Ethnomathematics in the Classroom

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Mathematics was created by people who needed to solve problems; it was not ordained from on high. The questions posed years ago, and the solutions found, are still useful today in motivating our students to learn various mathematical topics. A careful reading of the history of mathematics generates numerous pedagogical ideas which can and should be used in today's classrooms. In particular, since many important mathematical ideas grew out of the needs of various cultures around the world, it is vitally important that students in Western nations be exposed to the fact that mathematics is a universal phenomenon. What we call mathematics was—and is—present in many civilizations, although not always in explicit form In particular, various mathematical ideas which arise in today's courses were considered by other peoples in the context of their own experiences and values. We can and should use these ideas out of ethnomathematics whenever possible to illuminate the concepts discussed as well as to demonstrate the universality of mathematical ideas In this paper we will consider several examples of important mathematical ideas taken from combinatorics, arithmetic, and geometry, which can be considered in the context of their development in various societies around the globe.

The mathematical ideas to be considered developed out of specific needs in the cultures involved And although the ideas were not developed by people we would call "mathematicians," all of these concepts were explored further, either in the original civilization or in later ones, far beyond the immediate context of the original problem. Students also learn in this way, by beginning with the consideration of a specific problem and then following the ideas further to try to answer new questions They can only benefit by seeing that many interesting problems were first considered in the context of civilizations far different from their own. It is vitally important that modern curricula, particularly at the secondary level, the level at which most of the ideas here are first introduced, incorporate materials like the ones presented in order to broaden our students' understanding not only of mathematics but of the world in which we live.

We begin with combinatorics, in particular, the rules for counting permutations and combinations. Among the earliest reasons for an interest in the question of permutations was the problem of determining how many words could be formed out of the letters of the alphabet. In fact, this questions was discussed over a thousand years ago in both Hebrew and Arabic works and provides a context useful even today for introducing combinatorics to a class.

First, we consider the mystical Hebrew work, the *Sefer Yetzirah,* a work whose date is unknown but most probably is around the third century C E. One of the main purposes of this brief work, whose title can be translated as *Book of formations,* is to explore the formation of words out of the letters of the Hebrew alphabet: "The twenty-two letters ... [were] appointed and established by God ... He combined, weighed and changed them, and formed by them all beings which are in existence and all those which will be formed in all time to come." [Kalisch, 1987, p. 11] Thus, the letters of the alphabet were considered to be the building blocks of all things and it was therefore important to understand how they could be combined to form "all beings"

The author then noted that God "fixed the twenty-two letters on the sphere like a wall with two hundred and thirty-one gates ... But how was it done? He combined ... the א (aleph) with all the other letters in succession, and all the others again with א; (bet) with all, and all again with ב, and so the whole series of letters." [ibid.] In other words, the author has found that there are 231 combinations of two elements taken from a set of 22 ($22 \times 21/2$). A tenth century commentator, Saadia Gaon (892-942), who was born in Egypt but spent much of his life in Babylonia, noted that children in Palestine in his time still learned spelling and pronunciation by considering all $22^2 = 484$ possible ordered pairs of letters. He noted further that the author of the *Sefer Yetzirah* dropped all 22 repeated pairs from consideration (e.g. אא) and then halved the remaining number to get his result

For the case of permutations, the author of the *Sefer Yetzirah* wrote that "Two stones build two houses, three stones build six houses, four build twenty-four houses, five build one hundred and twenty houses, six build seven hundred and twenty houses, and seven build five thousand and forty houses. From thence further go and reckon what the mouth cannot express and the ear cannot hear." [Ibid., p 23] Thus, the number of possible words which can be formed from 2, 3, 4, 5, 6, and 7 letters respectively are $2!$, $3!$, $4!$, $5!$, $6!$, and $7!$. Saadia Gaon explained the rule further and calculated that there are 40,320 permutations of 8 letters, 362,880 permutations of 9 letters, and, since the longest word in Scripture has eleven letters, that there are 39,916,800 permutations of 11 letters.

Shabbetai Donnolo (913-970), who lived in southern Italy, derived the factorial rule explicitly by noting that "the first letter of a two-letter word can be interchanged twice, and for each initial letter of a three-letter word the other letters can be interchanged to form two two-letter words—for each of three times. And all the arrangements there are of three-letter words correspond to each one of the four letters that can be placed first in a four-letter word: a three-letter word can be formed in six ways, and so for every initial letter of a four-letter word there are six ways—altogether making twenty-four words, and so on." [Rabinovitch, 1973, p. 144] He even noted that if one used

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all twenty-two letters and rearranged them in various ways one would get all the words of all the languages on earth. "But the number is too great for flesh and blood to calculate."

Interestingly, the earliest Islamic writers who dealt with permutations also considered them in relation to letters of the alphabet, here the Arabic alphabet. For example, al-Khalil ibn Ahman (717-791), was able to calculate the actual number of possible words one could get by taking 2, 3, 4 or 5 letters out of the Arabic alphabet of 28 letters. (See [Rashed, 1974] for more details.) Some five hundred years later, a more detailed study of this entire subject was made by Ahmad al-Ab'dari ibn Mun'im (early 13th century), who lived at the court of the Almohade dynasty in Marrakech (now in Morocco). Ibn Mun'im gave a slightly different argument from that of Rabbi Donnolo in his own determination of the number of permutations of letters: "The problem is: We want to determine a canonical procedure to determine the number of permutations of the letters of a word of which the number of letters is known and which does not repeat any letter. If the word has two letters, it is clear that there will be two permutations, since the first letter may be made the second and the second the first. If we augment this by one letter and consider a three letter word, it is clear that, in each of the permutations of two letters of a two letter word, the third letter may be before the two letters, between the two letters, or in the final position. The letters of a three letter word therefore have six permutations. If the word is now augmented by another letter to make a four letter word, the fourth letter will be in each of the six permutations (in one of four positions). The four letter word will thus have twenty-four permutations. ...." [Djebbar, 1985, pp. 55-56] Ibn Mun'im concluded that no matter how long the word, the number of permutations of the letters is found by multiplying one by two by three by four by five, etc., up to the number of letters of the word.

Since ibn Mun'im is interested in counting words, he considers several other problems besides the mere counting of permutations of distinct letters. First, he notes that words may have repeated letters. Thus, if one letter is repeated k times and there are a total of n letters in the word, the number of possible permutations is $n!/k!$. His argument is simply that there are $k!$ permutations of the repeated letter, but that none of these permutations affects the word at all. Similarly, if each of p letters is repeated $k_1, k_2, \ldots k_p$ times (with $k_1 + k_2 + \ldots + k_p = n$), then the number of distinct words which can be formed is given by $n!/k_1!k_2!\ldots k_p!$. Ibn Mun'im next considers variations in vowel signs—which change the pronunciations and therefore the words. After several more similar calculations, he is able to show, for example, that the number of words of nine letters, each word having two non-repeated letters, two letters repeated twice, and one letter repeated three times, is 5,968,924,232,544,000.

Interestingly, in both the Hebrew and Islamic contexts, the initial work in permutations dealing with letters was eventually followed by an abstract treatment of the entire subject in terms of permutations of arbitrary objects. The chief contributors were Abu-l-'Abbas Ahmad al-Marrakushi ibn al-Banna (1256-1321) of Marrakech and Levi ben Gerson (1288-1344) of Orange, in France. [For details, see [Djebbar, 1981] and [Levi ben Gerson, 1909].)

We next turn to some questions of arithmetic, specifically the calculations of the Mayans in relation to their number system. Although it is not known why they developed their system, the Mayans during the height of their civilization in the first millennium used a place value system with 20 as a base in their astronomical and calendrical calculations. In ordinary usage, the individual places had numerical values of 1, 20, 400, 8000, etc., but in calendrical calculations these were modified so that the third place had a value of 360 (= 18 × 20), the fourth 7200 (= 20 × 360), and every place after that a value of 20 times the previous place. It is this calendrical place value system which will be used in what follows. And although the Mayans used a dot to represent 1, a line to represent 5, and combinations of these two symbols to represent numbers between 1 and 19, we will represent their base 20 numbers by using ordinary digits with commas to separate the places. Thus [3,5] represents $3 \times 20 + 5$ or 65, while [2,3,5] represents $2 \times 360 + 3 \times 20 + 5$ or 785. Note that in $[a,b,c,d,e]$, all letters except d can take on values from 0 to 19, while d can only be as large as 17.

The necessity for detailed calendrical calculations was partly related to the fact that the Mayans used two different calendars at the same time. First of all, there was the 260-day almanac which was the "product" of two independent cycles, one of length 13 and the other of length 20. Thus we can specify a calendar date by a pair (t,v), where t is a day number between 1 and 13 and v is a number between 1 and 20 representing one of the 20 day names. For example, since the list of day names begins with *Imix* and *Ik*, while the fourteenth and fifteenth day names are *Ix* and *Men*, the day (1, *Imix*) will be written as (1,1) while the day (5, *Men*) will be written as (5,15). (Note that as far as the succession of days is concerned, the day (1, *Imix*) is followed by (2, *Ik*), the fourteenth day is then (1, *Ix*) while the fifteenth is (2 *Men*). In other words, the two cycles are independent.)

The second Mayan calendar was the 365-day year. This calendar was divided into 18 months of 20 days each and an extra period of 5 days. For our purposes here, it is sufficient to designate a day on this calendar by its number y. Thus, because *Zac* is the eleventh month, the third day of *Zac* will be designated by $y = 203 = 10 \times 20 + 3$. The three cycles of 13 day numbers, 20 day names, and 365 days of the years were traversed independently. Thus the complete cycle of triples (t,v,y) was repeated after $\text{lcm}(13, 20, 365) = 13 \cdot 20 \cdot 73 = 18{,}980$ days, or 52 calendar years, or 73 almanacs.

The chief calendrical problems which the Mayans needed to solve were first, given a date (as a triple) and a specified number of days later, to determine the new date and, second, given two Mayan dates, to determine the least number of days between them. Let $[m,n,p,q,r]$ represent a Mayan number of five places, where $0 \leq m,n,p,r \leq 19$ and $0 \leq q \leq 17$. The first question in modern notation is then given an initial date (t_0,v_0,y_0) determine the date (t,v,y)

which is $[m,n,p,q,r]$ days later. To answer the question, we put $t \equiv t_0 + [m,n,p,q,r] \pmod{13} \equiv t + 144000m + 7200n + 360p + 20q + r \pmod{13} \equiv t - m - 2n - 4p + 7q + r \pmod{13}$ Similarly $v \equiv v_0 + r \pmod{20}$ and $y \equiv y_0 + 190m - 100n - 5p + 20q + r \pmod{365}$. For example, if the given date is (4,15,120), the new date [0,2,5,11,18] days later is (10,13,133). To solve the second problem, one needs to determine the smallest intervals between the dates in each of the three component cycles, to combine the first two to determine the smallest interval in the almanac, and then to combine this value with the third to determine the complete number of days. It turns out that the minimum number of days between the dates (8,20,13) and (6,18,191) is [1,8,15,18], or 10,398. (See [Closs, 1986] and [Lounsbury, 1978] for more details.)

It is typical in various elementary texts not only to include the arithmetic of place value in various bases, but also to include "clock arithmetic" as the example of calculating with a given modulus. A study of Mayan modular arithmetic, in its natural context, would give students an opportunity to understand why such ideas were developed. They naturally would also learn something about the Mayan civilization and about the brilliant scholarly effort which decoded the few extant Mayan codices, necessary because the Spanish conquerors had destroyed most Mayan works.

We finally turn to certain geometric ideas, beginning with the theory of graphs. The standard story is that this theory had a definite starting place in a paper published in 1736 by Leonhard Euler. Euler here considered a problem posed to him by the citizens of Königsberg, whether they could take a continuous walk over the seven bridges connecting the mainland and two islands in the River Pregel without retracing their steps. Euler not only answered the specific question (in the negative) but, as mathematicians often do, abstracted this problem into the more general problem of deciding whether it was possible to draw an arbitrary finite graph in one continuous line without retracing any part of it. What Euler did not know was that this same question of tracing a graph in a continuous line is found in several non-European cultures. For example, in the Bushoong culture in Zaire, children often trace graphs in the sand. Although they perhaps could not verbalize the conditions which determine whether a given graph is traceable, according to the reports of a European ethnologist in 1905 the children certainly were aware of these conditions and also knew the procedure which permitted its drawing most expeditiously. For the Tshokwe, a cultural group centered in northeastern Angola, figure drawing is not a children's game, but is part of a storytelling tradition among the elders. To draw the curves which form part of this tradition, the general procedure is to set out a rectangular grid of dots on which the curve is superimposed. Often, a series of similar drawings is made in different sizes. For example, one series starts with two rows of three dots and continues with three rows of four dots, four rows of five dots, etc. In each case a similar procedure is used to construct the curve itself (Figure 1) (For more details, see [Gerdes, 1991]).

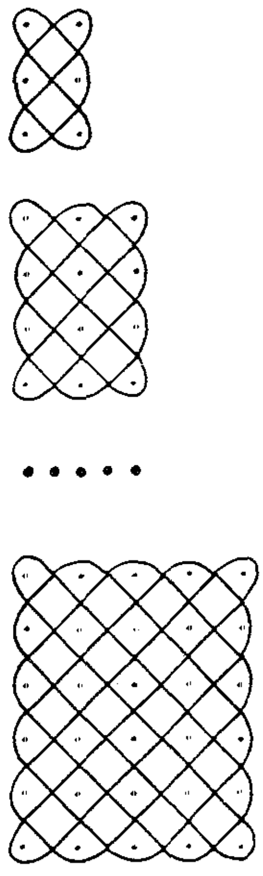

Figure 1

Graph theory ideas were also evidently considered in the Inca culture by those who developed the quipus, the data recording device used by the Inca leadership to monitor necessary data on taxes, population, numbers of workers needed for certain projects, etc. A quipu is a collection of colored knotted cords, where the colors, the placement of the cords, the knots on the individual cords, the placement of the knots, and the spaces between the knots all contribute to the meaning of the recorded data. Every quipu has a main cord, to which are attached other cords, called pendant cords, to each of which may be attached still other cords, called subsidiary cords. Data is recorded on the cords by a system of knots, clustered together in groups separated by spaces and using a base 10 place-value system.

As far as graph theory is concerned, however, the quipus are representative of a type of graph called a tree, a connected graph with no cyclic paths and therefore one in which the number of edges is one fewer than the number of vertices. Such graphs are often studied in courses in applied algebra, since trees are today an important form of graph useful in the modern study of data structures, sorting, and coding theory. In western mathematics, trees were first defined in 1857 by Arthur Cayley. In particular, Cayley dealt with the notion of a rooted tree, a tree in which one particular vertex is designated as the root. Cayley was able to develop a recursive formula by which he could calculate the number A_r of different trees with r branches (where "different" is defined appropriately). He applied his results to the study of chemical isomers. We have no direct information on whether the Inca quipu makers attempted to answer such questions about their quipus, but they certainly needed to understand the various ways in which trees could be constructed. And since the Incas associated both numbers and colors with each edge of their quipu trees, the questions they needed to answer in designing them to be useful were not trivial ones. One could easily make a detailed study of quipus part of a course in graph theory. (For more details, see [Ascher and Ascher, 1981].)

A course in geometry could also consider some of the mathematics of the Anasazi people, the native Americans who developed one of the most advanced cilivizations of pre-Columbian North America. The Anasazi occupied parts of the current states of Colorado, Utah, New Mexico, and Arizona from about 700 B.C.E. until about 1300 C.E. and are the ancestors of today's Pueblo Indians. The ruins of many of their magnificent buildings can still be seen in various places in the southwest, the largest collections being in Mesa Verde National Park in southwestern Colorado and in Chaco Canyon National Historical Park in northwestern New Mexico.

Many of the buildings in Chaco Canyon, in particular, were evidently constructed for ceremonial purposes. One of the more significant features of these buildings is their alignment, that is, the directions of the axes of the buildings. Some are oriented along an accurate north-south axis—and a road from Chaco Canyon was built in a due north direction some 50 km, keeping to that direction even in the face of topographic obstacles. Other buildings in the Chaco complex are apparently oriented to significant positions of the two most visible heavenly bodies, the sun and the moon.

We note here only some geometrical questions concerning Casa Rinconada, a 63-foot diameter circular building constructed in isolation from the other major buildings in Chaco Canyon, undoubtedly for ceremonial purposes. The first question to be asked is how did the Anasazi build such an accurate circle of that size. But the more important question is how were they able to align the building so carefully. In fact, the primary axis of the Casa Rinconada is due north and the four pillars that once supported the roof form the corners of a square, the parallel sides of which are north-south and east-west. A definitive answer to that question cannot be given, since the Anasazi did not leave written records. However, it is evident from various aspects of their astronomical knowledge that they knew that the sun rose directly in the east and set directly in the west on the dates of the spring and autumn equinoxes. Now that particular piece of information is not sufficient to construct an east-west line, since the question of on which day the equinox falls would still need to be answered. But since the Anasazi were also aware of the symmetry in the motion of the sun across the sky on any day, the following construction method is a possibility. We take a vertical pole and mark, with stones, say, the path of the pole's shadow from sunrise to sunset on any given day. If we then construct a circle centered on the pole, using a string with an attached stick, the two intersection points of this circle with the arc of the shadows will determine the east-west line. The north-south line can then be found by determining the midpoint of this line and connecting it to the pole. (Figure 2) This method, certainly within the capacity of the Anasazi, was used by Roman surveyors a millennium earlier. Students could be encouraged to consider other possible geometric methods of determining due north, methods connected to a knowledge of the heavens, because it is certain that the Anasazi used some such geometrical method. (For more on the Anasazi, see [Ferguson and Rohn, 1986].)

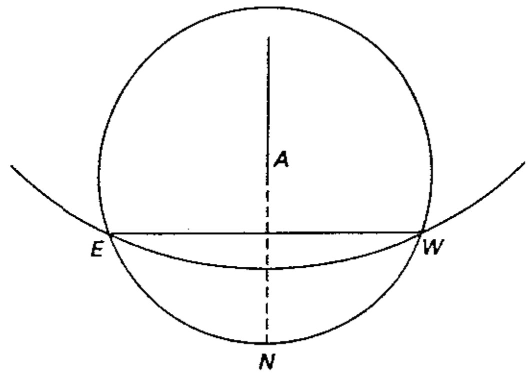

Figure 2

Like many other cultures in the Americas and in Africa, the Anasazi were also known for their use of geometric patterns for decoration. The Anasazi decorated their pottery while, for example, the artists of Benin (Nigeria) decorated their bronze castings. In these and many other cases, the artists used the general idea of symmetry and demonstrated, at least intuitively, the knowledge of various strip patterns and plane patterns. Although we cannot go into these ideas here, a study of these patterns in a geometry class today would certainly profit from the consideration of their use in various cultures around the world.

The use of geometry by the Anasazi, like the use of congruences by the Mayans or combinatorics by Jews and Arabs in the Middle Ages, demonstrate how mathematical ideas grow out of the needs of various peoples. It is clear that in all of these cases, important attributes of mathematical thinking, such as logic, pattern recognition, and application of previously known results, existed even if the participants in their development would not be called "mathematicians." But it is vitally important in the present day to convince students in North America in particular that such ideas were considered all over the world. Mathematics is not only a Western cultural phenomenon, but one which appears in many diverse civilizations. Our students will come away from a study of examples such as the ones presented here with a better understanding not only of mathematical ideas but also of the cultures out of which mathematical ideas have grown.

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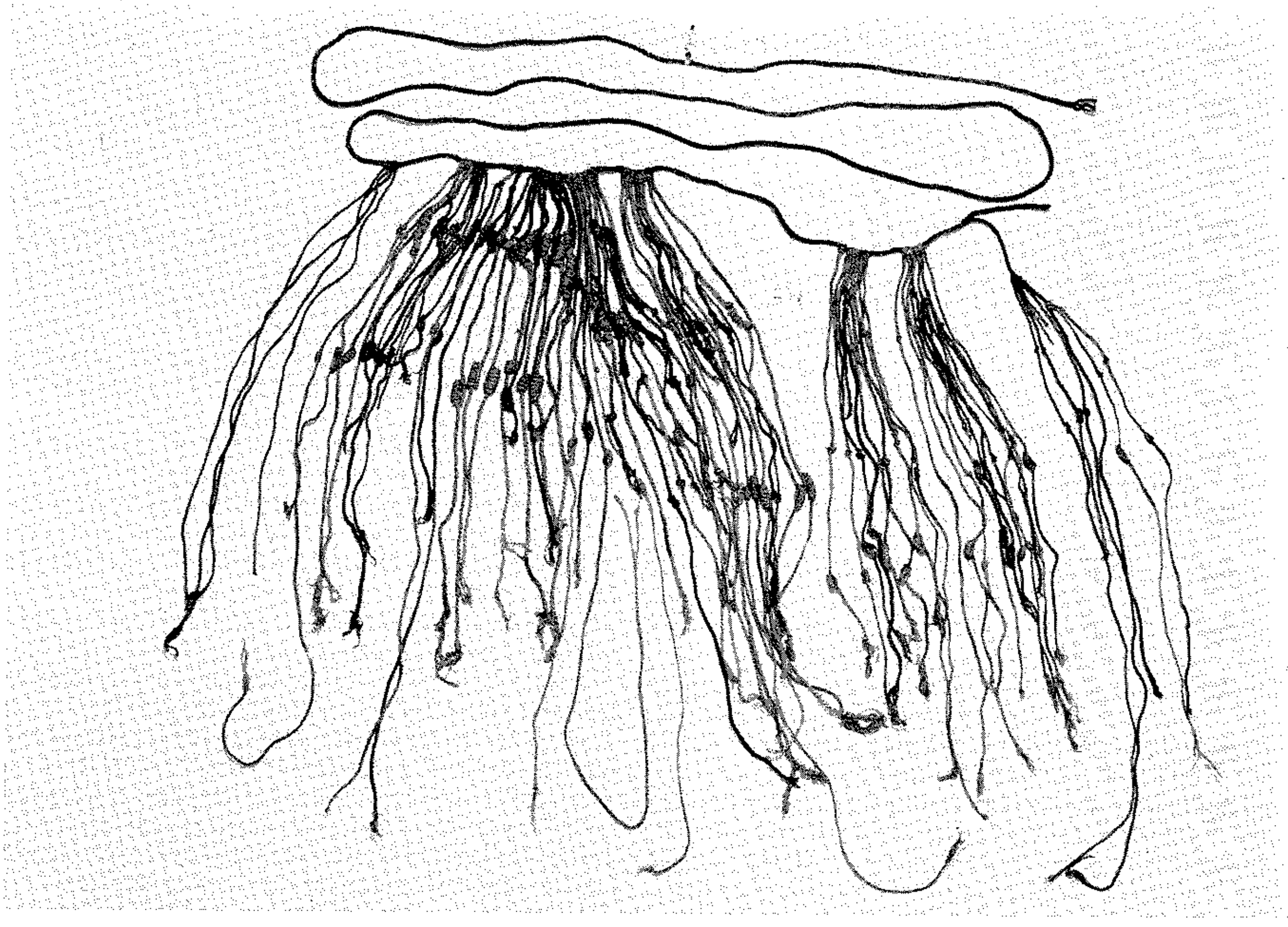

A quipu in the collection of the Peabody Museum of Archeology and Ethnography, Harvard University
Photo by Marcia and Robert Ascher