Solving Equations: the Transition from Arithmetic to Algebra

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Recent researches have pointed to certain conceptual and/or symbolic changes which mark a difference between arithmetical and algebraic thought in the individual. Some concern the different interpretations of letters [Booth, 1984], the notion of equality [Kieran, 1980, 1981], and graphic or symbolic conventions for coding operations and transformations in the solution of equations [Matz, 1982]. These observations make it feasible to hypothesize certain lines of evolution from arithmetical to algebraic language which correspond to the notions and the forms of representation of the objects and operations involved in the changeover. The changes which the learner has to make to gain access to algebraic language can then be visualized, on each of these lines, as cut-points separating one kind of thought from the other

One of these cuts is particularly interesting for the theme of problem solving: it is suggested by an analysis of the strategies and methods of solving systems of equations found in the pre-symbolic algebra textbooks of the 13th, 14th and 15th centuries [Boncompagni, 1954; Arrighi, 1974; Arrighi, 1970; Hughes, 1981; Benedetto & Arrighi, 1967]. We can observe, for instance, that the solution strategies for equations such as $x^2 + c = 2bx$ and $x^2 = 2bx + c$ are completely different from each other [Hughes, 1981; Filloy & Rojano, 1984]. This difference would not exist if the authors had had recourse to the rule of transposing terms from one side of an equation to the other for, at a syntactic level, the two equations would then be similar. But this facility would already imply an advanced ability to operate on the unknowns in the equations.

The operational insufficiency exhibited at the pre-symbolic stage of algebra suggests the presence of a cut, i.e. a break in the development concerning operations on the unknown, now at the level of individual thought. In our clinical study with 12-13-year-old children, "Operating on the unknown", as part of our more general study, "The acquisition of algebraic language", we show that operating on the unknown appears as an action which is required for the solution of certain first degree equations with at least two appearances of the unknown. To solve these equations it is not enough to invert the operations on the co-efficients. Examples of such equations are

$$38x + 72 = 56x\,; \quad 3x + 20 = x + 164$$

According to our study, the transition from the operational resolution of equations such as $x + 27 = 58$ or $4(x + 11) = 52$ to the resolution of the above equations is not immediate. It is necessary to construct, or acquire, some elements of an algebraic syntax, properly speaking. The construction of these syntactic elements is based on arithmetical knowledge which has worked well up to a certain point, but it must also break with certain arithmetical notions—hence the presence of a cut.

Consider the concept of equation. In arithmetical terms, the left side of the equation corresponds to a sequence of operations performed on numbers (known or unknown); the right side represents the consequence of having performed such operations. This is what we might call the "arithmetical" notion of equality. From such a notion, an equation such as $Ax + B = C$ can be solved by merely undoing, one by one, the operations given in the left hand sequence, starting with the number C. We shall call this type of equation "arithmetical."

The arithmetical notion does not apply to an equation of the form $Ax + B = Cx + D$; its resolution involves operations drawn from outside the domain of arithmetic —that is, operations on the unknown. For these operations to become meaningful to the learner it is necessary, in turn, for such an equation (which we call "non-arithmetical") to be provided with some meaning. This, in *its* turn, implies a change in the concept of equation or of the equality of numbers. With respect to the "meaning" of a non-arithmetical equation, the learner must at least understand that the expressions on both sides of the equals sign are of the same nature (or structure), and that there are actions which give meaning to the equality of the expressions (for instance, the action of substituting a numerical value for the unknown)

Profound changes in arithmetical habits and concepts do not occur spontaneously at the moment the learner is faced with the need for such a change. Children in our study, when faced for the first time with equations of the form $Ax + B = Cx$ approached them with trial-and-error methods, giving no indication that they spontaneously attempted to operate on the terms containing x [Filloy & Rojano, 1984, 1985]. Suitable interventions from a teacher at the point of transition may be crucial for students learning algebra for the first time.

On the other hand, although some modification of the arithmetical notions must take place in order that the new algebraic notions may be acquired, it is also necessary for the arithmetical knowledge to be preserved. Even in the case of the single example we have presented, it is necessary that arithmetical operations continue to be recognized as such so that all the previously acquired operationality in the resolution of equations is preserved What is required is an operational level of knowledge which can be placed between the arithmetical and the algebraic one, i e a level of pre-algebraic knowledge.

The transition stage

As we have said above, the students' conceptions of operations performed on numbers have to change in order that the concept of operating on objects other than numbers (such as unknowns) and the conception of these "new" objects themselves (what they represent, or may come to represent) may be developed. The teaching of algebra at this point must have recourse to suitable didactic means which bring into play all the related elements which are involved in the change

One of two opposed positions about the type of didactic resource to be used proposes *modelling* in some *concrete* context (i e familiar to the students) the new operations and objects so that they become endowed with meaning. Taking this as a starting point, the first elements of an algebraic syntax are constructed on the basis of the behaviour of the model. The contrary didactic position proposes beginning at the syntactic level, learning the appropriate syntactical rules and applying them, later on, to the resolution of equations. The latter is the traditional teaching approach based either on the Viète model (transposition of terms from one side of the equation to the other) or the Eulerian model (operating on both sides of the equation with additive and multiplicative inverses).

If we adopt the first position, we find it necessary to know something about the processes which intervene between the actions performed at the concrete level (i e on the model) and the corresponding elements of algebraic syntax which are obtained from them. These processes, which we call "abstractions from the operations", show certain standard characteristics in the course of their development by individuals, but they can also be greatly influenced by differences between subjects and they also vary with the model

To clarify some of the ideas we are putting forward we go on to illustrate them by reference to our clinical study.

"Operating on the unknown"

Our analysis of texts that antedate Viète's *The analytical art*, together with our own development of experimental teaching sequences, suggested to us the existence of a didactic cut along the child's evolutionary line of thought from arithmetic to algebra This cut corresponds to the major changes that took place in the history of symbolic algebra in connection with the conception of the "unknown" and the possibility of "operating on the unknown"

In terms of the curriculum, the cut is located at the transition between:

("Before") The students know how to solve arithmetical equations of the types

$Ax \pm B = C$
$A(Bx \pm C) = D$
$x/A = B$
$x/A = B/C$

In order to solve these it is sufficient to invert or to "undo" the indicated operations. *It is not necessary to operate on or with the unknown.*

("After") Students have received no instruction on how to solve equations of the types

$Ax \pm B = Cx$
$Ax \pm B = Cx \pm D$

To solve these it is not sufficient to invert the indicated operations. *It is necessary to operate on what is represented*

The population involved in our study consisted of three consecutive classes in the second year of an experimental secondary school in Mexico City. A written "pre-algebra" test was given to all the students containing three types of questions: arithmetical equations with letters (e.g. $5x + 3 = 90$), arithmetical equations without letters (e g. $\square - 95 = 23$), and word problems leading to simple arithmetical equations. Children whose scores on all three parts of the test were comparable, falling into either the bottom third, middle third or top third of possible scores, were selected for interview and the interviews videotaped

During the interviews the students were presented with sequences of five or more items to work on, illustrated by the following examples.

Sequence E. (verification of the pre-test)
e.g $x + 5 = 8$; $13x = 39$; $(x + 3)\,6 = 48$

Sequence C. (the equation as equivalence)
e.g $x + 5 = 5 + 2$; $x + x/4 = 6 + x/4$; $x + 5 = x + x$

Sequence I (operating on the unknown)
e.g $3 + 2x = 5x$; $7x + 2 = 3x + 6$; $10x - 18 = 4x$

Sequence A (word problems of "find a number type)

The analysis of the interviews, which confirmed the presence of the didactic cut (especially in the work of the students in the top stratum), is described in detail in Filloy and Rojano [1984]. The analysis also brings out the characteristic approaches of each stratum to the problem posed by the cut, i.e. their spontaneous approaches to the solution of the non-arithmetical equations in Sequences C and I.

In the next section we will look at our attempt to study the effects of instruction using concrete contexts to model non-arithmetical equations. We focus on students with high scores on the "pre-algebra" test in order to reduce the chance that the "transition phenomena" are due to shortcomings in the students' basic arithmetical knowledge.

Concrete models for operating on the unknown

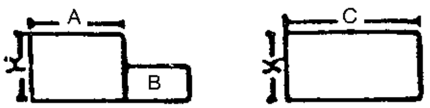
Although there are theoretical reasons for believing that a semantic approach to learning algebra is more likely to lead to good algebraic performance in later years than a purely syntactic one, this does not mean that the construction of algebraic syntax can be immediately and easily derived from such an approach Certain processes of abstraction have to be gone through; for example, of the operations which are performed on the elements of the "concrete" situation, where new objects and new operations are being modelled. Such processes presuppose in turn others, such as the process of generalising the actions performed on the model, and the process of discriminating the various cases that can be modelled.

We used two models in our research: the balance and a geometrical model.

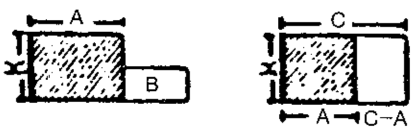
The geometrical model

As applied to $Ax + B = Cx$, where A, B and C are given positive integers and in this case, $C > A$. (The children were presented with particular numerical instances.)

Step 1: translating the equation into the model



Step 2: comparing areas



Step 3: producing the new equation, $(C - A)x = B$

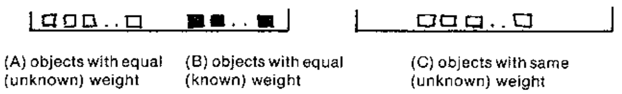
Step 4: solving the new equation

Step 5: verifying the solution

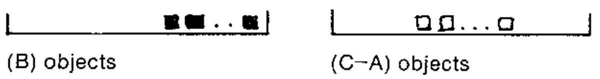
The balance model

Again, as applied to an equation of the type $Ax + B = Cx$

Step 1: translating the equation into the model



Step 2: repeated removal in pairs of objects of unknown weight, maintaining equilibrium, until none remain in the left hand pan



Step 3: writing the new equation, $(C - A)x = B$

Step 4: solving the equation

Step 5: verifying the solution

Those children who displayed a high level of pre-algebraic proficiency were provided with only the first step of either model, leaving to them the development of the subsequent steps with as little help as possible from the interviewer. Once they had mastered the use of the models for the type $Ax + B = Cx$, they were given increasingly complex types: $Ax + B = Cx + D$, $Ax - B = Cx + D$, $Ax - B = Cx - D$, etc., so that we could observe how they made the transference to the more difficult cases, and also how they abstracted the processes from the repeated operations carried out on the models.

Analysis of the abstraction processes

Over the course of the interviews, processes of abstraction of the operations on the "new" objects (the unknown, in this case) become manifest. We detected a number of phenomena.

1 *The temporary loss of previous abilities, coupled with behaviors fixated on the models.* The most frequent instances were the loss of operativity in solving the simplified (arithmetical) equation, i.e. a failure to recognize $(C - A)x = B$ as an equation which the students could solve syntactically on the basis of their earlier experience. We diagnose a "centration" or "fixation" on the model which precludes the students detaching the simplified equation from the concrete meaning that the model confers on it.

Example. VT, age 13, who showed considerable proficiency in solving arithmetical equations, including those with negative solutions, is working on the equation $8x + 30 = 5x + 9$. Using the geometric model, VT derives the simplified equation $3x + 30 = 9$.

VT: "How . . . ?"
I: "Do you think you can solve this one?"
VT: "Nine minus . . . nine minus a number larger than nine is going to give a negative number, divided by three . . ."

There is a long pause, and when the interviewer requests an answer, VT begins to reexamine the geometrical configuration. Eventually the interviewer asks her to try and solve the equation without using the diagram.

I: "Can you solve it now?"
VT: "No . . . (pause) . . . Well, if you want me to, I'll . . ." She takes out her calculator, inverts operations and obtains x= −7.

VT resorts to the geometric model, though its application is not pertinent for this example, and in spite of the fact that she already knows that the solution will be negative. The automation of her actions on the elements of the model hides its meaning; even the performance of the actions, which do not work, is not sufficient to make her voluntarily abandon the solution path. Since VT has handled the model swiftly and fluently in many previous items, her adherence to it here looks like inertial behavior.

2 *The modification of the arithmetical notion of equation.* (a) The modification arises because examples are encountered whose structure does not correspond to that of the examples used in the instructional phase of modelling.

Example. MT, also age 13, is working on a sequence of items. She has already abstracted from actions on the model and ignores it in the resolution of the equations. At first she does not perceive the equivalence of the equations $2x + 3 = 5x$; $3 + 2x = 5x$; $5x = 2x + 3$; $5x = 3 + 2x$.

Item 16: $2x + 3 = 5x$
MT: Writes: $2x + 3 = 5x - 2x = 3x$. "So x equals one."

Item 17: $3 + 2x = 5x$
MT seems unsure what movement of x must be effected. The interviewer recalls the equation in Item 16 MT now recognizes their equivalence but admits she had not previously notice it

Item 18: $5x = 2x + 3$
MT modifies it to solve it as in the previous item. She writes:
$5x = 2x + 3$
$2x + 3 = 5x$
$2x + 3 = 5x - 2x = 3x$

Item 20: $5x = 3 + 2x$
In spite of the fact that MT recognizes the equation as the "same" as the one in item 19, she does not assign it the same solution, only the same method of resolution:
$5x = 3x + 2x \quad 3x \quad x = 1$

2 (b) Another way in which the arithmetical notion of equation becomes modified is through the endowment of the "new equations" with meaning. One way of providing them with sense is through the verification process

Example After solving 25 items in Sequence I, the first few with the model and the rest at a syntactical level, MT spontaneously gives a "more algebraic" interpretation of the equation $10x - 18 = 4x + 6$ (Item 26)

MT: "In other words, they are equivalent "
I: "What do you mean by equivalent?"
MT: "If I find what x is and do this to it (pointing to the left side of equation) and get a result, it has to be equal to this result (pointing to the right side).

3. *The use of personal codes to indicate actions to be performed in the solution process.* This is an intermediate stage on the path to the development of a fully algebraic syntax as the personal code becomes inadequate when the examples become more complex.

Example: MT rapidly gives up the use of the concrete model and generates her from codes to indicate actions on the elements of the equations.

Item 13: $129x + 51 = 231x$
MT writes: $129x + 51 = 231x - 129x = 102x$

"Therefore x equals two " (sic!)
Item 15: $x + 5 = 2x$
MT writes: $x + 5 = 2x - x = 1x$

"One x must be equal to five, x equals 5."

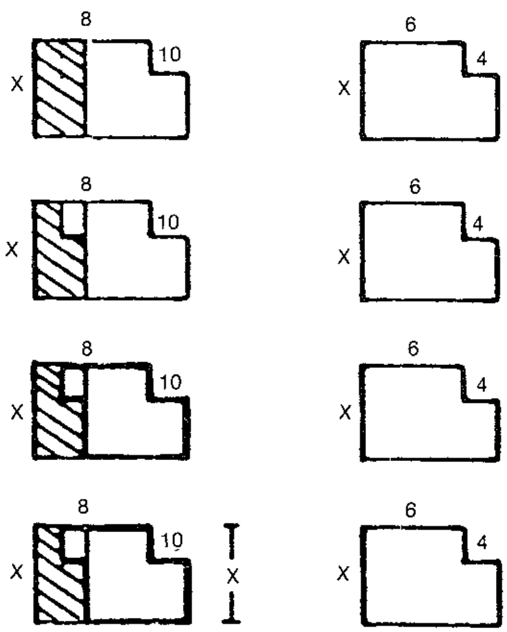
In the verification process relating to Item 13, MT also exhibits the "temporary loss of previous abilities" that we have already referred to. In spite of the fact that she has the simplified equation ($51 = 102x$) in front of her, she loses her previous facility and begins to ask herself (reverting to a more primitive level) "What number times 102 will give me 51?" Since such a number is not an integer, she anticipates that it will be negative. It is not until the interviewer focuses her attention on the multiplicative operation linking x and its coefficient that she regains her former operativity and evokes the inverse operation to reach a solution. In other words, it was not sufficient for her to have reached an effectively algebraic statement; it was also necessary for her to have meanings for the operations embedded in the statement

4. *The fixation on the model which may persist underneath an apparent algebraic operativity on the equation elements*

Example. VT, after 28 items in the non-arithmetical sequence, is still using the geometric model and shows no intention of giving it up.

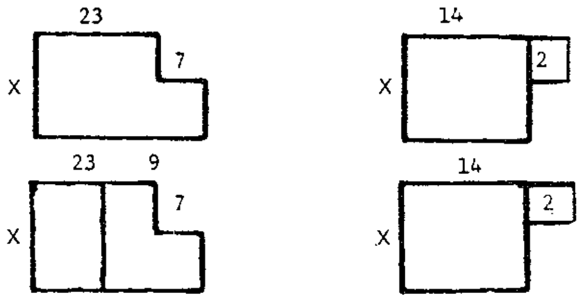
Item 29: $8x - 10 = 6x - 4$
VT: "Perhaps if I remove these parts first. " (indicating the "gaps" representing the -10 and the -4).

VT's second attempt:



There is some confusion between lengths and areas.
VT: "Well, x . . . minus 10 . "
After some help from the interviewer:
VT: "Two times x, minus 10, plus 4 (writes $2x - 10 + 4$) must be equal to . "
I: "Nothing!"

Item 30: $23x - 7 = 14x + 2$
Without any help VT produces:



With the interviewer's help, VT now realizes that the remaining area is not zero.
VT: "Nine times x minus 7."
I: "What is this equal to?"
VT: "It equals two." Writes $9x + 7 = 2$

Item 32: $10x - 3 = 4x$
Without any help, VT says, "That would be six times x, minus 3, equals zero", and writes $6x - 3 = 0$

5 *A breaking away from the model, transferring the operativity on the coefficients to the terms containing the unknown.* At the time of transition this can lead to the familiar error of combining terms of different degrees.

Item 11: $15x + 13 = 16x$
MT: "$16x$ minus $15x$ equals one. So one and 13 equals 14."
Her attention is drawn to the geometric model.
MT proceeds again: $15x + 13 = 16x - 15x$
MT: "One x . . . h'm. There should be one times x"
I: "And what does that have to be equal to?"
MT: "Thirteen." Writes $1x = 13$

6 *The presence of obstacles peculiar to each model*

7 *The recognition, through the use of models, of the diversity of first degree equation types*

We discuss Results 6 and 7 in the following sections.

Semantics versus syntax

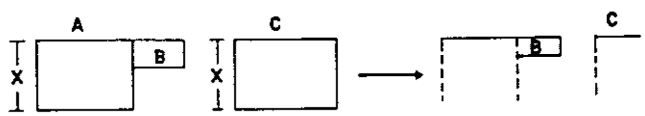
Results 1-7 given in the previous section arise from the description and analysis of certain general patterns observed in the performance of the students in the high stratum before and after they received instruction in operating on the unknown. This analysis ignored individual differences in the children's attitudes or tendencies. These differences, however, are interesting when we come to the "semantics vs. syntax" theme as we find that individual students exhibit preferences towards certain solution methods, ranging from the most operative and algorithmic to the most semantic and analytic.

Taking account of individual students' preferences or tendencies to choose certain solution methods helps us clarify the interactions between the syntax and the semantics of algebraic language. By differentiating the interaction phenomena that are strongly linked to a preference for algorithmic or analytic approaches, we minimize the risk of making false generalizations about the evolution of certain operations from concrete to syntactical form. We have analyzed in detail the interview protocols of two girls after the instructional intervention—two girls with extreme and antagonistic tendencies: MT, in the high stratum, age 13, with a marked algorithmic tendency, and VT, in the same stratum, the same age, with a strong semantic tendency. For both students, the same model was used in the instructional phase on operating on the unknown.

Our most relevant findings were:

1. *The spontaneous development of the use of a concrete model in order to operate on the unknown in an equation is not uniform* even in students with similar levels of pre-algebraic proficiency. The form of the development is strongly influenced by the individual student's tendency to choose a particular approach. Extreme cases were detected in which developments in the use of the same model were quite dissimilar. In one case (VT) with an operative tendency, the development adhered to the use of the model context even when the equation types required very complicated modelling procedures. In the other case (MT) there was a constant search for the syntactic elements present in the actions on the model as they were repeated in equation after equation and in type after type. The subject broke away from the semantics of the model and started associating actions on the model with a more abstract language through the creation of personal codes, belonging neither to the model nor to algebra.

2. *Obstacles to the abstraction of the model operations exist which do not depend on the particular model being used nor upon the student's personal preferences.* They arise from the degree of emphasis on the modelling aspects which permit the use of the subject's previously acquired knowledge and operations. The reduction of the new to the already known carries with it the risk of hiding the difficulties of arriving at and using the new elements that must be learned. In the process of abbreviating and automating actions on the two models in use here, the operation on the unknown can become hidden. In the geometric model, abbreviation leads to an ignoring of the linear dimension which represents the unknown; the operations on the areas are reduced to operations on the "data" of the equations, so preventing the unknown from playing any role in the solution process.



In the balance model, due to the discrete nature of the coefficients of the unknown "x" as well as those of the constant terms, operations of the same type can readily be performed on both kinds of numbers.

Automation (in both models) leads students to commit later on the errors typically associated with algebraic syntax, such as attempting to add and subtract coefficients of different degree. Even subjects with a strong attachment to the model commit these errors through their introduction of personal codes. Such personal codes, incidentally, may be produced by automating actions:

$$Ax + B = Cx + D \longrightarrow (A + B - D) + C$$

or by introducing "artificial" parentheses when substituting a numerical value for the unknown,
e g $B + Ax \longrightarrow (A + B)x$.

Comparing the two models

A third phase of our clinical study undertook a comparative analysis of the use of the balance model in the resolution of non-arithmetical equations and, for the geometric model, of the processes of abstraction from the operations on the model to the syntactic level. The comparison allowed us to identify the phenomena relating to the abstraction processes which do not depend on the particular model being used. We were also interested in variations between the models which we should take into account in recommending teaching strategies.

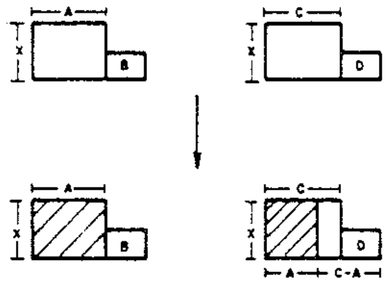
The most relevant results we found were the following:

1. *There are ways, specific to each model, of translating equations into terms of the model that become obstacles to the further use of the model.* In the case of the geometric model, the obstacle stems from the breaking down of the constant term (B) into two linear factors (the dimensions of the rectangular area representing B), i e one finds a b and an h such that $b \times h = B$ and so that either $b = C - A$ or $h = C - A$. This method fails when B is not a multiple of $C - A$ and becomes an obstacle to the extension of the solution method to non-integral equations

   In the case of the balance model, assigning values to the unknown weights can be an obstacle to the development of the "natural" strategy of cancelling pairs of identical weights. Also, in the context of the concrete model, the notion of the unknown may be weakened: with an example such as $4x + 6 = 2x$, the value of the first x must be less than the value of the second

2. *Some transferences in the uses of the model are more "natural" in one form of the model than the other.* With the balance model, the step from solving the $Ax + B = Cx$ to solving the type $Ax + B = Cx + D$ is a small one since "iterated cancellation" will reduce both types to arithmetical equations. In this model, too, the simplified equation is automatically expressed in terms of the model and does not require further translation into an algebraic code.

   In the geometric model, on the other hand, it is not a trivial matter to realize that the resolution of the type $Ax + B = Cx + D$ requires the overlapping of the rectangles representing the first degree terms Ax and Cx)



However the transition to types $Ax - B = Cx$ and $Ax - B = Cx + D$, while impossible for the balance model, can be accomplished in the geometric model with the introduction of the operation of removing areas when negative term are involved. This extension does not do violence to the semantics of the model

Of course, in the case of equations with negative solutions, there is no way, with either concrete model, of producing a meaningful reduction of the equation to an arithmetical one.

Concluding discussion

The results we have obtained allow us to assert that the correction of syntactic algebraic errors, and of the operational difficulties that occur in resolving complex problems or equations, cannot be left to be spontaneously resolved by children on the basis of their initial grasp of operational algebraic behaviour. This is because the path of such spontaneous developments does not go in the direction of what algebra is intended to achieve. The correction of these errors is a task for education and, if we want to introduce certain algebraic notions by means of models (including purely syntactic models), we would do well to bear in mind the main components of modelling.

Modelling has two fundamental concepts One of them is "translation", by means of which objects and operations in abstract situations are endowed with meanings and senses by being given more "concrete" manifestations. That state of things at the concrete level represents another state of things at the more abstract level. In our geometric model, for example, the equality of two areas corresponds to the equivalence of two algebraic expressions. Starting from our knowledge of how problems are resolved at the concrete level, we introduce operations which have analogues at the more abstract level and which will also lead to a resolution there For this reason, the translation must be a two-way process so that it is possible to identify operations at the abstract level with operations at the concrete level

A second component of modelling is the "separation" of the new objects and operations introduced by means of the model from the details of the meanings appropriate to the concrete context This is what MT attempts to do in the example we have discussed: to detach herself from the semantics of the concrete model since, ultimately, what is sought is not the solution of a situation she already knows how to solve, but the way to solve a more abstract situation by means of more abstract operations. This second component is what drives modelling towards the construction of an extra-modular syntax.

The study we are reviewing here shows that mastery of the first component can weaken or inhibit the second Subjects like VT achieve good control of the concrete model but, because of this, develop a tendency to stay and progress within the concrete context Fixation on the model can delay the construction of an algebraic syntax since this requires breaking away from the semantics of the concrete model.

In cases of a more syntactic tendency, like MT's, we note that obstacles may be generated in the course of abbreviating the actions and producing intermediate codes (intermediate between the concrete level and the fully syntactic algebraic level). These obstacles hinder the abstraction of the operations performed at the concrete level and are due to a lack, in the transition period, of adequate means of representing the states to which the various operations lead.

The obstacles arise from a sort of "essential insufficiency" in the sense that modelling (when it is spontaneously developed by children) tends to hide what it is meant to teach. When either of the two components is strengthened at the expense of the other, the *new objects and operations* become harder to see.

The dialectic between the two components of modelling should be taken into consideration by teachers and an attempt made to develop the two processes harmoniously so that they do not obstruct each other. From our analysis of the cases discussed here, it is clear that this is indeed a task for education The second aspect of modelling—the breaking away from previous objects and operations—is a process which negates part of the semantics of the model. These partial negations take place during transferences of the model from one problem situation to another (in our geometric model they occur in the transfer from one type of equation to another), but when the generalisation of the model is left to spontaneous development by the children, they are just as likely to negate essential parts of the model itself (in the geometric model, they may negate the presence of the unknown, or the operation on the unknown, for instance). The intervention of teaching is necessary to the development of the processes of detachment from, and negation of, the model, in order to lead towards the construction of the new notions.

The transfer of the problem situation (semantics *versus* algebraic syntax) to a level of actions on a model permits the closing of gaps between teaching and the problem situation. Through the analysis of the interactions at this new level, didactical phenomena come to light which point to the need for instructional interventions at key points in the processes which are unchained during the initial stages of algebraic language acquisition.

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The first author who explicitly described the method that according to most contemporary philosophers of science, characterizes physics was Christiaan Huygens He prefaced his *Treatise on light* by stating that in presenting his theory of light he had relied upon demonstrations of those kinds which do not produce as great certitude as those of Geometry and which even differ much therefrom since whereas the Geometers prove their Propositions by fixed and incontestable Principles here the Principles are verified by the conclusions to be drawn from them . . What I wish to suggest is that to a far greater extent than is commonly realized mathematicians have employed precisely the same method—the so-called hypothetico-deductive method Whereas the pretense is that mathematical axioms justify the conclusions drawn from them the reality is that to a large extent mathematicians have accepted axiom systems on the basis of the ability of those axioms to bring order and intelligibility to a field and/or to generate interesting and fruitful conclusions In an important sense what legitimized the calculus in the eyes of its creators was that by means of its methods they attained conclusions that were recognized as correct and meaningful Although Hamilton Grassmann and Cantor to name but a few, presented the new systems for which they are now famous in the context of particular philosophies of mathematics (now largely discarded) what above all justified their new creations both in their own eyes and among their contemporaries were the conclusions drawn from them This should not be misunderstood; I am not urging that only utilitarian criteria have determined the acceptability of mathematical systems although usefulness has undoubtedly been important Rather I am claiming that characteristics of the results attained—for example their intelligibility—have played a major role in determining the acceptability of the source from which these results were deduced To put it differently calculus complex numbers non-Euclidean geometries etc were in a sense hypotheses that mathematicians subjected to tests in ways comparable in logical form to those used by physicists

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