Two Comments on "Didactical Phenomenology"*

* Hans Freudenthal celebrated his 80th birthday in October 1985. We publish two responses to his most recent book as a gesture in recognition of the gift from a distinguished mathematician to those who have chosen to study and practise the "educational task."

H. Freudenthal. *Didactical phenomenology of mathematical structures* Dordrecht, Holland: Reidel, 1983.

ALAN BELL

This is a valuable and important book, but the price will restrict its circulation to wealthy people or institutions. In style, too, it is not a book for everyone. But it is to be hoped that some people will be found to work on turning its ideas into classroom realities. For me, it expresses exactly (or nearly so) the right approach to mathematics for learners, that is, the development and codification of relational aspects of our physical and mental environment, the symbolisation of them then the manipulation of them, to the end of gaining deeper insights.

It is a unique book. On the epistemological dimension, it stands midway between books like Newman's *The world of mathematics*, and the work of Piaget. The former book aims to put mathematics into perspective by showing how it provides the means of solving problems in particular fields; and it draws attention to its logical, representational and symbolic processes. Piaget, on the other hand, exposes a number of key points in the growth of people's mathematical awareness He makes no claim to mathematical comprehensiveness — indeed, he is using mathematical and scientific material to illustrate a theory of the development of reasoning.

Freud nthal's book gets closer to mathematics in the making (and the remaking through which learners pass), but treats the human experience in a generalised way. There are many detailed ideas, but few actual insights into people's perceptions.

On the didactical dimension, this book stands between the mathematically oriented critiques of the curriculum, and the now burgeoning research on mathematical understanding, with its mapping of normal conceptual development and common misconceptions, derived from extensive and careful study of pupils

Freudenthal's didactical phenomenology draws from his knowledge of mathematics, its applications and its history; from an analysis of textbooks to distill the wisdom of previous generations of teachers regarding how the learning of particular topics may be helped; and from his own observations of children's learning. His aim is to teach not for "concept attainment", (for example, by choosing important concepts and finding suitable embodiments through which to teach them), but to aim at the "constitution of mental objects". This involves the drawing together of all the aspects of the concept and its context which are actually associated with the concept and its applications. For example, the apprehension of counting number includes the direct recognition of the number of small sets of objects (say up to 5), the recognition of the invariance of the numerosity of a set over change of time, change of place, over "shaking transformations", over the splitting of a set and the rejoining of its parts; it involves the awareness that if a set covers more space, but less densely, the number may be preserved; and that the things counted may be physical objects, systematically arranged or not, rhythmically repeated sounds, mental "steps" or markers, and so on. Thus one may construct a learning situation by reversing the usualy mathematical movement towards abstraction, by re-embedding the bare mathematical ideas in contexts again.

The book starts quite appropriately with a chapter on length, discussing its invariance under transformations of bending and break-and-make, and the relation of addition of measures to juxtaposition of the objects. The Number chapters discuss the range of problem types for addition, subtraction and so on which have been developed by Vergnaud and Carpenter and their colleagues.

In discussing algebra, or, rather, "the algebraic language", Freudenthal starts by noting different degrees of structure and of formality in the various languages we use. He notes linguistic transformations which even young children learn to apply — changing you to I, with the corresponding change in the verb, negating, changes of perspective such as "are you over there?", "yes, I'm here". (Some writers have taken this to imply that very young children possess algebra; but this needs to be qualified by the recognition that the act of transformation is unconscious I would say that the mathematical element enters when the action becomes conscious, when the change is recognised as similar to others.)

Natural or verbal language has a number of syntactical rules — sentences start and finish with conventional signs and contain one verb. The everyday algebra of generalised arithmetic has rather more formal rules, and the almost entirely formal language of the foundations of mathematics consists largely of formal rules; in this case verification of truth is independent of the meaning of any concepts which the symbols might denote. The tighter, more formal, structure of algebraic language has didactic implications; children used to the looseness and redundancy of natural language are often not aware that if an algebraic expression

For the Learning of Mathematics 6, 1 (February 1986)
FLM Publishing Association, Montreal, Quebec, Canada

is to convey unambiguous meaning to a reader, certain details are essential (for example, brackets).

The distinctions among names denoting particular objects (known, and unknown), names denoting indeterminate members of a set, and names denoting variables, which are now familiar in mathematics, have their counterparts in natural language: compare "the mouse is a rodent", "we have mice in the house," "which mouse is the fattest?". The symbols used for these meanings have also evolved historically from words; (Babylonian methods for solving equations were given as verbal descriptions of the solution of paradigmatic numerical examples), through conventional terms (like the Italian "cosa") through to letters.

Freudenthal points also to the importance of formal substitution in algebra, whereby new relations, either more elaborate or simpler, can be made from old ones, and to a variety of characteristic algebraic strategies and tactics, such as concept extension with a principle of preservation of algebraic properties (e.g. the system of positive and negative integers), the gaining of insight by translating from words into symbols (or vice versa) and the development of algorithms for making chosen transformations, as in the solution of equations. These are worthwhile comments, and they touch on several aspects of pupils' understanding which have been studied in recent years — note for example Lesley Booth's concretisation of the notion of an algebraic expression as a "machine program" They still need further work to turn them into coherent approaches to the curriculum

In his long chapter on the Function concept, Freudenthal traces its origin in the awareness (1) of change — in time, position, moons, wind, temperature, aims, length of day and so on, and (2) of dependence, first qualitative, then crudely quantitatively, then more precise — the more the merrier, seesaws one side up, other side down, I gain marbles, you lose some. Usually one quantity is seen as changing independently, the other depending on it.

The set-theory definition of a function as a special kind of relation is mentioned, and rightly exposed as misleading from a didactical-phenomenological standpoint — indeed, it may be said to base the definition of function on its *least* characteristic properties. On the other hand, the freeing during the 18th and early 19th centuries of the notion of function from analytic expressions, as became necessary to deal with Fourier series and with solutions of the differential equations of vibrating strings, was a genuine clarification of the essence of the function concept.

Freudenthal gives three illustrations to show differing degrees of operationality of the function concept — the examples in which the function *idea* and not just the notation becomes necessary. The first of these comments that the notation $C(M,r)$ for a circle, centre M, radius r, exhibits a correct but rather empty functionality, but if M runs along a curve K, and r is fixed, then the functional character becomes clear, and can lead further to give a system of circles with an envelope.

For us, the idea of function is closely related to the Cartesian graphical representation; this is a language to learn, to move fluently between graphical features and the corresponding functional properties. (It might be added that *different* global aspects of the function are displayed in some cases by a "mapping" graph, such as a set of arrows joining points on two parallel lines.)

Geometry is treated at considerable length — eight chapters, in which Freudenthal grapples with the major task of putting this many-faceted field into perspective. The first of these chapters starts with some of the simplest geometrical structures — networks consisting just of nodes and links, like a stylised railway map. The introduction of distance then allows consideration of networks regarded as invariant under affine, projective, similarity and congruence transformations. In the next chapter, taking a new standpoint, we consider the geometry which develops out of our perceptions of physical objects and their properties of flatness, straightness, roundness, symmetry — the familiar properties of Euclid; and in the next chapter again, continuing from the same standpoint, we move into the concepts of topology — continuity, general continuous transformations, connectivity and dimension. The next geometrical chapter tries to capture the essence of "the topographcial context", that is, the concept of three dimensional space, the locating of positions and objects. As Freudenthal puts it, in a typical paragraph, trying to convey meaning by words in a poetical way,

> the catching of space (or of the space) as a mental coexistence of places, that is, of
> places of objects
> places of objects and perceivers
> places of perceivers
> in their mutual physical and mental relations.

Chapter 11 returns to the standpoint from which we started three chapters ago, picking up the study of physical objects but moving into their more abstact properties — plane, direction, perspective, polygons, circles... surfaces of revolution; and then considering the various mappings by which figures and spaces may be transformed. The last two geometrical chapters consider how geometrical knowledge is used to enable measurement of objects to be codified, and position fixing to be algebraised.

In this book Freudenthal has attempted something quite new. He recounts in some detail how he cast around when trying to determine an appropriate style. It is still difficult to read, locally rather than globally, perhaps because many of the points are made obliquely, through remarks or expressions of opinion. As an attempt to trace the origins of important mathematical ideas in our more primitive understandings of the world, it is richly allusive, widely comprehensive, but, inevitably suggestive rather than definitive. In part, it originated as background papers on which the curriculum developers of the IOWO institute could base their search for appropriate teaching situations. As such, it can provide sound and suggestive ideas for our own curriculum construction and search for material.

This book is so much *sui generis* that it tantalises one with the feeling that — yes, this is the intimate link-up between mathematics and experience which should be the basis of our curriculum — but leaving one wondering what kind of assertions these are, and on what basis one might test or challenge them. For me, this is more akin to poetry than to science; one gains vivid flashes of illumination from some remarks, while others say nothing to one, or maybe conflict with one's own apprehension. (For example, for me, the treatment of negative numbers in *Mathematics as an educational task,* based strongly on notions of reflections, did not carry conviction, whereas the story in the present book is quite helpful.) I hope I have conveyed some impressions here which will encourage others to tackle this book and find some of the gold that it contains.

W.M. BROOKES

What matters about Freudenthal's *Didactical phenomenology of mathematical structures* is that someone has had the confidence to assert that mathematics only exists in terms of double structures. It is not the detail of the book that is significant, rich though it often is, but the fact that his analysis is based on the implicit dialectical relationship between the experiencing of mathematics and the asserting of mathematics. There has been increasing recognition that the words "mathematical activity" have been vital in raising teachers' awareness of how mathematics happened. But the power of mathematical symbol forms to become separated from the contexts which gave rise to their invention has continually and effectively acted to devalue the processes through which the forms become acceptable (established, even?) So it becomes a principle of mathematics to know, for instance, the properties and uses of matrices yet seemingly irrelevant to be familiar with, or even conscious of, the complexity of experiencing that led to their development. The history of mathematics is a story of forgetting the creating process once the end product becomes effective in all its operational senses. It is the efficiency of a mathematical object as a decontextualised operator which makes its presence felt. At the primary level of first existence no-one has to be interested in the events of its creation. Its virtue lies in its effectiveness as an operator on the world or as an explainer of the world. The more sophisticated creation of generalised descriptions and justifications which characterises intensive mathematical activity — that is, mathematics generated from itself — is similarly disconnected from the process of invention.

The history of the teaching of mathematics has always had to face a reversal problem. In an essentially natural sense only those mathematical objects and associated skills were to be taught which were of significance. There was absolutely no sense in which anyone needed be taught how to create mathematics, for this would be done only by those who needed to — and they would not necessarily have a teacher to call on. Consequently, experiences were offered to pupils, students and apprentices which, as far as possible, resulted in them being able to operate in the way the mathematics indicated. Those who responded positively clearly possessed mathematical ability; those who didn't were unfortunately lacking. Not until the need for the numbers of people skilled at some level of mathematics started to increase did questions really begin to be asked about process and experience. We have a huge back-log of ignorance to overcome; ignorance of what experiences actually lead to the real control of mathematical phenomena. And before having solved the problems implied by this we have been overtaken by clear needs to teach more people how to create mathematics, how to ask questions, how to sort out ill-defined situations, how to accept a diverse set of directions in which to go. It is not surprising that we are in difficulties. At present most courses in which the control of mathematics objects is the aim rely on hope and the skill of identification. That is, we offer an experience and then find out who has benefitted. It is uncommon to find teachers who can put their hands on their hearts and say that they will guarantee that the students in their class will all be able to hand a particular and appropriate object, whether these are fractions, percentages, matrices, differential equaitons, existence theorems, functors, etc. We tend to know how to recognise success when they achieve it. But that is *post hoc* rather than *propter hoc*.

The attempt to grapple with phenomenology of experiencing which results in mathematical skill needs a strong theory. And first it needs strong-minded people to intitiate matters. It is not that we should have done it before, it is only now that the need has become pressing.

Most of our current insights depend on the implicit rejection of an "ability" theory of people. Where there is a presumption of the diversity of valid responses to experience there is a readiness to observe, react, re-interpret and then reconstruct. There is in these circumstances no sense of denying mathematics or sacrificing its rigour and precision on the altar of personal choice. Rather it is recognised that the facilitating power of the teacher is increased when he knows that his pupils or students are already orientated on some path or other, or are likely to see things this way rather than that, or that today they cannot, tomorrow they can, and so on. The diversity of learning phenomenology is at odds with the single-minded appearance of mathematical phenomenology.

The assertion of the didactical phenomenology of mathematical structures gives us some space to think and breathe. We do not have to identify "the correct method". We can allow the space to expand as we see different ways in which phenomena can be experienced and then work with those differences to expand the number of people who, by virture of such empathy, can approach mathematical objects with confidence. The space begins to allow us a place to stand in order to be more critical of the symbolism which manifests the existence of mathematical phenomena. The study of mistakes (error?) becomes much more productive when it is underpinned by the dialectical relationship between the phenomenology of perception and the phenomenology of mathematics.

Freudenthal's conceptualization is of central importance to our further enlightenment.