Proto-mathematics and/or Real Mathematics*

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I wish to contribute to the discussion about the teaching of mathematics in context, situated, as ethnomathematics, and humanistically as contrasted with what one might call traditionally (in one sense of that word), from the perspective of one who has no particular enthusiasm for old-fashioned teaching methods and is a supporter of the humanistic attitude but fears that the baby may be going out with the bathwater in some places. In particular, I am concerned to demarcate what is mathematics and what are often useful preliminaries but are not mathematics, what Chevallard [1990] called, also in a critical mood, proto-mathematics. There does not seem to me to be a clear enough view of just what does constitute mathematics in much of the literature that I read, and I hope to contribute to a discussion of that important question here. To do this I begin by discussing ethnomathematics both more strictly and very loosely defined and then pass on to a discussion of what mathematics is about in order to make as clear as I can why mathematics must be what has been labelled "decontextualized". Only when that fact, as I see it, has been acknowledged can a profitable discussion take place on the usefulness or even possibility of teaching mathematics "in context". If one can't tell the baby from the bathwater, the baby is in serious danger. What follows is meant as a baby picture.

Proto-mathematics

For a long time, children at school in English-speaking countries studied arithmetic and later geometry, algebra, and perhaps trigonometry. This persisted until quite recent times, with the result that, thirty years ago where I was brought up, these three last were thought by high-school graduates to compose mathematics. The writers of schoolbooks had by the 1950s already begun calling them mathematics books rather than books on the branch of mathematics they treated. The displacement of the specific terms by the general and more sophisticated term in education is a historical matter that I do not propose to go into. It seems to me that since then there has been a further watering down of what is called mathematics to the point where anything that might lead to mathematical considerations is now being called mathematical, for example, "practices which are typically mathematical such as observing, counting, ordering, sorting, measuring, and weighing." [1] This would have been news to Plato, for whom the analogous term referred to theoretical study of arithmetic, geometry, and astronomy of a certain sort. [2] There is nothing unique in D'Ambrosio calling things mathematical that are merely places where we can see mathematics; P.J. Davis does the same thing in his quite different essay on art, "Mathematics and Art: Cold Calipers against Warm Flesh." [3]

D'Ambrosio is of course aware that he is using the term mathematics "in the *latu sensu*" [p. 231]. In coining the term ethnomathematics, he means by it something even broader, "distinct modes of explaining and coping with reality in different cultural and environmental settings" [p. 232], "the study of the generation, organization, transmission and dissemination and use of these jargons, codes, styles of reasoning, practices, results and methods" [p. 233], in other words, rational thought and its practical adjuncts. The other founder of this notion, Marcia Ascher, began from an anthropological start and recognized "implicit mathematics" [4] elsewhere, in what she calls "mathematical ideas", having to do with "number, logic, and spatial configuration and ... the combination or organization of those into systems and structures" [p. 37]. The founders, both mathematicians, are concerned that for their study it is important to see mathematics wherever the kind of insight is present that mathematics concerns itself with. A scientific analogy would be to see physics wherever there is anything physical. Physics is of course concerned with the physical, but only with certain insights into the physical world, other insights belonging to other sciences, and some to mathematics. Ascher uses symmetry as an example of a mathematical idea [p. 37], when I think that symmetry is a subjective notion where mathematics may someday have more worth saying, but at present has rather little to say [cf. Devlin, 1994; Radin, 1995]. She is rather clear that, while real mathematics and these notions of implicit mathematics "are linked", they are also "different" [p. 38] because of the former's transcending the corporeal basis of the latter.

The variety of notions now covered by the term "ethnomathematics" is brought out in the special issue of *For the Learning of Mathematics* [5] devoted to the subject. The short article by Paulus Gerdes ["Reflections on ethnomathematics", pp. 19-22] is particularly concerned with this variety. Among roles for ethnomathematics, or what is better called proto-mathematics [6] than ethnomathematics, are two that are certainly legitimate and mutually reinforcing: the research topic archaeology/anthropology of mathematics, which is well named ethnomathematics and has nothing immediately to do with children, and the use of historical or present-day popular "implicit mathematics" as the motivation and starting point for the study by children of explicit mathematics. The historian Victor Katz dwells on the latter—so obviously having considerable potential for good—in his contribution to the special issue ["Ethnomathematics in the classroom", pp 26-30]. Rik Pinxten ["Ethnomathematics and its practice", pp 23-25] directs attention to the former. He attributes to Alan Bishop [1988] a stress on the difference between what I am calling proto-mathematics, "the set of skills and procedures for counting, measuring, and the like, that a group or an ordinary individual knows and uses in life", and what I am calling (real) mathematics, the scientific discipline, admitting the possibility for some overlap. In the term I have borrowed from Ascher, the real mathematics is "implicit" in proto-mathematics. Being of a somewhat philosophical cast of mind, Pinxten thinks that it is important to see just what proto-mathematics looks like, recognizing that mathematical knowledge, like other knowledge, "is contextual and cultural in nature" so that real mathematics "builds on and is somehow guided by the concepts and intuitions" of proto-mathematics whether we like it or not, whether we care to study the phenomenon or not. As the opposite of the new-math approach that dismissed proto-mathematics without a look and attempted to teach real mathematics from the beginning, this has something to recommend it. And he can point to Ascher as having accomplished what he points to. He suggests as a third-world strategy making explicit the proto-mathematics of children's culture (when it differs from that of urban westerners) before the introduction of the explicitness of any real mathematics, a suggestion of apparent paedagogical merit

> My proposal is to make this conscious and explicit, and to actively train these (other) native insights of the subjects in order to reach a level of understanding and sophistication in them which is already largely available (subconsciously) in the preschool knowledge of the Western child entering the curriculum. The exploration of functional relations and spatial concepts in the vernacular, using the contexts of the experience of the children allow us to do this. Through this exploration, linguistic mapping, and the actual training of visualization, the child will reach the insights needed to come to grips with the (implicit) world view of [real mathematics]. Hence the move from [proto-mathematics] to [real mathematics], and the use of [proto-mathematics] notions in the successful development of [real mathematics], will have to pass through the conscious, systematic, and explicit exploration of the largely subconscious and ill-developed outlook in [proto-mathematics] in the preschool child of another culture [p 25]

It is not clear to me why this advice does not apply—with somewhat less emphasis—also among urban westerners. I do not know where else one could reasonably begin.

All of this thoughtful and helpful activity could, of course, be cast aside. The teaching of real mathematics could simply be replaced by an encounter with proto-mathematics. This seems to be a notion of ethnomathematics that has surfaced in the popular press. When D'Ambrosio arrived in England to speak at a conference in 1994, he was greeted co-incidentally by a column in *The Observer* of London, which touched upon ethnomathematics within a somewhat better informed discussion of the regrettable "teaching" of illiteracy. Melanie Phillips wrote,

> This is the maths we pick up by chance in day-to-day life, said to be as valid, if not more so, than the maths we're taught in school. So it follows, as the guide [a course guide in mathematics education in connection with an education diploma course of the University of London] helpfully explains, that classroom teaching merely confuses and demoralizes the pupil. Education is thus reduced to no more than the serendipity of random experience. [7]

There is another aspect to the ethnomathematical notion that I want to air. I see a different danger from the one imagined by the writer in *The Observer*. Not that nothing will be taught, but that something other than mathematics will be taught under its good simple name, or under the more user-friendly term "humanistic mathematics". Like ethnomathematics, humanistic mathematics as I understand it is a legitimate approach to our many-faceted subject. The emphasis upon the human-createdness of mathematics is fundamental to my own thinking about mathematics as the most delightful of the ways that we understand the world. The notion, popular from ancient Greece to at least early in the present century, that the Creator does geometry, is, I think, exactly the mistake I was pointing out at the beginning, of seeing real mathematics where mathematics is only implicit. I do not understand why Kronecker attributed the integers to God and the rest to us; I am quite prepared to claim the integers too! One of the ways in which the human-createdness of mathematics can be emphasized most effectively is through the history of our creating it, and the teaching of the history of mathematics is just as suitable for this as the ethnomathematical approach, which does not need to look backward in time. I subsume history in ethnomathematics because such history is always in a cultural context and so is always ethnomathematics. What worries me is the possibility of teaching the anthropology or history or philosophy of mathematics *instead of* mathematics. I have been reminded of this possibility by several things I have read lately. One was a published student essay recounting the satisfaction of a "math requirement", put off from semester to semester and finally satisfied by "independent study" (no examination, presumably) of Marcia Ascher's *Ethnomathematics*. Now the student's study of quipus, the grammar of number words, sand tracings, the bridges of Königsberg, and games is exactly the study of what Ascher calls "mathematical ideas", not mathematics. As public relations, the treatment was a great success. The essay concludes,

> Didn't you ever wonder why people, even very young children, are able to sing a song and remember each and every word? Perhaps that is because we view singing a song as pleasurable—it makes us feel good. Maybe a similar approach to teaching math and sciences is the answer. When the fear is removed and one finds learning pleasant and no longer intimidating, one can begin to explore new worlds that once were thought to be unreachable. [8]

As an approach, as I said earlier, I think this is excellent psychology and gives hope of learning some mathematics to someone who did not cope adequately with the introduction of letters at the beginning of algebra But this task was supposed to satisfy a math requirement. It is a prelude; as the main event, it is like satisfying an English requirement by reading literary criticism—except that criticism is often harder than the literature rather than easier and is itself literature of a sort. An isolated course of scant mathematical content is not something to be alarmed about, but later in the same issue of the *Humanistic Mathematics Network Journal* the Director of the University of Chicago School Mathematics Project, Zalman Usiskin, commenting on the NCTM Standards, wrote,

> With all this ability to generate examples and confirm patterns with examples, I worry about the future of deduction, that aspect of mathematical reasoning that is unique Induction may generate patterns but it does not tell us that the patterns hold
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> . . Reasoning using deduction needs to be in the curriculum of all students, from grade 1 up It is the way we decide whether something is true in mathematics, and to avoid it is akin to teaching science without experiments We need to look again at the roles of assumptions, logic, definitions, theorems, and proof in an exploratory environment It is not enough to say that students will want to confirm the patterns they find: our research indicates that many *PDM* [unexplained] students consider confirmation by example as powerful as confirmation by proof [9]

In studying what their schools are calling mathematics, these *PDM* students have had the subject hidden from them. I am disposed to take this matter seriously enough to write about it because the little reading that I have done in science education suggests to me that something analogous may be happening there. School science, the study of certain aspects of the physical world, could be replaced by what is quite legitimately called in university circles "science studies", meaning history, philosophy, and sociology of science. If the little science in schools were replaced by the study of how other cultures understood and understand the physical world, it would further the political agenda of multiculturalism, but it would teach the children even less about what the whole world has built upon the insights of Galileo, Copernicus, Kepler, and Newton It would be infinitely easier and more fun for those with no antecedent interest in the physical world, but it would deny them and all the others the information that is meant to be conveyed about the way the world works, which I take to be one of the aims of teaching science in schools. If there really is such a tendency in science, it is more important and will be more difficult to combat a similar tendency in mathematics

The danger that I see in looking at mathematics "in context", which is another way of putting proto-mathematics on offer as mathematics, is that one will see only the context and miss the mathematics. This theme, from a very different point of view, was the subject of a recent paper by Anna Sierpinska [1995].

Ethnomathematics

Having mentioned what I take to be misuses of the fine ideas of ethnomathematics and humanistic mathematics, I want now to think for a moment humanistically in ethnomathematics terms

The mathematics of Plato, to which I referred above, was the ethnomathematics of the Hellenistic culture. There is nowhere else for mathematics to start but as ethnomathematics, unless one hypothesizes tablets of stone containing formal expressions and their miraculous understanding What distinguishes the Hellenistic ethnomathematics from that of other cultures? It may be proof, which G.E.R. Lloyd considers distinguishes it from Indian ethnomathematics. [10] Or it may be axiomatic method, as G. Joseph attempts to show. [11] Whatever features of Hellenistic ethnomathematics distinguished it in its day, Euclid's *Elements* is the definitive text of western European ethnomathematics in the early modern period, displaying proof in a context of axiomatic method. Whether the axioms in Hellenistic ethnomathematics were considered by the Greeks to be true and obviously so, this was a common assumption of western European ethnomathematics, conferring upon mathematics enormous prestige One of the features of western European mathematics from Newton to Cauchy was a widespread lack of rigour. Almost simultaneously with a start being made at putting this apparent lapse right, the advent of non-Euclidean geometry put a permanent but slowly recognized end to the illusion that the axiomatic method requires axioms that are true and obviously so [12] The rigorizing of European mathematics proceeded through Weierstrass and Frege, culminating in *Principia Mathematica,* and continues in foundational studies to this day, but present-day mathematics the world over is axiomatically based and proved informally but is not based upon a uniquely agreed-upon set of axioms. What we mean today by the axiomatic method is different from the old prestigious axiomatic method, and with that difference we are no longer doing the ethnomathematics of eighteenth-century Europe (whether it was continuous with Hellenistic mathematics or not). We have been driven back to building on the ethnomathematics of India, or wherever it was had proof without universal axioms. We are now doing world-wide mathematics, whose paradigm emerged in the period from Bolyai, Gauss, and Lobachevsky to Noether and Gödel. It is this, which is no local society's ethnomathematics, only the global society's, that I mean by *real* mathematics This is definitely not to say that earlier mathematics could not be real in this sense, but, real or not, it was all ethnic.

Real mathematics

To explain my view of real mathematics, I am going to begin with a famous remark of Russell [1901] that, like D'Ambrosio's, is a trifle broad in its compass:

> Pure mathematics consists entirely of assertions to the effect that, if such and such a proposition is true of *anything,* then such and such another proposition is true of that thing It is essential not to discuss whether the first proposition is

> really true, and not to mention what the anything is of which it is supposed to be true . . *If* our hypothesis is about *anything* and not about some one or more particular things, then our deductions constitute mathematics. Thus mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true

This view has by no means been given up in the intervening century. Fundamental to the discussion by Sierpinska [1995] of mathematics "in context" is the recognition that some educators are offended by the reality that mathematics is decontextualized, as it is described by those who deplore it. All calculation is decontextualized; calculation is boring trivial routine deduction.[13]

Michael Dummett argues, in a paper called "What is mathematics about?" [1994], that the Russell (Frege, Whitehead) view quoted above is "closer to the truth than any other that has been put forward", it was

> brilliant because it simultaneously explains various puzzling features of mathematics. It explains its methodology, which involves no observation, but relies on deductive proof. It explains the exalted qualification it demands for an assertion: in other sciences, a high degree of probability ranks as sufficient grounds for putting forward a statement as true, but, in mathematics, it must be incontrovertibly *proved*. It explains its generality; it explains our impression of the necessity of its truths; it explains why we are so perplexed to say what it is about. Above all, it explains why mathematics has such manifold applications, and what it is for it to be applied. It allows that mathematical statements are genuinely propositions, true or false, and hence accounts for what is manifestly so, that mathematicians may be interested in determining their truth-values regardless of the uses to which they may be put; at the same time, it explains the content of those propositions as depending on the possibility of applying them, and thus justifies Frege's dictum that it is applicability alone that raises arithmetic from the rank of a game to that of a science [p. 14]

Since our definitions and axioms do not specify exactly what they may apply to, it is not possible for us to be certain what is true without proof. Interpretations can give us only counterexamples. This is a difference between the proto-mathematical study of patterns in which mathematics is only implicit and the explicit exploration of those and other patterns in what is these days called (to some explanatory effect) "the science of patterns", the subtitle of a popularizing book by Keith Devlin [1994]. But Devlin does not explain to anything like my satisfaction what he means by "science" in his subtitle. For me, science means that physics, for instance, studies the insights that we have into physical phenomena and the logical connections among the insights, which correspond to what we regard as explanatory connections among the phenomena. It is important that mathematics is not just the study of embodied patterns but the *scientific* study of patterns: that is, the study of insights that we have into patterns and that are detachable from the embodiment of the patterns, and the explanatory logical connections among those insights. If the insights are into embodied patterns, then we can use the embodiments in the study as scientists do when they "confirm" a theory (though strictly speaking one can maintain that they only fail to falsify, a lot of such failures adding up to corroboration, which passes for confirmation). Detached insights cannot be so confirmed: only the logical connections are available to *explain* in an intellectually satisfying way.

While I see therefore no danger that we shall have to abandon proof,[14] we have abandoned something, and, since the abandonment occurred before I was born, I have never regarded it as a loss. It is only in thinking about it lately that the impact of the loss of the old meaning of axioms has come home to me, even intellectually. It is the change that Morris Kline [1980] wrote of as "the loss of certainty", though contemporary mathematicians cannot recapture the emotional impact. Holding more or less to the Dummett version of what mathematics is about, I see no harm in multiple "axiom" systems. We know, for instance, that a model for one axiom system may exist within a model for another system although the two contradict each other: for instance, the non-euclidean geometry of great circles on a sphere inside three-dimensional Euclidean geometry. While this does not trouble us and need not trouble us, the old meaning of an axiom system has been destroyed. From this perspective, it is not surprising that the event gave rise to the search for "certainty", for a unique satisfactory foundation for all of mathematics *à la* Frege. Nor is it surprising, from our later perspective, that the result was merely the creation of the subject of "foundations", none of them more certain than the others and some of them leading to interesting and useful mathematics.

I have an understanding of the Russell view, a modification of it that I have not found elsewhere and that avoids the reduction of mathematics to logic, one of the reasons why Russell's view is, as Dummett writes, "how generally discredited". If logic is sufficiently widely construed, then everything intellectual can be reduced to logic: I am thinking of logic as the science of general deductive argument. Mathematical structuralism (not structuralism in the sense of Althusser, Barthes, Lacan, and Levi-Strauss, the sense in which something may be post-structuralist) holds that it is not the mathematical objects that are being studied in mathematics but the structures that they compose.[15] This way of approaching the subject matter deals more justly with the structural than with the calculational aspects of mathematics. I suppose that constructivism deals more justly with the calculational aspects than the structural. One needs to deal with both

What is mathematics about?

As mathematical structuralism points out, mathematical objects, since they are not even specified, are not what mathematics is about. This is not to impugn the objectivity or mind-independence of mathematics.[16] As Russell argued, mathematical statements, whether calculations or structural theorems, are not about anything in particular. On this account, mathematical objects are sometimes considered to be fictions [17]; this is a possible description, but confusing and unnecessary because it is too specific. One ought not call "whenever" a fictional time, "wherever" a fictional place, "whoever" a fictional person, or "whatever" a fictional entity; they are just words that are indefinite because they lack definite reference or antecedent. Mathematical objects are the grammatically (and psychologically) necessary posits that allow us to speak of the relationships that they have. Relationships, including how things behave together (not just in what are normally called structures), offer, I believe, a sufficiently reasonable way to describe what mathematics is about. It is hard to speak of the determination of a line by two points on it without mentioning the points and the line, but what is important in the determination is that one kind of thing has one kind of relation to (in this case "contains") the other and that two of the latter determine one of the former. The relations of containing (which need not be just containing, since there are sets of axioms in which lines and points are interchangeable) and determining are what that axiom is about. The ontological status of points and lines is of no consequence, since real points—if one thinks that there are such—determine the corresponding real lines no more and no less than imaginary points—if there are no real ones—determine imaginary lines. This is because of an apparently universal convention of using ordinary kinds of real relations when imagining unreal things. In science fiction, where this is done all the time, only a few things are changed. The science fiction world is different from our own, but only in a few respects. In ordinary novels, an imaginary child has the same relations to its parents as a real child to its parents; we signal this by using the words "child" and "parent". Those with a determinedly ontological bent will of course be disturbed by any emphasis on relations, for ever since the Greeks relations have been "poor relations" of things, for no good reason I can see. A son of mine is no more real than his relation of sonship to me, since his very coming into existence is by virtue of that relation. Since I consider relations no less real than things (whatever "real" may mean), I am not in the least bothered by the extension of relations holding among real things to holding among things that may or may not be real, or even to things that definitely are not real like the characters in novels. Fiction works only because we *do* extend to the imaginary characters the relations that hold among ordinary persons. Fiction and mathematics do not lack objectivity on account of being about mixtures of real and imaginary things or real and unspecified things, respectively.

As Dummett points out so succinctly, the Russell view of mathematics has many virtues but also the vice of reducing mathematics to logic. I believe my narrowing of Russell's view avoids the vice. Because it is narrower it does not include all of logic. Logic, I take it, studies how arbitrary statements about objects lead to other statements about the same objects, independently of the objects that are meant and what precisely the former and latter statements say about them. By restricting the statements to be statements about *relations* among unspecified objects, there is no obvious danger that all statements whatever will be included. To put this another way, the only way that the connections among arbitrary statements we call logical can be brought into mathematics, as I have sketched it, is for them to be recast as relation statements, and that is exactly what has been done in order to render much general logic mathematical. Predicates acquire mathematical notation and become subject to a calculus.

In emphasizing the relations among mathematical objects rather than the objects themselves I am, I think, being truer to the way we think and learn. Important among the relations are functional relations, functions. As Saunders Mac Lane has pointed out to me, [18] there is "a strong mathematical tendency to replace relations whenever possible by functions", but this is just our favorite kind of relation. We do not, for instance, care whether positive integers are complex numbers, real numbers, rational numbers, or natural numbers because the relations that they have among themselves are the same regardless of which of those systems (structures) contains the numbers we are concerned with. From the learning perspective, Anna Sfard has been pointing out lately [19] "that the operational (process-oriented) conception emerges first and that the mathematical objects (structural conceptions) develop afterward through reification of the processes" (from the abstract of the latter paper) that is not easily achieved. This illustrates that learning mathematics does not proceed by access to mathematical objects, since notoriously we do not have access to such objects (P. Maddy [1990] thinks we have access to sets), but by access to the relations from which mathematics is constructed. We clearly do observe and even participate in membership, which is what is important about sets. The fundamental relations from which mathematics is constructed are everyday relations rendered what is sometimes called transcendent by detaching them from the everyday things that they relate (where they are already often mind-independent). I am not happy with the word "transcendent" both because it is vague to the point of mystification and because it suggests that the relations by being detached, are altered, when their usefulness depends upon their *not* being altered. By this detachment, nevertheless, they are rendered not only mind-independent but also world-independent. Then our axiomatic approach to those relations works only with those aspects of the so-called transcendent captured in an axiom system. While they are intended to represent the transcendent, except in trivial cases they are a mere projection of what is beyond the grasp of any particular axiom system.

Mac Lane [1986] seems to me to be on this track when he asserts "that subjects of Mathematics are *extracted* from the environment; that is, from activities, phenomena, or science—and that they are then later applied to that—or other—environments." He explains,

> I have deliberately chosen this [word] "extraction" to be close to the more familiar word "abstraction"—and with the intent that the Mathematical subject resulting from an extraction is indeed abstract. Mathematics is not "about" human activity, phenomena, or science. It is about the extractions and formalizations of ideas—and their manifold consequences. [p. 418]

In my opinion mathematics is not just about these extractions and their formulization, but about the re-attachment of these ideas to "human activity, phenomena, or science". The whole enterprise involves the detachments of insights, the study of the insights, and the re-attachment of the insights, or as it is normally called, the application of the mathematics. Purity in mathematics is just the limitation on the scope of the detachment and reattachment to mathematics itself.

I am sufficiently impressed with the virtues of genetic and evolutionary epistemology to need a plausible origin for such extraction. There must have been a situation in the past where, for the first time, persons did this extraction of what they regarded as the same system of relations from different parts or aspects of the environment. My suggestion of what this may have been is the two sources of numbers in counting and measuring. I have never seen any pedagogical material that made anything of this, but it is obvious as soon as one thinks about it that counting and measuring are different. Counting uses the integers from two up, and measuring usually needs fractions. Measuring without fractions is actually a special case of counting, and one can get away with it in volumes, where one measures out in a container of standard size. But in linear measure, one measures with a length of standard size and typically the count does not come out even. The Pythagoreans' horror of the incommensurable was their reaction to the counterintuitive fact that there are lengths that cannot be made to be integral no matter how small one makes one's measure. It represents the permanent splitting (in theory) of measuring from counting, two activities that they had thoroughly merged in their thinking. Having these two different places where numbers are used, it was natural to detach numbers and think about them, and so we have arithmetic with its own rules not overtly connected to either counting or measuring, but available for either use. These rules are the precursors of axioms, expressions of the arithmetic relations.

Axioms indicate the kinds of relations among the objects and how the relations are related. In thinking, one can use specific objects so long as one does not import into one's conclusions or reasoning relations among the objects not specified by or deducible from the axioms. If one knows how to do this, then one can probably give a proof (according to some style of inference or other—a variety are available) based solely upon the axioms and concerning abstract objects since one has ignored the specific aspects of the less than fully abstract objects reasoned about; that is the only way to be sure in what is usually a very tricky situation. Note that the implicit appeals to diagrams in proofs in Euclidean geometry and eighteenth-century analysis do not necessarily lead to error. Axiomatic relations that we have extracted are the "structure" that we have found "in" the objects that motivated the axioms. I suspect that the only way in which we understand most things, and especially the only way we can share understanding, is by this sort of transition from what is mind-dependent to what is mind-independent, as is normal with language. It is also the only sense I can make of the use of the term "structure" in this context. As Mac Lane acknowledges [p. 123], "some may hold that "abstract notions are difficult to understand", but he is right to "hold with G. Kreisel that these notions "in fact are usually introduced to make concrete situations intelligible" (*Math Reviews* 37 [1969] #1234)

It is, incidentally, by emulating this detachability feature of mathematics (which seems to be what is called its formality [20], as well as by using mathematics that natural philosophy shifted to being what we now call science. We no longer wonder what inertia and gravity and electric charge and magnetism *are,* we study their quantitative relations in space-time. In the words of Gregory Bateson, gravity, for example, is an "explanatory principle", which means that it is an "agreement among scientists to stop trying to explain things at a certain point" [1972, p. 39]. A link is Wigner's famous question, "How can one account for the unreasonable effectiveness of Mathematics in providing models for science and knowledge?" in Mac Lane's formulation [p. 445]. If mathematics is regarded as the study of the implication relations among insights of a relational sort, then "effectiveness ... in providing models" is automatic. For Mac Lane this is puzzling. When phenomena have given rise to mathematics, there is "reason to expect that the results might be applicable in return, at least to the original phenomena. But the resulting application is often to other and different phenomena!" [p. 445]. Not so surprising if the mathematics is capturing precisely ways in which the relations among the "original phenomena" are like the relations among the "different phenomena". What is not known in advance (typically) is *which* "different phenomena", but that there could be some is not surprising on this account of what mathematics is about, not so different from Mac Lane's:

> (i) Various phenomena do have underlying similarities and regularities ...
> (ii) On the basis of millennia of experience, mankind has developed "ideas" [insights] about the phenomena which in turn are used to extract from the phenomena a conceptual description of some of these similarities and regularities ...
> (iii) ... Because these connections [among the formal descriptive statements] help the practical understanding of the regularities, they have been extensively examined and in some cases reduced to formal proofs, by rules of inference, from astutely chosen axioms.
> (iv) In many cases, the results of these proofs fit the facts. [p. 446]

If detachable insight is regarded as the subject matter of pure mathematics, then the whole mathematical process that needs to be studied and taught has three stages:

(i) The perception in the real world of what could be the subject of mathematics—the stuff D'Ambrosio wrote of in my quotation at the beginning.
(ii) the abstraction of relations that can be discussed mathematically as pure mathematics, and
(iii) The re-attachment of the insights gained back to the real world—so-called applied mathematics

From proto-mathematics to mathematics and back to application. One thing that the trip ought to reveal is the boundary crossed twice.

If I am right about mathematics being the science of detachable relational insights, it goes some way toward explaining why relevance is so serious a problem. Relevance is a theme of the current so-called reforms of mathematics education in the United States. [21] In order to achieve relevance, it is suggested, we should teach proto-mathematics, not detached real mathematics. But until you detach it, allowing you to prove mathematical things, it is not mathematics. As soon as you do detach it, its relevance will be missed by those who do not realize that insight can exist detached from subject matter. For such a person proof is pointless because it operates only at the detached level. If an anti-mathematical bias that denies this possibility is allowed to dominate the teaching of mathematics, so that "proof" becomes a dirty word, as Wu puts it, the results will be similar to those of the anti-grammatical bias in the teaching of English composition. Those who are not taught in schools the important things like mathematics and grammar will have to learn them for themselves, which is a great burden for the educational establishment to place on the children whose parents have entrusted the teaching of these things, and many others, to the schools. What is needed is not to pretend that real mathematics can be studied in the concrete things about which one may have antecedent insights (some of them proto-mathematics) but to exhibit, as well as the mathematics, the relating of mathematical insights to their concrete instances, just as one does not teach grammar for its own sake but for the sake of lucid reading and writing. It may be that the "reform" to which I refer is a backlash against the teaching of pure mathematics, the teaching, as it were, of grammar for its own sake instead of for the sake of communication. Wu rails against an emphasis on process combined with a denial of the process internal to mathematics, proof. If one *imbeds* proof, the middle part of the mathematical process, in the larger process that has detachment (abstraction) and reattachment (application) as its outer ends, it may be more palatable to its opponents; it may even teach them something. What is called the problem of knowledge transfer is a part of the larger mathematical process. it is heartening to see Gila Hanna [1995] courageously addressing the central mathematical process. For many, however, proving is out for children, not least because their teachers know so little about it. The only person whom I know to be working on this problem is Andrew Wohlgemuth of the University of Maine, who is trying systematically to introduce how to prove things in a course (and text—as yet unpublished) for intending teachers. [22]

Conclusion

I have no evidence that children are much interested in insight. But they love insights. Unfortunately, many children do not connect insights (as in science) with what they do in mathematics. Unsurprisingly mathematics has no charm for such children. They must somehow be encouraged to construct for themselves models in which to find insight. It seems to me, as a non-expert on education, that Sfard [1991] and Kieren/Pirie [1989] are on the right track. [23] The only way to foster detached insights is to have models and so to construct models, and the only way to construct models is to manipulate them, and the only way to manipulate them is already to have them; accordingly it seems logically inevitable that one is always going to have to work on a seriously deficient model in order to improve it. In arithmetic—to return to terms we know the meaning of—the model is so simple that to dwell on it is rather boring. But the model in geometry is inexhaustible, or at least unexhausted, and it includes the more interesting aspects of arithmetic and all of trigonometry and is helpful for constructing the models needed for algebra and then calculus. [24] Moreover it is easy to suggest aspects of the geometrical model with even the most primitive technology, a slate, and to improve upon it with even the most sophisticated technology. The relations in one's model are what one can think about, perceive, as it were. The relations among the relations are what one can talk about. When one talks convincingly about those relations one is proving something. The deduction of interesting logical consequences of common understandings of mental models is and will remain a core mathematical-education activity, however little some mathematical educators choose to emphasize it. There is beauty and implicit interest in a symmetrical figure, but there is *communicable* interest in virtually any geometrical figure, and communication is fundamental to education, however constructivist one's leanings. I wish to echo Usiskin's and Wu's calls to prove things in geometry.

I am aware that none of the above addresses the vexed question of what mathematics to teach. Such a merely implicit argument for teaching geometry is not intended to settle what is an important and political question that such insights as I have go no way toward answering. It is not even clear to me who should decide. I claim only that the question is often not seriously posed. Some educators have already eliminated the subject, a bizarre educational treatment of the central intellectual discipline. [25]

Notes

[1] D'Ambrosio [1994] p. 230

[2] S.v. in *μαφη* H.G. Liddell and R. Scott, *A Greek-English Lexicon* (H.S. Jones, ed.) Oxford: Clarendon Press, 1968

[3] Ernest [1994a] pp. 165-183

[4] Ascher/D'Ambrosio [1994] p. 36

[5] Vol. 14, no. 2 (June, 1994).

[6] Chevallard [1990], referred to in Pimm [1994]

[7] 1994 03 27. Reference supplied by John Fauvel in the Chair's remarks in the International Study Group on the Relations between History and Pedagogy of Mathematics Newsletter no. 32 (July 1994); text supplied by Ubiratan D'Ambrosio.

[8] Byerly [1994], p. 24

[9] Usiskin [1994].

[10] Lloyd [1990] p. 75, quoted in Joseph [1994].

[11] Joseph [1994]

[12] Joseph points out that we have fallen back to some weaker view of proof like that advocated by Lakatos. Infallable deduction from obviously

and certainly true axioms is just not what mathematicians are doing. That view of mathematics, it seems to me, is a quaint artifact anthropologically displayed like those Marcia Ascher shows us; we have moved on. I doubt that this has been adequately realized by mathematicians or philosophers of mathematics

13 This is not a new observation; Frege came to the same conclusion in §87 of *Grundlagen*

14 This has been suggested. See Hanna [1995]

15 Iymoczko [1994]; Resnik [1981] and [1982]; Shapiro [1983] and [1989]. The May 1996 special issue of *Philosophia Mathematica* is devoted to this sort of structuralism

16 G. Kreisel indicated in his review (*Br. J. Phil. Sci.* 9 [1958], 135-158) of Wittgenstein's *Remarks on the foundations of mathematics* that, "Wittgenstein argues against a notion of mathematical object . . but, at least in places . . not against the objectivity of mathematics" [p. 138, n. 1] I owe the reference to H. Sinaceur

17 Bunge [1992] for example.

18 Letter of 1995 9 21

19 Anna Sfard [1991] and Anna Sfard & Leora Linchevski [1994]

20 Mac Lane [1986], sect XII 1, "The Formal".

21 Wu [1995]. For several years, all intentional change has been labelled 'reform".

22 Daniel J. Velleman of Amherst College has something [1994] for a general audience

23 Apparently following the lead of van Hiele [1986]

24 Wu [1995]

25 Davis [1995] is more helpful, pointing out that mathematics is, in Bateson's [1979] terms. "the pattern which connects"

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