RE-SOLVING THE LEARNING PARADOX: EPISTEMOLOGICAL AND ONTOLOGICAL QUESTIONS FOR RADICAL CONSTRUCTIVISTS

ANDERSON NORTON

Over 30 years ago, a debate between Jean Piaget and Noam Chomsky revived an ancient learning paradox first introduced by Plato (Piattelli-Palmarini, 1980) Chomsky challenged Piaget to explain how one could construct more advanced knowledge from one's existing (less advanced) knowledge Chomsky argued that, without such an explanation for learning, we must assume the innativist position – that all potential cognitive operating is somehow encoded in the human genome

A different paradox motivated Piaget's development of constructivist epistemology. Since Plato, western philosophy had assumed that we could measure the truth of human knowledge by means of its match with an ontological reality; on the other hand, human experience does not permit the test of such a match (Piaget, 1971/1970; von Glasersfeld, 1991) How could we test whether the world we know, epistemologically, matches an ontological reality beyond our experience? "Radical constructivism is a theory whose roots lie in the rejection of illegitimate claims for epistemological certainty" (Confrey, 1994, p. 3).

This article examines the learning paradox from a radical constructivist perspective and in a mathematical context, introducing underlying assumptions and various solutions along the way. I begin by explaining constructivist positions on epistemology and ontology This discussion will distinguish the two branches of philosophy that often get entangled in discussions of the learning paradox. Indeed, we will see that the conflation of epistemology and ontology presents one of two thorny issues arising in previous literature on the topic. I then introduce several attempts to resolve the paradox in mathematical contexts. Many fruitful approaches rely on Peirce's idea of abduction, which warrants its own section Finally, we will consider the ontological paradox that remains once the epistemological one withers away

Constructivist perspectives on epistemology and ontology

Whereas ontology is the branch of metaphysics that examines the nature of being, epistemology deals with theories of knowledge. Radical constructivist epistemology emphasizes the role of experience in gaining "predictability and control over one's environment" (Confrey, 1994, p. 3) People act purposefully within the environments they experience and assimilate new experiences using their available schemes for acting. When perceived results of actions do not fit expected results, those schemes are modified (Piaget, 1971/1970; 1985/1975). Thus, "knowing an object does not mean copying it – it means acting on it" (1971/1970, p 15). von Glasersfeld (1991) further contended, "as long as the models we construct help us to solve the problems that concern us, their ontological status ought not worry us" (p. 23).

von Glasersfeld espoused an epistemology cleanly broken from ontology. He did not deny a reality external to human experience but recognized that we have no way of knowing it In fact, constructivist epistemology depends upon some external world in which we act to establish experience, but it posits that even time and space are constructed through such experience "Hence, whatever 'ontic' reality might be like, it makes no sense to think of it as anything that we could recognize as structure" (pp 114–115) Kant used "noumenal realm" to distinguish this reality from the "phenomenal" reality we experience, and the conflation of the two has contributed to confusion between epistemology and ontology (Campbell, 2002).

For radical constructivists, the distinction between noumenal and phenomenal realms establishes a clear demarcation between knowledge and belief Knowledge – the realm of mathematics, and science in general – is rooted in experiences that provide tests of our constructions. Metaphysical belief must be accepted on faith Although beliefs and knowledge may inform one another, an individual has no way to test her or his belief about any universe except the one he or she has constructed through personal experience (including social interactions) and through abstractions from those experiences.

In their rebukes of constructivism, Kilpatrick (1987) and Olssen (1996) argued against constructivism as anti-realist As such, they critiqued constructivism for a presumed ontological position rather than an epistemological one. In her defense of constructivism, Noddings (1990) claimed that constructivism breaks with epistemology, referring to constructivism as "post-epistemological" (p 20), but it is more accurate to say that constructivism breaks with ontology. Constructivism certainly deals with "the nature, sources, and limits of knowledge" (Wikipedia, 2008, under 'Epistemology') but does not make claims or counter-claims about metaphysics or any other mind-independent reality. It is no more realist or anti-realist than it is Catholic or anti-Catholic. Olssen (1996) hinted at this point when he quipped, "Even if we accept that constructivism does not deny the existence of a real world, it is a claim that must be accepted on faith for, given the constructivists' views on knowledge, it is a claim we can never know" (p. 287). But not denying the existence of a real world is no claim at all. Besides, the *existence* of a real world is not the issue but, rather, what do we know about it? On the other hand, claims that we know a mind-independent reality must be accepted on faith, because we can never know that we know it; hence, the ongoing debate.

Kilpatrick (1987) claimed "constructivism needs to come to terms with mathematical realism" (p. 20), but reflecting on this history of science, we can understand an aversion to considering the mind-independent existence of mathematical objects. After all, where would these objects exist? In the mind of God? Embodied in the fabric of the universe? Kilpatrick (1987) cited Gardner for an example implying the latter: "I have yet to meet a mathematician willing to say that if the human race ceased to exist the moon would no longer be spherical" (p. 20). But writing in the 1980s, neither author had opportunity to consider the consequences of modern physics, such as string theory, which implies the moon is certainly not spherical (not even approximately so) but presumably some higher-dimensional object (Randall, 2005). The history of science is wrought with such examples (Kuhn, 1995/1957). To glimpse the fallibility of metaphysical claims at any moment in time, we need only consider popularly accepted models of the world and compare them with the cutting-edge models of theoretical scientists. Davis and Hersh (1980) noted a similar phenomenon in mathematics; namely, the lack of a home for mathematical objects, independent of mind and faith, compels mathematical realists to retreat to formalism when pressed.

The learning paradox applied to mathematics and radical constructivism

Five years following the debate between Piaget and Chomsky, Fodor (1980) described the learning paradox as follows:

> Let's assume, once again, that learning is a matter of inductive inference, that is, a process of hypothesis formation and confirmation. ... It is *never* possible to learn a richer logic on the basis of a weaker logic, if what you mean by learning is hypothesis formation and confirmation. Yet I say again that learning must be nondemonstrative inference; there is nothing else for it to be. And the only model of nondemonstrative inference that has ever been proposed anywhere by anyone is hypothesis formation and confirmation. (p. 148)

Note that Fodor described learning, and hypothesis formation, as an inductive process. He seems to have overlooked Peirce's theory of abduction as an alternative (addressed in a separate section), even though Chomsky specifically referred to this method of inference earlier in the debate (Piattelli-Palmarini, 1980, p. 52). The assumption that hypothesis formation occurs through induction constitutes one of two flaws in applying the learning paradox to mathematical development and radical constructivism.

Since Plato, two forms of inference have dominated western philosophy: induction and deduction. Induction has been attributed with the creative power of producing conjectures and deduction attributed with the power of verifying them. If indeed all learning were to occur through induction, we would have to assume the innativist position. What Fodor unveiled was that, in fact, induction has no creative power at all (von Glasersfeld, 2000).

The second flaw involves a conflation of epistemology and ontology. In addition to the learning paradox, which concerns epistemology, constructivists still face an ontological paradox. Namely, if scientists construct their knowledge from personal experiences and abstractions from those experiences, how can we explain scientists' uncanny ability to formulate theories that not only explain observed phenomena but successfully predict previously unobserved phenomena? I will examine this question separately in the final section of the paper.

Approaches to resolving the learning paradox

Over the past few decades, several researchers have proposed solutions to the learning paradox (Abrahamson, in press; Bereiter, 1985; Hoffmann, 2003; Simon, Tzur, Heinz, & Kinzel, 2004; Steffe, 1991; von Glasersfeld, 2000). Others have articulated socio-cultural (*e.g.*, Neuman, 2001) or social constructivist (*e.g.*, Cobb, Yackel, & Wood, 1992) perspectives from which the paradox is mitigated or ceases to exist. Examining those perspectives goes beyond the scope of this paper. Here, I am concerned with questions for radical constructivists (von Glasersfeld, 1995), especially the following working definition of the learning paradox: How can we explain a student's development from less powerful ways of operating to more powerful ways of operating? I elaborate on a few related approaches as they apply to the context of students' fractions learning.

Noting Piaget's work, Bereiter (1985) suggested "chance plus selection" provides a possible path to resolving the paradox (p. 208). This solution relies on the repeatability of fortuitous actions. Simon *et al.* (2004) have described similar a phenomenon in terms of *reflection on activity-effect relationships,* which they refer to as "an elaboration of [Piaget's] reflective abstraction" (p. 324). The authors claim that when students reflect on the effects of their goal-directed activity, they can abstract relationships between goals, actions, and effects. The authors use the construct to address the learning paradox with the following example.

A 9-year-old student named "Micki" had been asked to compare the sizes of pieces from two identical squares that had been divided in half in two different ways: one square was divided into two equal pieces vertically, the other square was divided into two equal pieces diagonally. Micki provided a response typical of many fourth and fifth grade students – that pieces from one of the squares (in Micki's case, the diagonally-divided one) would be bigger than pieces from the other. Although such students may understand that each partition produces equal pieces (namely, halves of their respective wholes) and that the two wholes have the same area, they do not yet understand that the partitions produce "a new unit of quantity that has a specific size relative to the original unit" (p. 316). The authors go

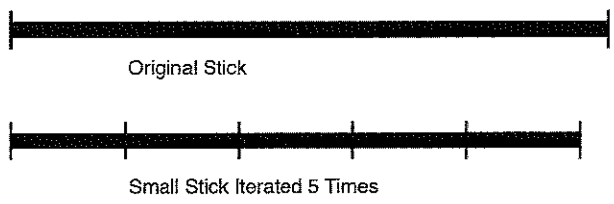

Figure 1 Results from Micki's action of iterating the first cut (adapted from Simon et al, 2004)

on to illustrate how Micki developed this more advanced conception through reflection on activity-effect relations.

Understanding Micki's existing conceptions of fractions and her available mental actions, the second author (Tzur) asked Micki to cut a given stick into five equal parts Tzur and Micki had been working in a computer-based fractions environment that enabled Micki to manually designate a cut from the original stick and repeat the resulting piece five times, as illustrated in Figure 1 Noticing the effect of her actions, Micki determined that the cut was too small and began making adjustments to the cut, repeating the cut five times, and comparing the result to the length of the original stick The authors argue that reflection on her activity and its resulting effects enabled Micki to abstract a conception of unit fractions that included their size relative to the whole

If we were to assume a socio-cultural perspective, we might attribute Micki's learning to the task the teacher posed or the tools Micki used, but radical constructivists assume a more student-centered perspective on cognition and learning: "Among the vast set of sensory motor signals that bombards the organism's senses, only those that can be incorporated into structures and operations already available in the mental system evoke particular cognitive responses" (p 310). Thus, success in teaching using the activity-effect construct depends upon designing tasks that the student can assimilate using available mental activities (operations) In Micki's case, the authors hypothesized that an iterating operation was available and instrumental in her development of a new conception of unit fractions. Tzur played an important role in supporting Micki's learning, by building a model of her ways of operating and designing tasks to evoke new uses of those operations, but the operating and subsequent learning were Micki's alone. On the other hand, we might say that Simon *et al.* (2008) have developed a construct that minimizes the *chance* in "chance plus selection."

We now ask how Micki developed available operations in the first place and what *psychological* processes account for her selection of the iterating operation for use in that particular problem solving situation. Further elaboration on Piaget's reflective abstraction provides an answer to the former question, and Peirce's theory of abduction provides an answer to the latter question

Reflective abstraction

Piaget (1985/1975) has described reflective abstraction as an innate mental operation (one of few innate human abilities Piaget admitted as necessary for the construction of all other knowledge) that enables the individual to reflect on her or his ways of operating and use those operations as material for further operating.

> Reflective abstraction includes two indissociable activities. One is "reflecting" or projecting onto a higher level something borrowed from a lower level . . The other is more or less conscious reflexion in the sense of cognitive reconstruction or reorganization of what is transferred. (p. 29)

Mysteries surrounding reflective abstraction prevented resolution of the learning paradox during the debate between Piaget and Chomsky. During the debate, Chomsky declared, "My uneasiness with reflective abstraction is . . that I do not know what the phrase means, to what processes it refers, or what are its principles" (Piattelli-Palmarini, 1980, p. 323) Understanding that "reflective abstraction must be operationally defined in particular contexts with respect to particular schemes before it has any clear meaning," Steffe (1991, p. 42) demonstrated how reflective abstraction could explain a student's construction of a more powerful number scheme from a less powerful one. He cited a teaching experiment he conducted with a student named Tyrone.

Tyrone began third grade with only a figurative scheme for counting, which meant that he could count collections in the absence of sensory material, but he was unable to count on For example, when told that two cloths hid sub-collections of 8 and 3 items, respectively, Tyrone could imagine the items and count his acts of pointing at these imagined items in order to determine that there were 11 items in the entire collection. However, he could not take the cardinality of either sub-collection for granted and always had to begin counting at 1. In the context of such problem-solving situations, Tyrone began using his unitizing operation in a new way, in order to keep track of his counting activity over the second sub-collection (*e.g.*, 9 is 1; 10 is 2; 11 is 3; 11!). Steffe explained how Tyrone's reflection on his unitizing and counting actions engendered a new counting scheme that included counting on. This reorganization constituted the development of a more advanced way of operating from a less advanced one, through reflective abstraction

To address the question of how students select particular operations to use in novel problem solving situations, we now turn to Peirce's theory of abduction. In fact, both Steffe (1991) and von Glasersfeld (2000) have suggested abduction, in addition to scheme theory, as a potential "key to the learning paradox," and Hoffmann (2003) has demonstrated the fruitfulness of such an approach. In particular, Hoffmann showed that abduction played a crucial role in the formation of new ideas through diagrammatic reasoning Though a full discussion of diagrammatic reasoning reaches beyond the scope of this paper, I note that Hoffmann eloquently addressed the same faulty assumptions addressed here, though from a non-constructivist perspective

The role of abduction

A late 19th-century logician (and the father of pragmatism), Charles Peirce recognized "man's truth is never absolute because the basis of fact is hypothesis" (Peirce, 1982, p. 7). This recognition focused his attention on the creative power of hypothesis formation, previously attributed to induction. As Fischer noted, Peirce's great insight was that "a valid induction already presupposes as a hypothesis the law or general rule it is supposed to infer in the first place" (2001, p. 366). Or, as Peirce himself put it, "induction makes its start from a hypothesis" (1998, p. 106). Thus, "abduction as the only knowledge-generating mechanism must needs become the central focus of epistemological discussion" (Fischer, 2001, p. 368).

Peirce described abduction as a kind of reverse deduction, in which one adopts a general rule in order to explain a surprising observation. For example, suppose that after computing the sine of 67 degrees, your calculator displays a negative answer. You might jump to the conclusion that your calculator was set to compute radians rather than degrees. That conclusion may be reasonable but does not deductively follow from the surprising observation of the negative answer; on the contrary, the negative answer deductively follows from the assumption about degrees and radians. Jumping to conclusions invalidates abduction as a strictly logical inference, although abductions can achieve logical coherence once they are tested via induction or deduction. As such, "all our new ideas, and all additional knowledge, are the results of violations of established laws of thought" (Fischer, 2001, p. 374).

As with reflections on activity-effect relations, the role of abduction in the learning paradox fits Bereiter's idea of chance plus selection. The abducted idea, once affirmed, becomes part of a larger cognitive system. "If abductively constructed hypotheses are corroborated inductively then the experience-prior rules (logic) of a conceptual system are transformed or adjusted - as is the case with all 'scientific revolutions'" (Fischer, 2001, p. 377). von Glasersfeld (2000) also attributed abduction with a central role in resolving the learning paradox, referring to abduction as the "mainspring of creativity" (p. 10). In the next section, I demonstrate how the abduction of mental operations provides a comprehensive response to the learning paradox, explaining both the origins and selection of novel ways of operating. However, to establish the theoretical integrity of such a response, we must reconcile Piaget's constructivist epistemology with Peirce's realist ontology.

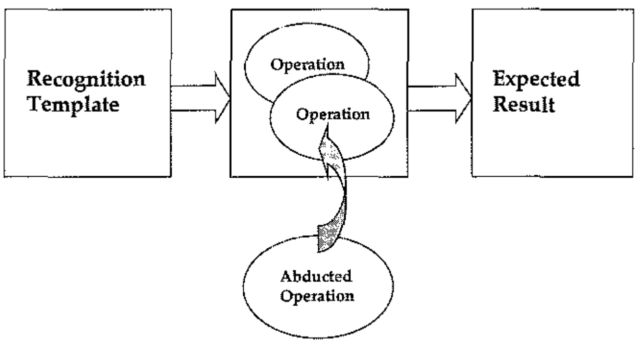

Figure 2. The abduction of an operation into a scheme

Peirce's abduction and Piaget's epistemology

Although Peirce understood that all human knowledge remains hypothetical, he held a realist ontological position that claims human knowledge tends toward an absolute truth. Hoffmann (2003) referred to this ontology as "evolutionary realism" (p. 121), and Radford (2008) determined this realism could not align with Piaget's constructivism because "the relations to be discovered are conceived of as *preceding* the semiotic experience" (p. 10).

Radford's point underscores the ontological paradox that we will discuss at the end of the paper - one that intrigued Piaget, but need not bother realists such as Peirce. However, I have argued earlier in this paper that there is no incompatibility between Piaget's epistemology and any particular ontological position. In any case, we can justify applying Peirce's theory of abduction within Piaget's epistemology for two key reasons. First, abduction can be understood as a pattern of reasoning that relies on Peirce's particular brand of realism no more than induction relies on Aristotle's particular brand of Platonism [1]; in fact, Peirce credits Aristotle as the author of abduction (Peirce, 1998). Second, several constructivist researchers have already used abduction in their explanations of learning (*e.g.* Cifarelli & Saenz-Ludlow, 1996; Fischer, 2001; Steffe, 1991; von Glasersfeld, 2000).

Following his analysis of children's processes of generalization and problem solving, Rivera (2008) concluded that, "learners simply cannot perform an operation on an object without first assuming an abductive claim" (p. 24). This view aligns with the constructivist perspective that "knowing an object means acting on it" (Piaget, 1971/1970, p. 15). The next section elaborates on this idea by exemplifying students' "operational conjectures."

The abduction of mental operations

In a previous paper, I have described a student's operational conjectures in the context of learning fractions (Norton, 2008). Operational conjectures involve the abduction of operations into an existing structure for operating - a scheme. The three-part structure of a scheme includes a recognition template (the goals or situations in which a scheme applies), a sequence of operations (mental actions), and an expected result of operating. Figure 2 illustrates this structure and the abduction of an operation into it.

For example, I asked the student, Josh, to name the size of a fraction that had been created by taking a half of a half (see fig. 3). Josh responded, "one third ... if it was even." After we measured the stick to find that it was actually, one-fourth, Josh proceeded to explain:

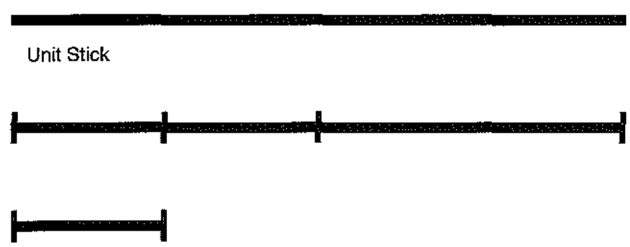

Figure 3. Josh's novel use of partitioning

"Let's see. Because . them two look the same [pointing to the two fourths in the partitioned whole]; you could put one more [partition] in there [pointing to the middle of the right half]." Josh's statement indicates an operational conjecture because he acted on the stick in a novel and uncertain way. He had a part-whole scheme available, which enabled him to name fractions based on the number of visible parts in the fraction and the number of visible parts in the whole. Using his part-whole scheme, he knew that one-fourth meant that the piece in question should be one out of four equal pieces making up the unit stick, but there were only three visible parts. His operational conjecture was to use his partitioning operation to create the fourth equal part in the whole from the three unequal parts, which explained the surprising measure of the piece (one-fourth).

This pattern fits the general pattern of operational conjectures illustrated in Figure 2 and demonstrates how this particular operational conjecture advanced Josh's fractions conceptions from that of a part-whole scheme to a more advanced fractional scheme. Whereas before Josh named fractions by comparing the numbers of visible partitions in the fraction and the whole, he began to create equal parts by partitioning the stick himself. In other words, the abduction of Josh's partitioning operation within his part-whole scheme enabled him to work more powerfully with fractions. Moreover, Josh's use of partitioning contributed to a novel use of iterating as well (Norton, 2008) [2]

As a potential resolution to the learning paradox, operational conjectures would remain limited to students' available operations if not for reflective abstraction. Cognitive growth is not limited to combining existing operations in all possible ways, because schemes of operating continue to grow through operating - through the self-regulation of the system (Maturana, 1988; Piaget, 1971/1970; 1985/1975). Furthermore, humans construct new operations through abstractions from physical action. Researchers of embodied cognition have made particularly convincing cases for such abstractions, in terms of metaphors (*e.g.*, Abrahamson, in press; Anderson, 2003; Núñez, Edwards, & Matos, 1999)

The paradox that remains for constructivists

Besides the epistemological one, I have described two ontological paradoxes: the dilemma of explaining how knowledge could be measured against a world beyond human experience; and the dilemma of explaining the predictive power of mathematics and science. The first paradox motivated Piaget and his articulation of constructivist epistemology. The second paradox has motivated realists like Peirce and Hilary Putnam (1975) who argued, "realism is the only philosophy that doesn't make the success of science or mathematics into a miracle" (p. 50). Several researchers have articulated the dilemma, including Piaget, Peirce, and Einstein

> How is it that man ever came by any correct theories about nature? We know by induction that man has correct theories, for they produce predictions that are fulfilled. (Peirce, as cited by Harrowitz, 1983, p. 196)

> Einstein said on one occasion that scientific theories were free creations of the human mind. .. Einstein also said that what marveled him was that, even though scientific theories were free creations of the human mind, they could be used to explain the world. (Maturana, 1988, p. 38)

Maturana responded, "In fact, scientific explanations do not explain an independent world, they explain the experience of the observer, and that is the world that he or she lives" (p. 38). However, the power of prediction within one's lived world should still marvel us, and these marvelous predictions apply as much to mathematics as (any other) science. For example, it was Newton's mathematical equation for "universal gravitation" that accurately predicted the Earth's gravitational pull on the moon. And yet Einstein later refuted the universal truth of that otherwise useful equation

Peirce too marveled at predictive power. This power, no doubt, motivated his evolutionary realism: "an accord between mind and course of events [that] is more evolutionistic than rationalistic" (Eco, 1983, p. 218). Should constructivists - as Kilpatrick (1987) has suggested - embrace some form of realism as well?

Realism provides an alternative to accepting the idea that, "truth comes down to an act of faith" (Olssen, 1996), but from the constructivist perspective presented here, faith permeates daily life as well as religion. Even mundane acts such as placing a glass on a table depend upon faith that our previous experiences with glasses and tables will hold true. Eco (1983) has described the role faith played in the less-mundane acts of Sherlock Holmes' "deductions"; he more accurately referred to these inferential leaps of faith as *meta-abductions*. Beyond the kind of abduction previously described, meta-abductions remain untested but accepted with confidence. Perhaps Piaget provided the best possible explanation for the uncanny success of such abductions:

> The steady agreement between physical reality and the mathematical theories employed in its description is of itself amazing, since the mathematics so often antedates its physical application, and even when the mathematical apparatus is devised to fit certain newly found facts, it is nevertheless never derived from these facts but constructed as a deductively elaborated match for them .... It is a correspondence of human operations with those of object-operators, a harmony, then, between this particular operator - the human being as a body and mind - and the innumerable operators in nature - physical objects at their several levels. (Piaget, 1970/1968, pp. 40-41)

This explanation should resonate with researchers of embodied cognition and radical constructivists alike. As with operational conjecture, it attributes the power of mathematical knowledge generation to mental action. Furthermore, it emphasizes the dependence of the developing mind on the environment (social or otherwise, but always including the body) in which it grows. On the other hand, there need be no criterion of match - not even approximate match - between creations of the mind and a mind-independent reality. The only necessary criterion is usefulness in predicting perceived outcomes from action.

Notes

[1] Aristotelianism grew out of Platonism, and discussions persist over whether the former ontology should be considered a form of the latter one.
[2] The abducted operation actually identified in the study was "splitting," which is defined as a composition of partitioning and iterating

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Round trip

If you drive to the top of a hill at an average speed of 50 km/h, what must your average speed be on the drive down in order for the average speed of the round trip to be 100 km/h?

(unknown origin; selected by Peter Liljedahl)
