness and irregularity In each case, the global meanings emerge out of a conditional connection between the local meaning and a corresponding cause-and-effect situated abstraction, forged out of the specific features within the Chance-Maker microworld

As I mentioned earlier, Piaget and Inhelder argued that children come to recognise random mixtures as those which cannot be explained by operational thinking: in particular, randomness can not be reversed, a defining characteristic of operations. There is support for this notion in this study. It seems that the children characterised their meanings for the stochastic through the absence of properties which are apparent in deterministic phenomena If a phenomenon were *not* predictable, *not* controllable, *not* regular in its behaviour, then it was random The one exception was fairness, which for some children was deterministically based in the sense that fairness was sometimes inherent in the symmetry of the device and this symmetry would *cause* fair results. Presumably all these notions were abstracted from the everyday use of terms like 'fairness' and 'randomness' when experiencing short-term, unpredictable and uncontrollable behaviour.

The analysis in this study suggests that the children's local meanings are used initially to make sense of long-term behaviour. The local meanings then appear to be misconceived. Yet, meanings such as unpredictability, unsteerability, and so on, are the only resources available to the children as they attempt to make sense of the behaviour of the gadgets By paying particular attention to the role of the structuring resources, this piece of research not only finds support for the weak implication of the heuristics research but also suggests how new global meanings for randomness might emerge

Acknowledgement

I should like to thank Professor Richard Noss for his insightful contributions during this research and for his valuable comments on the first draft of this article.

Notes

[1] This article is based on aspects of my doctoral thesis (Pratt, 1998)

[2] From a constructivist perspective, mathematics cannot exist independently I use the phrase here metaphorically to mean that the software designer could envisage certain plausible actions upon the intended structures In carrying out this virtual experiment, the microworld designer selects those structures where plausible actions might lead to the construction of mathematical meanings.

[3] The names of children used in this article are pseudonyms.
[4] Professor Andy diSessa heads a team at the University of California-Berkeley which is developing what is termed a *computational medium*, called Boxer. This project is seen as extending the notion of literacy to a new domain, one where users express themselves in various ways, including mathematically, in various modalities (graphic, literal, computational, ...) within the Boxer medium. Boxer is particularly well-suited to iterative design, because of the ease and flexibility with which one can reconstruct the interface
[5] The children in this study were part of a computer-rich project, the Primary Laptop Project, and so were quite sophisticated in their use and appreciation of computer technology
[6] diSessa's theory of conceptual change (diSessa, 1993), based around the notion of phenomenological primitives and causal nets, is well-suited as a framework for the co-ordination of meanings for randomness. I intend to elaborate on this theoretical model in another article

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