# Supercalculators and the Curriculum

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In April of 1986 the board of directors of the National Council of Teachers of Mathematics [1988] included the following statement in a recommendation regarding the use of calculators in the mathematics classroom:

> *The National Council of Teachers of Mathematics recommends that publishers, authors, and test writers integrate the use of the calculator into their mathematics materials at all grade levels* [p. 6]

This is a strong statement regarding the use of calculators and the implications must be addressed. Given the current capabilities of calculators, the implications would seem to have multiplied 100-fold from their original intent. The National Council of Teachers of Mathematics [1989], in the 9-12 standards has not relented, however, but made the assumption:

> *Scientific calculators with graphing capabilities will be available to all students at all times* [p. 124].

Few teachers have had the opportunity to explore the use of such calculators. Consider what such an assumption might mean in terms of mathematical power for students *in the present curriculum*.

## Supercalculator capabilities and possible teaching implications

Reading documents regarding machines such as the Hewlett Packard HP-28S [e.g. Hewlett Packard, 1988; Michel, 1987, 1988; Muciño, 1988; Nievergelt, 1987; Tucker, 1987, 1988; Wickes, 1988], or *supercalculators* [à la Nievergelt, 1987], leads one to conclude enormous changes in both fundamental school mathematics topics and fundamental ways of doing mathematics are upon us. Before examining the impact such changes may have on the curriculum, consider a few examples of the capabilities of the supercalculators and possible implications for teacher-student interactions.

General capabilities of a "personalized" machine include:

a) two dimensional graphs and zooms;
b) vector and matrix computations;
c) numerical equation solving;
d) symbolic manipulation commands and tests;
e) structured programs in RPL *(Reverse Polish Lisp)* FORTH-, PASCAL-, BASIC-, or LISP-like languages;
f) substantial memory (128K + 32K);
g) symbolic differentiation and simplification [Wickes, 1988];
h) keys labeled, say SARX, that, given a function and an interval return the proper integral for surface area of the solid of revolution about the x-axis, set up in symbolic form, and the numerical value for the surface area to the nearest 0.01;
i) keys labeled PDIV or PROOT that give, respectively, the symbolic result of the division of two polynomials or the algebraically computed real and complex roots of up to a fourth degree polynomial [Hewlett Packard, 1988];
j) a key, CEQN, that gives the characteristic equation of a matrix [Wickes, 1988]; or
k) symbolic and numeric solutions to classes of differential equations [Hewlett Packard, 1988];
l) curve fitting routines [Hewlett Packard, 1988].

These examples are not exhaustive, but simply illustrative. Further power is now available in the Hewlett-Packard HP-48SX.

In order to see some specifics, let's look at a few problems:

**Problem 1 (Foley's favorite function [Foley, 1987]):** Suppose you wish to maximize the volume of a topless box that can be made from a sheet of material 15 cm by 20 cm by cutting out the square corners as illustrated in Figure 1

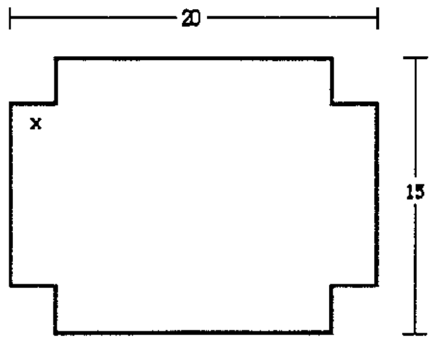



Figure 1
Plan for a topless box

The volume of the box can be represented as the function:

$$V(x) = (15 - 2x)(20 - 2x)x,$$

where $x$ is the length of the side of the square removed from each corner. As Foley [1987] points out, using a graphics calculator with the standard screen range produces what appears to be an invisible graph. That is, the graph of the function does not appear to be visible, because the range of values for the function is too large for a standard default scaled screen. A supercalculator can have an autodraw key called, say AUDRA [see e.g. Tucker, 1989]. The faithfully reproduced screen displays in Figure 2 illustrate the use of such a key followed by the INS key to digitize a rough estimate for the relative maximum, followed by an INSM key that computes the derivative of the function, finds the


*For the Learning of Mathematics* **10**, 2 (June 1990)
FLM Publishing Association. Montreal, Quebec, Canada

zero of the derivative close to the INS estimate, and reports back the $x$ and $f(x)$ coordinates for the relative maximum of the function

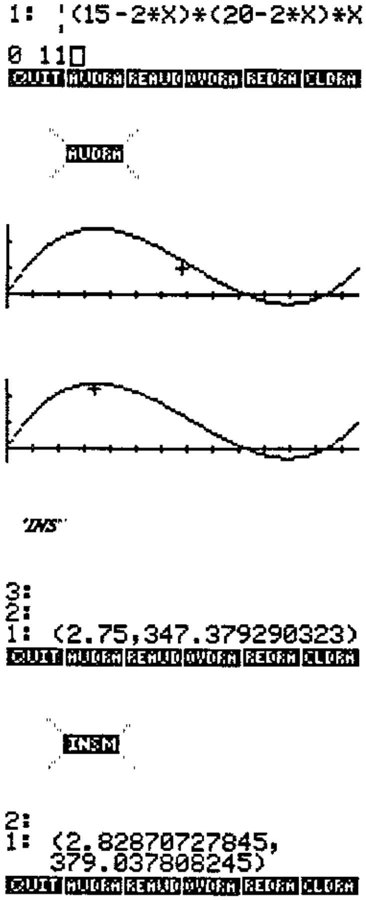



Figure 2
Autodraw and finding a relative maximum

Thus, as Figure 2 illustrates, with such keys on a supercalculator, one simply enters the function, indicates the range of $x$-values of interest, and presses AUDRA for a representative graph of the function Moving the cursor close to the maximum point reveals the maximum is higher than the point (2.75, 247.37), explaining why most first graphs do not usually produce a representative graph (default scaling rarely includes $y$-values larger than 10) One *could* zoom and trace to estimate the maximum with more precision, but the INSM-key gives the relative maximum of the function with full, available precision in a single keystroke

Thoughtful teachers will recognize that student difficulties with topless-box type problems lie with correct interpretation of the problem, representation of the volume of the box as a function of $x$, and deciding how to determine the maximum of the function over the appropriate domain of values for $x$. Contrary to first impressions, the availability of a supercalculator does not trivialize such a problem for students, it only trivializes the computations.

Many teachers using supercalculators are going to feel students do not seem to know anything The reason for such feelings is that we used to reveal the conceptual part of the problem as hints to students so that students could begin work on the computations. Teachers will discover now they must teach those formerly revealed concepts or find they are only teaching button-pushing. The result of such conceptual emphasis is that thoughtful teachers will find it will take students longer to do fewer problems

If students use such supercalculators to do mathematics, we must redirect their interest from computing derivatives and finding zeros of the resulting derivatives to:

a) the meaning of a representative graph,
b) the relationship between the function graphed and the actual problem,
c) the proof that the result is correct,
d) what to do when such methods produce erroneous solutions or no solutions at all, and
e) how to design other keys for other common problems

**Problem 2 (Tall's favorite function [Tall, 1985, 1990]):** Consider the graph of the following function and determine its behavior:

$$T(x) = \cos x + \sin(100x)/100.$$

Figure 3 and Figure 4 illustrate one possible sequence of investigatory moves a student might consider, using a supercalculator

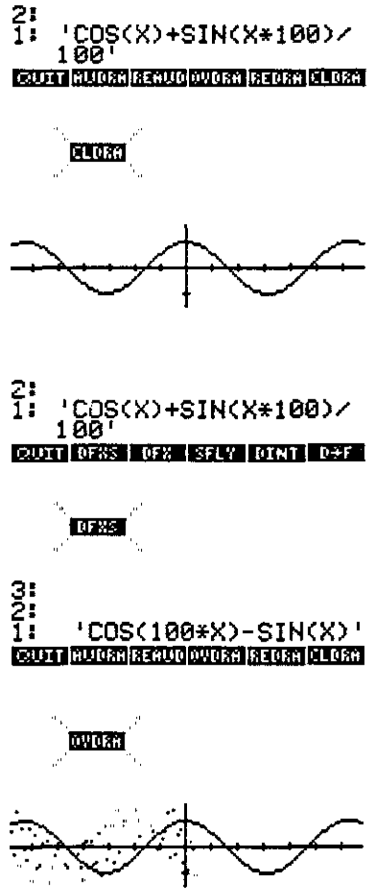
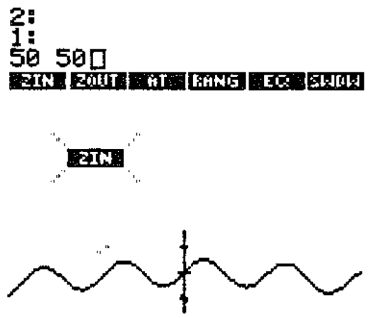


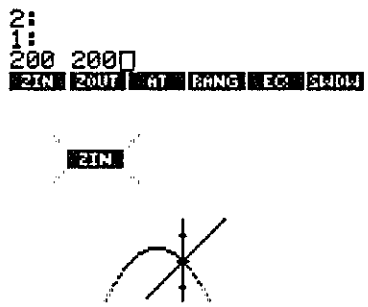

Figure 3
Graphing a function and symbolically computed derivative

First, in Figure 3, the student graphs the function using the clear/draw key, CLDRA, finds the derivative with the differentiate and simplify key, DFXS, and graphs both functions on the same display with the overdraw key, OVDRA, stopping before the complete graph of the derivative, the random looking dots, has been completed The calculator does not connect plotted points with lines so the derivative looks a little strange, but the graph is an honest representation for the domain and range specified (some calculators erroneously connect such plotted dots and imply incorrect graphs)

The next step, shown in Figure 4, was to zoom in by a factor of 50 on both dimensions at the point (0, 1). The result is a graph of $T(x)$ and $T'(x)$ in a range representing about three pixels from the previous graph. The function, $T(x)$, reveals a different nature now, with a slope of one at a point where the first graph in Figure 3 had led one to believe the slope would be zero Regrettably, there are only a few dots representing $T'(x)$, so the student orders another zoom, this time by a factor of 200 on each dimension. The result is shown as the second graph in Figure 4, representing about one pixel of the previous graph given in Figure 4 The second graph in Figure 4 reveals the local straightness of $T(x)$ and the nature of the now continuous appearing derivative, $T'(x)$. Whether or not one has produced a faithful, representative graph of a function is a critical issue for the study and use of such graphs




**RANGE (– .136, .97)**
**(.136, 1.032)**




**RANGE (–.00068, 99985)**
**(.00068, 1.00016)**

Figure 4
Zooming to explore function and derivative at a point

In this sequence, we have an opportunity to illustrate that first graphs may be deceiving, but that with a total magnification of a factor of ten thousand, a representative graph of both the function and the derivative is possible Functions which are differentiable eventually magnify to smooth curves and, finally, straight lines It is this *local straightness* [see Tall, 1990] that defines a differentiable function

What questions might you expect from students regarding such functions, T and T′? There are many, but let's take one: "Why does the T′ have more bumps than T? I thought differentiation smoothed-out a curve, yet the final graph of Figure 4 shows T′ has more "curviness" than T Why?" What have we done here? We seem to have opened a Pandora's Box Do we really want to explore these issues? I hope the answer is yes, but are you ready and do you have the time?

How should students use supercalculators to study such functions? It seems the focus would not be on computing derivatives, but rather on the geometric interpretations of continuous and differentiable functions and their relationship to symbolic definitions of these fundamental concepts of continuity and differentiability If local straightness is a graphical representation of differentiability, what is a graphical representation of continuity (Tall suggest flatness when stretching on the $x$-axis alone [Tall, 1990])?

**Problem 3 (Bundy's favorite problem [Bundy, 1975, p. 12, as related in Rissland, 1985]):** Solve the following equation:

$$\ln(x+1) + \ln(x-1) = 3$$

Using a supercalculator, the problem could be investigated symbolically, numerically, or graphically Originally, substantial focus was on the algebraic manipulations necessary to solve the problem. Today our interest might be redirected to other considerations Consider the following supercalculator possibilities:

*Symbolic method:*

$$\ln(x+1)+\ln(x-1) = 3$$

***EXP,***

***SFLY:***

$$-1+x^2 = 20.0855369232$$

***QUDX:***

$$\pm 4.59189905412$$

**Only + of ± appropriate**

The student enters the equation, presses the exponential function key, EXP, which automatically acts on an equation by taking the exponential of both sides of the equation Seeking simplification, the student uses the simplification key, SFLY [Wickes, 1988], to obtain a quadratic equation The quadratic equation solving key, QUDX, yields two roots, found algebraically. Only the positive root yields a solution. Since the symbolic computations are easy to carry out, one would want students to investigate the extra root and try to determine which symbolic manipulation produced the extra, "incorrect" solution, as well as think about how one decides to apply EXP to both sides of the equation.

*Numerical method:*

**ln(x+1)+ln(x−1) = 3**

***SOLVR,***

***X = 1***

**4.59189905411**

**Just to check meaning of negative root found symbolically:**

***X = −4.59189905411***

***LEFT=***

**(3, 6.28318530718)**

The numerical strategy uses the root-finding routine, SOLVR, which accepts an initial guess of $x = 1$, where the function on the left side is actually undefined! The solution is immediate and requires no algebraic manipulations. The student checks the result of substituting the negative root found in the algebraic method and finds the result is the complex number $3 = 2\pi i$, rather than 3.

*Graphical method:*

**ln(x+1)+ln(x−1) = 3**

***−10 10 AUDRA***

***3 OVDRA***

***INS***

***INSX***

**(4.59189905413, 3)**

The graphical method has the advantage of combining graphical and numerical methods to provide a geometrical representation of the left- and right-sides, graphed as functions, and the intersection of the two curves using the INSX key.

In examining these potential symbolic, numeric, and graphic approaches to solving the equation, $\ln(x + 1) + \ln(x - 1) = 3$, one finds the possibility of drawing student attention, not to the various computations, but rather to an investigation of what the function is representing, perhaps a question about why one might want to solve such equations, the effects of EXP on logarithmic functions, the potential complex-valued logarithmic functions that apparently exist, how one determines the appropriate methods for solving various equations, and finally, how does one prove the solutions are correct and that no other solutions are possible. It appears the computations are trivial, but again, conceptual issues abound. One fear is that, instead of treating these conceptual issues, our solution will be to make problems more computationally complex. Then the computations will be difficult, even with supercalculators, and one won't have to face the conceptual questions because everyone will want help with computations. That's what we've been doing for years isn't it? My hope is that we will have the strength to focus on conceptual issues and not simply make the computations more difficult.

**Problem 4 (Nobody's favorite problem):** Solve the following system of equations:

$$\begin{aligned} 2x + 3y + 4w + z &= 7 \\ x - 2y + 5w - z &= 2 \\ x + y + w + z &= 1 \\ 7x + y - 3w - z &= 5 \end{aligned}$$

Using matrices:

$$\begin{bmatrix} 7 \\ 2 \\ 1 \\ 5 \end{bmatrix} \qquad \begin{bmatrix} 2 & 3 & 4 & 1 \\ 1 & -2 & 5 & -1 \\ 1 & 1 & 1 & 1 \\ 7 & 1 & -3 & -1 \end{bmatrix}$$

***÷, ARRY−>, DROP***

**.461538461541**

**1.79999999999**

**.646153846152**

**−1.90769230768**

**or, ÷, *VD−>F***

**'6/13'**

**'9/5'**

**'42/65'**

**'−124/65'**

**For fractions, otherwise,**

***2 FIX***

**0.46 1.80 0.65 −1.91**

Consequently, using matrices one enters the constant vector, the coefficient matrix, and presses divide. Decimal solutions may be changed to fractions using the vector decimal to fractions key, VD→F, or 2 FIX to produce rounded decimals. It is clear that substantial computational time is saved by supercalculators.

Would you talk about anything here? Some would want to discuss the appearance of 65 in the fractions. Others might wish to discuss the original problem from which these equations were derived. Why should all the coefficients be integers? What happens to errors when there are errors to be accounted for in each of the coefficients? The reader can think of many other possibilities. Should such question be discussed? The problem is pretty boring if you do not do so, but do we know how and do we have time? What about singular matrices that can be "jiggled" to produce solutions? What mathematics can profitably be represented by matrix notation? With practical matrix manipulation so readily available it is clear it is now feasible to consider such representations

**Problem 5 (Polynomials):** Find the algebraic roots to the equation:

$$4x^4 - 5x^3 + 3x^2 + 7x - 5 = 0.$$

To solve algebraically a polynomial of degree 1 to 4 on a supercalculator, simply enter the list of coefficients and apply PROOT [Hewlett Packard, 1988] followed by 4 NRND if you wish rounded solutions, as follows:

**{ 4 -5 3 7 -5 }**

***PROOT***

***4, NRND***

**(0.81,-1.15)**

**(0.81,1.15)**

**(0.63,0.00)**

**(-1.00,0.00)**

**(solved with radicals)**

There are two complex roots, two real roots, and all were found algebraically using radicals. Abel's theorem and Galois theory tell us that such algebraic solutions will be possible only for general polynomials of degree less than 5 Numerical solutions will be necessary for higher degree polynomials. Symbolic polynomial manipulations such as the following are also possible:

**{ 4 -5 3 7 -5 }**

**{ 1 1 }**

***PDIV***

**{ 4 -9 12 -5 }**

**{ 0 }**

**{ 1 1 }**

**or**

**{ 4 -9 12 -5 }**

**{ 1 2 3 }**

***PMUL***

**{ 4 -1 6 -8 26 -15 }**

The key PDIV produces the quotient, the remainder, and the dividend, while the PMUL key simply multiplies polynomials all in list format [Hewlett Packard, 1988].

Some student will observe that fifth-degree equations must have a real root, find it numerically, divide by the appropriate term and produce a fourth-degree equation that can be solved symbolically Have they extended Abel's theorem? Do we want to talk about it? I hope so. Imagine the other questions that could be asked

**Problem 6 (Elementary calculus):** Find the volume of a solid formed by rotation of $y = x^3$ about the $x$-axis from $x = 0$ to $x = 1$.

On a supercalculator, one uses the volume of rotation about the $x$-axis key, VRX, as follows:

**'X^3' 0 1**

***VRX***

**'π*X^(3*2)'**

**{X 0.00 1.00}**

**0.45**

***D->Fπ***

**'1/7*π'**

One enters the function, the lower and upper limits, and presses VRX The calculator responds with the symbolic function that should be integrated, gives the variable of integration and limits in a list, and provides the numerical approximation to the integral, accurate to 4 decimal places, but rounded off to 2 decimals Suspecting a multiple of $\pi$ for an answer, the decimal to fraction times $\pi$ key, D→F$\pi$ [Tucker, 1989] is pressed, yielding $\pi/7$. Not only can students compute definite integrals, but the proper integral "setup" is provided as well

Should students write such routines? Sure, if they are going to get asked to do a lot of volume of rotation problems But why do we ask for a lot? Do we need to? Consider the possibility that no conceptual mathematics is being asked of students if a single key can be designed to do the job.

**Programming procedures:** The following two programs illustrate the programming of procedures [Wickes, 1988]:

***SFLY***

```
« COLCT
  « EXPAN
  » OANO
  « COLCT
  » OANO
»
```

```
OANO
«     -> p
      «
         DO DUP p EVAL
         UNTIL DUP ROT
SAME
         END
      »
»
```

The notable properties of these procedures are that they simplify expressions, illustrate procedures calling procedures, and in the case of OANO, illustrate that programs themselves can be inputs to other programs OANO is designed to repeat whatever program is input for *p* until the expression on the stack no longer changes with the repeated application of the input program, *p* Thus, if repeated applications of EXPAN followed by repeated applications of COLCT, both built-in procedures, will simplify an expression, then SFLY successfully simplifies the symbolic expression. The programming language is powerful when variables can take on values such as algebraic expressions and procedures as well as the usual numbers, vectors, matrices, complex numbers, strings, and lists

SFLY is a powerful program, but it does not always work Why? What is intermediate expression swell? What does simplify mean? Is simplify a clearly defined process? What does it mean for programs and expressions to be variables? How can we use such generalizations of the meaning of variable? How do we know we have correct results from such programs?

These examples are designed to suggest mathematics texts that place major emphasis on numeric or symbolic computations with numbers (including complex numbers), graphs, polynomials, vectors, matrices, derivatives, integrals, Taylor series, trigonometric identities, and zeros of functions are measures of supercalculator capabilities, not of student learning. Random samples of exercises from textbooks used for K-14 mathematics in the United States reveal few exercises that should remain with supercalculators in the hands of teachers and students. Perhaps as much as 90% of the exercises and explanations at all grade levels should be removed. Of course, the issue is then, what shall we teach? The issue of the impact of computers on mathematics is not new.

Since the early 1950s, numerical computations, structured programming and symbolic manipulations have been available on computers [Hamming, 1980]. Mathematicians have called for mathematical programming [Kemeny, 1966] and computer mathematics systems [Birkhoff, 1972] in mathematics courses for some time. Today, the addition of user-friendly graphics capabilities and the psychological impact of a hand-held, personal carrier of mathematical ideas (a supercalculator) makes curricular changes mandatory [Steen, 1988]

Pollak, in prophetic articles about calculators and computers [Pollak, 1977, 1982], noted substantial changes were needed in two partial orderings of the curriculum (i.e., those based on mathematical prerequisites and those based on social importance) and consequently fundamental changes were needed in the curriculum in the United States One could argue there are partial orderings from cognitive development that influence the curriculum too, but nevertheless, it is likely massive changes are needed

## Research

Past learning research has made some progress in the supercalculator-related areas of variable, concept development, problem solving, representational systems, and cognitive development. Significant work has been done with computers and the meaning of variable [e.g Chomsky, 1988; Clement, Lochhead, & Soloway, 1982; Dubinsky, Elterman, & Gong, 1988; Küchemann, 1981; Krutetskii, 1976; Oprea, 1988; Shumway, 1989a; Wagner, 1981] with children ranging from age 5 to age 20. However promising this work has been, we need to extend the universe of the *concept of variable* to include variables defined over objects such as: real numbers, complex numbers, strings, vectors, real arrays, complex arrays, lists, global names, local names, programs, algebraic objects, and binary integer numbers. Supercalculators take a unified approach to these objects; calculator operations apply whenever meaningful, and all such objects can be inputs to programs, including programs themselves There are dramatic, mathematical generalizations of the meaning of variable available on supercalculators. Systematic exploration of the development of such generalized concepts of variable is needed As an extension to variable research, Kieran [1988] notes research on the systemic structure of equations reveals many student misconceptions. Can supercalculators, with symbolic manipulations of equations and graphical representations of such manipulations, provide a vehicle for careful study of such misconceptions?

Computer coding and its impact on mathematics learning has been studied and seems to be most related to concept development and problem solving [e.g. Blume & Schoen, 1988; Johnson & Harding, 1979; Suydam, 1986]. Arguments regarding relative merits of computer languages are often made on the basis of structured programming, recursion, global and local variables, graphics, and the ease of naming and writing procedures Supercalculators offer flexibility of programming, graphics, procedures, lists, symbolic manipulation, and recursion. Computer coding on a supercalculator becomes much more procedure-oriented than prior student coding in BASIC and allows programming habits desired by computer scientists and mathematicians. The personalization of the supercalculator is an important psychological factor Computer programs are coded and then executed by a single keystroke Programs become a part of the supercalculator capabilities and are always available. Algorithm design becomes highly personal but also very important for repeated application by the author. Systematic study of the impact of such availability of authored programs for use, modification, and refinement is needed.

Representational systems have gained deserved attention [e.g. Janvier, 1987; Kaput, 1988] and the research is directly applicable to supercalculators since supercalculators provide access to many of the representational systems being studied. Continued efforts in representational systems and investigation on supercalculators is desirable.

Cognitive development research needs to direct some long-term efforts towards study of fundamental concepts of mathematics, their representations, and their development in children using supercalculators as tools for exploring mathematics. The supercalculator advantages for such efforts are low cost, portability, personalization, storage capability, and generalized mathematical power.

Teaching experiments and clinical studies exploring supercalculator representations of important concepts of mathematics rarely studied with young subjects (ages 3-20) are needed. Research has begun with efforts such as Dick's project to revise and test calculus materials designed for students using supercalculators building on prior experiences with younger subjects [Dick & Shaughnessy, 1988] and Michel's year-long teaching experiment with 15 year-old students studying mathematics, physics, and science for 13 hours per week using supercalculators [Michel, 1988]. Significant study of generalized variables, complex variables, matrix representations, differentiation, integration, probability distributions, zeros of functions, Taylor series, computer arithmetic, and theorems such as those of De Moivre, Bolzano, Galois, Euler, Gauss, Cauchy, and Gödel are called for and the concept of proof is considered basic mathematics [Shumway, 1986b]. Estimation concepts must be developed for algebraic computations as well as numeric computations. Further identification and exploration of fundamental mathematics is needed.

Most analyses of the impact of calculators and computers call for a de-emphasis of many traditional computational skills [e.g. Pollak, 1982; Steen, 1988]. Research involving paper-and-pencil skill development associated with graphing, solving simultaneous equations, finding roots of functions (e.g. factoring or simplifying), polynomial arithmetic, differentiation, integration, matrix arithmetic, differential equations, characteristic equations of matrices, and hypothesis testing without the use or knowledge of supercalculators should be terminated. Substantial collections of research efforts in mathematics education have become moot.

Certainly there are important uses of technology in school mathematics that might not be done effectively on supercalculators. Simultaneous presentations of mathematical representations [Kaput, 1986], educationally designed function graphers [Tall, 1989], microworlds [Thompson, 1985], supposers [Schwartz & Yerushalmy, 1987], computer proof writing [Devitt, 1988], three-dimensional looking interactive graphics [Kuhn, 1988], spread sheets [Arganbright, 1984], and complex symbol manipulations may be some examples. Such computer uses will have impact on the way mathematics is learned but are unlikely, until readily available, to have the impact one might expect of supercalculators.

## Curriculum

Require supercalculators for *all* mathematics? A first sweep at philosophical analysis suggests:

*Concepts and proofs may now be basic mathematics* because, relieved of the computational burdens, conceptual understanding and proofs of the correctness of results are the remaining essential elements of doing mathematics [Steen, 1988; Shumway, 1989b].

*Deeper treatment of fundamental conceptual learning is necessary* for effective use of supercalculators. History suggests, when computational power is increased, mathematical understandings must necessarily increase as well. Mathematical applications without conceptual understanding are likely to be the mindless manipulations of current curricula [Steen, 1988].

*De-emphasis of many forms of skill learning* once thought to be essential for mathematical development seems important and likely. Researchers must test the premise that supercalculator-based computations will produce the number sense and symbolic intuitions needed for deeper study of mathematics [Steen, 1988].

*Substantial, earlier treatment of fundamental concepts of mathematics, if possible, is desirable.* For example, if earlier conceptual understanding of matrix representations of mathematics by children is possible, then such understanding can be used in a supercalculator environment to study such mathematics and its applications. The question unresolved by philosophical research is the question of the possibility of earlier treatment. Experimental research with children and supercalculators is needed.

*Reanalysis of fundamental mathematical concepts and fundamental mathematical thinking must be done.* Prior efforts [Bourbaki, 1950; Lakatos, 1976; Mac Lane, 1986] must be studied and modernized to account for changes in mathematics, societal needs, and supercalculator computational power [Pollak, 1982]. Developmental work [Inhelder & Piaget, 1958] must be repeated for mathematical thinking in the context of supercalculators.

*Best-evidence decisions regarding these issues are needed now.* Researchers are obligated to lead, offer evidence, and help make best-evidence decisions repeatedly. Children in school today will be adults in 1990-2003. The tools available to *do* mathematics today must be the tools used to *learn* mathematics today. Decisions must be made now, and repeatedly revised as new data and philosophical analyses are available.

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