INTUITIVE AND FORMAL MODELS OF WHOLE NUMBER MULTIPLICATION: RELATIONS AND EMERGING STRUCTURES

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The role of intuitive models in mathematical reasoning has been studied deeply by Fischbein (1987) for whom intuitive models are mental tools that connect what is intellectually inaccessible with what can be manipulated. For Fischbein, "a system B represents a model of a system A if, on the basis of a certain isomorphism, a description or a solution produced in terms of A may be reflected consistently in terms of B and vice versa" (p. 121). Even when a model is consciously used, there are features of the model that remain tacit. These tacit features influence or even distort the interpretation of a concept. This is important for the teaching of mathematics, as when a teacher introduces a mathematical object through a model, she may be not aware of the influence of tacit features of the model on students' conceptions. A careful consideration of models and their features is useful to develop such an awareness.

Multiplication of whole numbers, for example, can be defined and studied through different models, including repeated sum, grouping, skip counting, hopping along a number line, stretching a number line, grid making, area generation, scaling and linear functions (Davis & Renert, 2013).

Different models and their features are not used equally by teachers in school practice. Repeated sum and grouping are often considered first (Davis & Renert, 2013). According to the *repeated sum model* (RSM), the multiplication $a \times b$ is the repeated sum of b taken a times (or vice versa, in some languages). This operation is often represented as putting together a groups of b objects. This does not mean that grouping and addition are exactly the same 'thing' but as quantification of the total amount of objects sorted into equal groups can involve repeated addition, they are closely related.

Fischbein, Deri, Nello and Marino (1985) conducted an extensive study on implicit models of multiplication. They claim that RSM is a *primitive* model, that is "it reflects the way in which the corresponding concept or operation was initially taught in school" (p. 15) and for this very reason it "tacitly affects the meaning and use of multiplication, even in persons with considerable training in mathematics" (p. 6). Mulligan and Mitchelmore (1997) maintain that this model is so interiorized because children are trained to look for the *equal-sized-groups* structure of multiplicative problems. Also, Freudenthal claims that multiplication should be introduced as a repeated sum to interpret equal-groups problems (1973). So, RSM and grouping are considered as tightly connected.

Limits of RSM were highlighted by several researchers. Davydov (1992) evidenced the problem of giving meaning to multiplication by 1: "if given 1×2, then what is added to what" (p. 13, as quoted in Boulet, 1998, p. 12). Another is the impossibility of a meaningful extension of this model to rational numbers. These observations explain the necessity for other models of multiplication. The use of the array or rectangular model has been proposed as a possible solution (Freudenthal, 1973; 1986). In the *array model* (ARM) the multiplication $a \times b$ is interpreted as a rectangular array of b rows with a elements in each row. In recent literature, the use of ARM is more and more encouraged.

According to Vergnaud (1983), multiplication can be conceived as a unary or a binary operation. In the unary interpretation, factors play different roles: "I read $a = q \cdot d$ as a equals q times d, the factors q and d are not exactly the same thing: d is the thing that is taken q times" (Freudenthal, 1986, p. 26). The former is a magnitude, while the latter is a pure number. In the binary interpretation, three variables are involved. All numbers represent magnitudes and the unit of the product derives from the units of the factors (*e.g.*, $1\ cm^2 \times 1\ cm = 1\ cm^3$). Barmby, Harries, Higgins, and Suggate (2009) claim that RSM encourages unary thinking while ARM portrays multiplication as a binary operation. In fact, RSM and ARM are two possible representatives of Vergnaud's two structures.

In previous literature (Freudenthal, 1973, 1986; Barmby *et al.*, 2009; Larsson, 2015), these two models for multiplication (ARM and RSM) have been often contrasted, and some studies paid attention in determining which model is more intuitive according to different problems (Larsson, 2015). Even if the RSM appears widespread as the first approach to multiplication, some researchers suggest that an early introduction of the ARM could present advantages. In this article we intend to contribute to the discussion of the two models by focussing on the mathematical structures underpinning the models and their mutual relations. Our purpose is to clarify the didactical implications coming from the use of each of the two models. They work as paradigmatic examples of different models to be compared.

In order to elaborate on the didactical perspective, we assume what we consider a widely shared assumption: that relational understanding should be preferred over instrumental understanding (Skemp, 1976). In other words, teaching/learning of multiplication should not be just about

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how to calculate products but also why we can calculate them with certain procedures. Different models of multiplication can serve in different ways to justify algorithms, showing the 'why' of arithmetical properties. Thus, our comparison of RSM and ARM will focus on how multiplication properties can be justified within these models.

We assume that consistency or inconsistencies between formal definitions and intuitive models that are used in education are at the base of students' acceptance of mathematical properties (Fischbein, 1987). Thus, in the following, we explore the coherence between formal proofs and their corresponding informal explanations of commutativity and distributivity, referring to each intuitive model. In the next sections, formal counterparts of RSM and ARM are defined. Afterwards, the commutative and the distributive properties are considered: the analysis of mathematical structure of formal proofs within each formal model is alternated with the discussion about educational implications with respect to the corresponding informal model.

Formal models of whole number multiplication

We will refer to Zermelo-Fraenkel set theory, within which whole numbers can be defined as follows:

$0 = \emptyset$ (empty set)

$n^+ = n \cup \{n\}$

where n^+ denotes the successor of n. So, we define the set $1 = \{0\}$and similarly $2 = \{0,1\}$, $3 = \{0,1,2\}$ and so on. A generic set A is considered as having n elements when there is a bijection between A and n; we write $|A| = n$.

On the base of this definition of whole numbers and following Peano (1889), we can define addition recursively:

$n + 0 = n$

$n + m^+ = (n + m)^+$

Subtraction is defined as the inverse operation of addition:

$n - m = p$ when $m + p = n$

Peano also defines multiplication recursively:

$0 \cdot n = 0$

$m^+ \cdot n = m \cdot n + n$

Division is defined as the inverse operation of multiplication:

$n : m = q$ with a remainder $0 \leq r < m$ when $m \cdot q + r = n$

Peano's recursive definition of multiplication is a repeated addition of n. This arithmetical operation can be enacted through the repeated union of groups of n objects. The multiplication $m \cdot n$ is conceived of as the union of m disjoint sets of n objects and its result consists of the cardinality of the set that is obtained at the end of such activity (See Figure 1). In this case, the two factors play different roles: one is multiplied and the other is the multiplier. We can notice that if multiplication is conceived of as repeated addition, its inverse, division, can be conceived of as repeated subtraction, that could be enacted by repeatedly taking away equal groups of objects. In the case of a division $n : m$, groups of m objects are taken away from a group of n objects. The result of such operation is given by the number

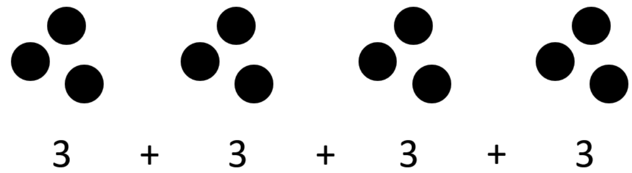

Figure 1. Repeated addition as equal grouping.

of subtracted groups and the remainder is represented by the remaining objects when they are less then m.

We consider Peano's definition of multiplication as a formal model of multiplication as repeated addition. Hence, we will refer to this model as the *formal repeated-sum model* (FRSM).

Turning now to our other model, following Vest (1971), we claim that ARM is coherent with the Cartesian product. We use the symbol $n \times m$ to represent the Cartesian product of n and m, where n and m are two sets according to the definition given above. In other words, $n \times m$ is the set of all the pairs (a,b) where $a \in n$ and $b \in m$ [1]. The process of generating and counting all these pairs can be represented through a double-entry table in which each column corresponds to an element of one set, while each row corresponds to an element of the other one. In this way, each cell contains one possible pair. For instance, in the case of 3×4 we have 12 pairs (See Figure 2, left).

We define $n \star m$ as the cardinality of the Cartesian product of n and m, so $n \star m = |n \times m|$. We will refer to this definition as the formal array model (FARM). This multiplication consists in finding all the ordered pairs made of elements from two sets: one set with n elements and the other with m elements. This process can be performed in a systematic way. We can first count all the pairs having as first element a certain element of the first set. Then, we count those pairs in which the first element is another element of the first set and so on, until we have considered all the elements of the set. We can represent the process with the 'crossing lines method' (See Figure 2, right) in which vertical lines stand for the elements of one set and horizontal lines stand for those of the other one.

Formally, as we did for FRSM, we can define division in FARM as the inverse operation of multiplication. If we refer to the intuitive representation by double-entry table or crossing lines (See Figure 2), performing $n \div m$ means to arrange n objects in rows (or columns) made of m objects. The result of this operation is given by the number of rows (or

	0	1	2	3
0	0; 0	0; 1	0; 2	0; 3
1	1; 0	1; 1	1; 2	1; 3
2	2; 0	2; 1	2; 2	2; 3

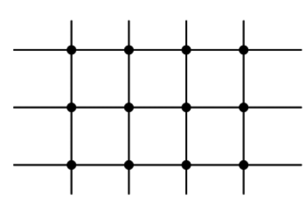

Figure 2. Cartesian product represented by double-entry table and crossing lines.

columns) and the remainder is represented by the remaining objects when there are not enough to complete the row/column.

The commutative property

Let us consider the proofs of commutative property of multiplication within each of the formal models defined above, as well as informal explanations corresponding to them.

The formal proofs

In FRSM, the commutative law $n \cdot m = m \cdot n$ can be proved by mathematical induction with respect of each of the two variables, m and n. If we start with m, the base case will be $m = 0$; so, we must prove that $n \cdot 0 = 0 \cdot n$. This relation can be proved in turn by induction on the variable n. In the induction step on m we must again resort to mathematical induction on n [2]. In the end, the proof depends on three applications of induction.

Proving commutativity in FARM means to prove that $n \star m = m \star n$. It is sufficient to define the function f from $n \times m$ to $m \times n$ so that $f(x,y) = (y,x)$. It is trivial to prove that this function is a bijection, so the commutative law is proven.

The informal explanation

Let us now consider intuitive explanations of the commutative property, related to the formal proofs presented above, in order to discuss possible cognitive implications from an educational point of view.

Within FARM, we proved commutativity by defining the function f. Such a function inverts the order of the elements in a pair. Therefore, interpreting the Cartesian product as a double-entry table (as in Figure 2), the first row becomes the first column, the second row becomes the second column, and so on. Thus, we can interpret f as a rotation of the array. Such a rotation changes the position of the array, preserving the number of elements in the array (the area of the rectangle) as shown in Figure 3. In short, the symmetry of rectangular models supports the explanation of commutative property of multiplication (Freudenthal, 1973).

The immediacy of such an informal interpretation of the formal proof provides a convincing support for the commonly shared claim that ARM provides "a useful representation for making certain properties of multiplication as a binary operation, such as commutativity, intuitively acceptable" (Greer, 1992, p. 277).

In contrast, Fischbein *et al.* (1985) noticed that, if multiplication is conceived of as repeated addition, commutativity is not immediately acceptable; the argument is based on the fact that multiplicative situations often do not bear exchanges between the variables: if c is the cost of one cake, it is quite easy to understand that adding c repeatedly, n times, gives the cost of n cakes, but why should adding n cakes c times also result in the cost?

The analysis of the formal proof within FRSM highlights other sources of difficulties, independent of any real-life interpretation. Actually, the formal proof in FRSM is based on a repeated (and nested) application of mathematical induction, and the corresponding informal explanation within the RSM will be based on a similar use of recursion. However, as research shows, recursive reasoning is not so intuitive, even for experienced mathematical students (Harel, 2002). Harel defines *quasi-induction* to refer to those cases in which the student can prove a theorem for a specific whole number and she uses this result to prove the theorem for the successive number. We can use quasi-induction to deal with commutativity: for instance, we can easily verify that $4 \times 3 = 3 \times 4$ calculating $3 + 3 + 3 + 3$ and $4 + 4 + 4$. Now, we could try to use this result to prove that $5 \times 3 = 3 \times 5$. Doing that, we have a procedure that could enable us to verify that $n \times 3 = 3 \times n$ for any successive whole number n. This task is not trivial. If we want to use $4 \times 3 = 3 \times 4$ to prove that $5 \times 3 = 3 \times 5$ we could first add 3 to both sides of the equality: $4 \times 3 + 3 = 3 \times 4 + 3$

The left side is 5×3 because 4×3 means putting together four groups of three objects and so, adding another group, we will have five groups of three objects. What about the right side of the equality? If we resort to the equal-groups model, 3×4 means three groups of four objects and the added 3 can be considered as three separated things. We take each one of these things to add it to the three groups of four things, as shown in Figure 4.

We finally have three groups of five things. The same procedure may be applied recursively to prove that $6 \times 3 = 3 \times 6$, that $7 \times 3 = 3 \times 7$ and so on. However, this re-grouping appears not immediate enough to make commutativity easily accessible. This can explain the difficulty students meet while explaining commutativity within the RSM, as Larsson (2015) noticed.

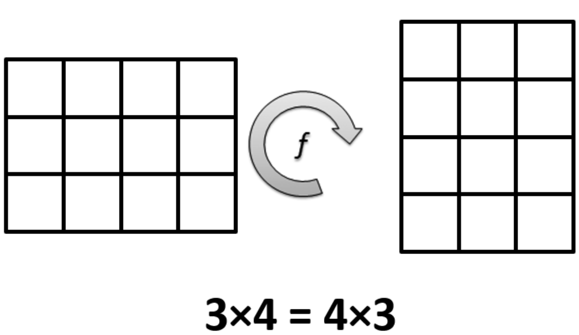

Figure 3. *Function f interpreted as a rotation.*

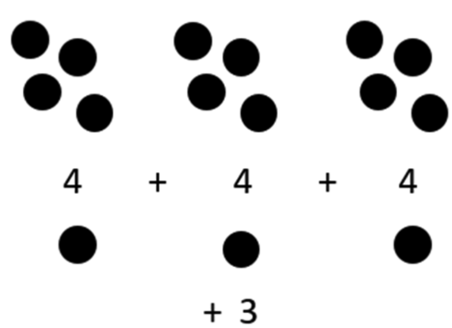

Figure 4. *Explanation of commutativity by 'inductive' regrouping.*

Distributive property

We now turn to proofs of distributive property within each of the formal models defined above, and informal explanations corresponding to them.

Formal proofs

Contrary to commutativity, proving the distributive property is quite straightforward in FRSM. As a matter of fact, in order to prove that for any triplet of whole numbers *n, m, p,* we have $n \cdot m + n \cdot p = n \cdot (m + p)$, we just need one induction on the variable *n*.

The property is easily verified when $n = 0$. As far as the inductive step is concerned, we assume that

$n \cdot m + n \cdot p = n \cdot (m + p)$,

and, by using the commutativity of the sum and the definition of successor, we obtain that

$n^+ \cdot m + n^+ \cdot p = n^+ \cdot (m + p)$ [2].

Let us now consider the proof in the FARM; in this case, we need to prove that $|n \times m| + |n \times p| = |n \times (m + p)|$. To do that, we define the function $g(\ (x,y)\ ) = (x,\ m + y)$ from $n \times p$ to $n \times (m + p) \setminus n \times m$. We use the symbol $A \setminus B$ to designate the complementary set of B in A, where $B \subseteq A$.

The function *g* is a bijection and so

$|\ (n \times (m + p)) \setminus (n \times m)\ | = |\ n \times p\ |$.

Then, we can prove that

$|\ n \times (m + p) \setminus (n \times m)\ | = |\ n \times (m + p)\ | - |\ n \times m\ |$ [2].

Because of transitivity, $|\ n \times p\ | = |\ n \times (m + p)\ | - |n \times m|$. This is equivalent to the distributive property, according to the definition of subtraction.

The informal explanation

The role of the two different models for learning distributivity is questioned in the literature: different contrasting positions can be found. For instance, Carpenter and colleagues (2003) suggest using rectangular arrays (*e.g.,* ARM) to introduce the distributive law. Also, Freudenthal claims that "only by means of the rectangle model of the product do properties of multiplication become visible: […] distributivity, two rectangles of equal height (or width) moved side by side" (Freudenthal, 1986, p. 26).

On the contrary, Larsson (2015) presents results supporting the claim that students can take advantage of the RSM in justifying calculation strategies based on the distributive property. She held that her findings "suggest that if we want students to learn and understand the [distributive property] we might better introduce the [distributive property] by equal groups and discuss the limits of its validity as well as how it can be used" (p. 301).

Let us now try to deepen the analysis of different possible explanations of distributivity resorting to the two different models, on the basis of the formal proofs described above.

The following intuitive argument can be drawn from the formal proof within FRSM: adding $m \cdot n$ to $p \cdot n$ is like adding *n* for *m* times and then again for *p* times, so we are repeating the same process $m + p$ times. From a cognitive point of view, we need to conceive a process (multiplying $m + p$ times) consisting of two sub-processes (multiplying *m* times; and multiplying *p* times). Following Sfard (1991), we can say that grasping distributivity means that the long sequence of operations (in our case the repeated sums) must be thought as a unique entity. She calls this process 'condensation': "The phase of condensation is a period of 'squeezing' lengthy sequences of operations into more manageable units. At this stage a person becomes more and more capable of thinking about a given process as a whole, without feeling an urge to go into details" (p. 19).

However, the process of condensation takes a long time and requires repeated experiences to be achieved. Thus, operating with numbers on the basis of arranging and re-arranging groups of elements may give pupils the opportunity to foster the condensation phase; "this is the point at which a new concept is 'officially' born" (p. 19).

The proof of distributivity within FARM requires the use of set subtraction on Cartesian products. The function *g* defines a correspondence between $n \times p$ and a set that is obtained by subtracting $n \times m$ from $n \times (m + p)$. An informal proof can be derived from that, interpreting function *g* in the ARM as 'cutting away' an array $n \times m$ (*n* rows of *m* elements) from an array $n \times (m + p)$ which leaves an array with the same size as $n \times p$ (See Figure 6).

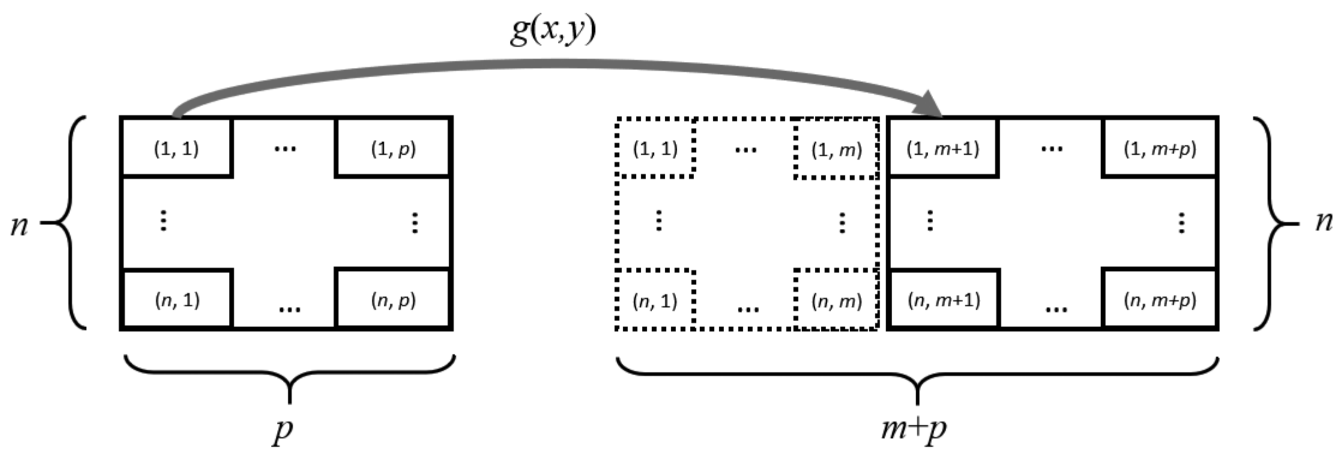

Figure 5. *Domain and codomain of g are shown with continuous lines; the dotted line designates n × m. The arrow relates an element to its image.*

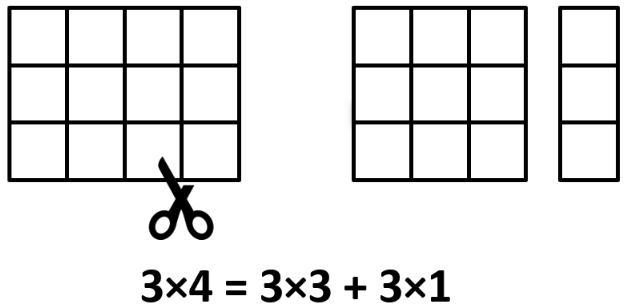

Figure 6. 'Cutting away an array' in the proof for distributivity.

The inverse of this function consists of 'gluing' side by side two arrays with same height to obtain an array that is as large as the two initial arrays together, as in the Freudenthal quote above. Also in this case, the informal explanation of distributivity, *i.e.,* the interpretation of the proof according to ARM, seems quite intuitive. However, cutting/gluing operations hide a possible difficulty: they correspond to adding and subtracting multiplications, not just numbers. In the same way, we previously noticed that proving distributivity in FARM required operations between Cartesian products of sets, not just sets. When using ARM to prove the distributive property, we are performing operations on arrays, that is performing operations on representatives of multiplication. So, from the cognitive point of view, explaining distributivity in ARM requires treating multiplications as single elements involved in other operations. When we interpret the intuitive explanation in ARM with Sfard's tools of analysis, we see that the distributive law requires going beyond the condensation phase and treating an operation as an object involved in other operations, namely what Sfard (1991) calls 'reification'.

These differences between the two models also become evident when using multiplication practically. When multiplication is conceived as a unary operation—for instance in situations involving just one unit of measure—the distributive property is immediately evident. If I add the cost of three boxes to the cost of five boxes, it is intuitive that I get the cost of eight boxes. I add dollars to dollars and I still get dollars. On the contrary, when multiplication is conceived as a binary operation—for instance in the case of determining the possible outfits composed of trousers and shirts—the distributive property is no longer evident. It appears intuitive that I can add together the number of outfits possible with a set of trousers and a set of short sleeved shirts, to the number of outfits possible with the same set of trousers and a set of long sleeved shirts, to obtain the number of outfits possible with either kind of shirt. It appears less intuitive that it works also if I first add the shirts and then I multiply by the number of trousers. A sum of shirts results in a sum of outfits.

These observations could explain why students resort more successfully to the RSM rather than to the ARM when they need to provide an explanation about calculation strategies involving distributivity (Larsson, 2015).

However, the ARM of multiplication can be represented by manipulative counterparts which could support the overcoming of this cognitive demand; in particular, the availability of specific concrete objects representing multiplication may support the condensation and reification involved in explaining distributivity. For instance, using rectangular pieces of squared paper to represent a multiplication allows one to enact specific transformations (rotating, cutting, gluing) representing those involved in our proofs (Maffia & Mariotti, 2015).

The discussion developed so far shows how differences highlighted within the respective formalisations (FRSM and FARM) correspond to differences, from the cognitive point of view, between the corresponding informal explanations in the respective models. Though both models can be appropriate to explain properties, we claim that the two models are not 'cognitively' equivalent. On the one hand, it is impossible to state which is the most appropriate because, depending on the situation, one model can be more evidently applicable than the other. On the other hand, it becomes clear why it might happen that when both models are introduced, children are not able to see them as models of the *same* operation (Davis & Renert, 2013). This last remark calls for further investigation. In the next section, we analyse the relations between the formal counterparts of the two models to study possible educational implications of proposing and combining the two models.

Relations between the two models

Formally, proving the equivalence of FRSM and FARM means checking that, for any pair of whole numbers, the two definitions of multiplication lead to the same result. That means, for any n and m, we should have $n \cdot m = n \star m$; in other words, $n \cdot m = p$ (in FRSM definition) if and only if $|n \times m| = p$. To prove this, we can define a function D from p to $n \times m$ such that, for any $x \in p$, $D(x) = (q, r)$, where q and r are the quotient and remainder of $x : m$ in FRSM. The equivalence between the two definitions of multiplication is proved by showing that D is a bijection. A different proof can be achieved using mathematical induction. This is quite straightforward after we have noticed that $m^+ \star n = m \star n + n$ [2].

The proofs show clearly that the two models of multiplication are formally equivalent; but what is more interesting for our discussion is the fact that they also suggest some relations between the two models.

The inductive proof is based on the relation $m^+ \cdot n = m \cdot n + n$, that is the relation used to define FRSM. Actually, multiplication as repeated addition can be represented also in ARM, as adding new rows (or columns) to the array. Such an interpretation makes clear the process of condensation (and reification) that is required to accomplish multiplication as repeated addition.

The bijection proof uses division to create a correspondence between the two multiplications. We can use division also in an informal explanation: To see that the result we obtain within RSM is the same as that we obtain within ARM, we can re-arrange the elements of our set in a way that gives us more information than just the result. For instance, let us consider the multiplication 'five times six'. Following RSM, we can calculate it by adding the number five, six times, and this corresponds to putting together six groups of five things. If we count these things, each object corresponds to a number between one and thirty. When we find that the last object corresponds to thirty, we have the result. If we now ask how many groups of five objects we have put together, we must start a process of division as repeated subtraction. Separating each group from the whole, we can rearrange the objects in rows; in so doing, we can show that this set of objects corresponds to a five by six array. We use the first five things for the first row of the array, then other five things for the second row. We end with six complete rows. Then, we can observe that by arranging thirty objects in rows of five we get six rows (30 : 5 = 6). The array appears as a model both for multiplication and division. Furthermore, while arranging the numbers, we also gain information about the division by 5 for numbers smaller than 30. We can label the rows and columns of the array so that the row and column labels correspond respectively to the quotient and remainder when divided by 5 (See Figure 7). The last column contains multiples of 5.

The comparison between the two models based on translating 'repeated addition' into 'constructing an array' highlights an essential feature of one definition with respect to the other.

When we look at an array, we have more information than in the case of a chaotic group of objects. In particular, specific arithmetical relationships are displayed and they can be made available via suitable interpretations. We can see the division as 'a whole' because the dividend, the divisor, the quotient, and the remainder are all visible at the same time. This additional information is gained thanks to the creation of, in Hoch and Dreyfus's terminology, a 'structure', which is, "a broad view analysis of the way in which an entity is made up of its parts. This analysis describes the systems of connections or relationships between the component parts" (2004, p. 50). This idea of 'structure' does not coincide with the formal mathematical one, but it is coherent with the distinction between *operational* and *structural* conceptions made by Sfard (1991). It is also coherent with the idea of structure that Mason, Stephens and Watson (2009) claim to be an essential part of mathematical teaching/learning because they consider it powerfully productive for bridging the gap between procedural and relational approaches to mathematics.

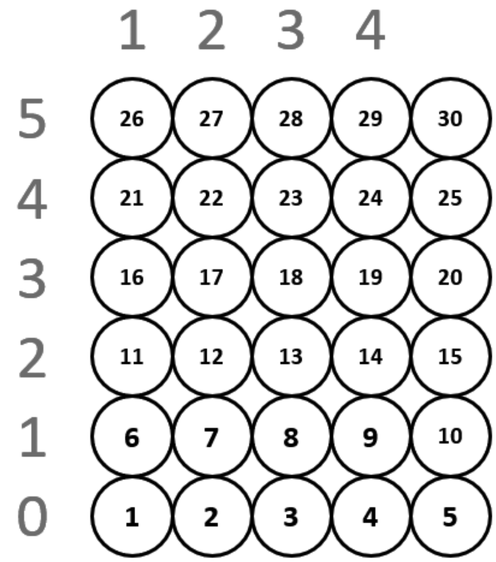

Figure 7. Five times six' objects arranged in an array 'five by six'.

Conclusion

The detailed analysis developed in the previous section aimed at investigating mutual relations between two models used to introduce multiplication for whole numbers. Formalization of the two models allowed us to prove that the two models are logically equivalent, however it is possible to notice that proofs of the properties follow different procedures in the FRSM and the FARM contexts. Induction is a necessary path for proving multiplication properties in the FRSM context, though it is not easily applicable in the case of commutativity. This suggests that, for informal explanation, ARM is a more suitable context for commutative property. The distributive law can be explained within ARM, but only by referring to multiplication as an object: this could be a source of difficulty for students. This mathematical analysis converges with empirical studies by Larsson (2015).

Furthermore, the use of multiple models for multiplication requires the passage from one model to the other. For example, n multiplied by m could be firstly introduced as the operation giving the total number of objects in an array of m lines with n objects per line. Then children could find out that it is the same as adding the n objects m times. We noticed that the process of passing from one model to another and informally checking their equivalence could be relevant, both because, in so doing, we can state equivalence of the models that children may not be aware of, and because such passage could create the opportunity to observe and state relational properties between numbers as results of different procedures, working to develop that sense of structures advocated by Hoch and Dreyfus (2004) and Mason *et al.* (2009). So an important conclusion is not just that presenting different models to pupils is relevant, but also that the operation of putting in relation the models has the potential to be particularly productive.

Here we address only whole number multiplication and this is a limitation of our work. The choice of the models that are used to introduce an operation should take into account also the possibilities of extending such models to other numerical domains. Concerning the models analysed in this article, literature shows us that starting from RSM can be problematic when introducing multiplication of fractions (Thompson & Saldanha, 2003). Simon and Blume (1994) argue that ARM is the basis for understanding the transformation of length into area measurements. However the shift from discrete to continuous lengths seems cognitively important. Possibly, as we have done in this article, a mathematical analysis of definitions of multiplication in other numerical sets may be put in relation to such findings. Furthermore, we cannot neglect that mathematics is characterized by *anontologisation*, that is "cutting the bonds with reality" (Freudenthal, 1983, p.81). Understanding properties of multiplication is an opportunity to understand theoretical necessities like that of preserving distributivity when defining multiplication between signed numbers and so having 'minus times minus gives plus'. Mathematical education aims to harmonise intuition and theory without neglecting the gap between spontaneous and mathematical thinking.

It is worth stressing the speculative nature of our conclusions. We have no reason to assume that the informal explanation corresponding to an easier formal argument would be always intuitively more acceptable for students. However, we think that this analysis would provide a support to readers for their sensing that repeated addition and array model multiplication are indeed different in their definitions and applications to mathematical and real problems.

The same kind of analysis could be performed to compare other models of multiplication or other arithmetical operations. The conjectures raised from such mathematical analysis could be starting points for further empirical studies about cognitive implications of using different models to explain arithmetical properties and for new studies on educational implications of relating these models.

Finally, we have shown how understanding the differences between the formal models can be helpful in understanding differences in didactical approaches and decisions that might be taken using the informal models. By noticing this, we can conjecture that the exploration of such models and their possible integration could also become part of teacher education programs.

Notes

[1] When used between numbers, as in '3 × 4 = 12', the symbol × is the usual 'school' multiplication sign without the technical meaning given it here.

[2] Complete proofs can be found in the appendix at http://flm-journal.org. See the link by this article in the online table of contents for this issue.

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