Intuition and Proof *

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1. Intuition and common sense

The commonsense interpretation of intuition is that intuition is commonsense. In other words, people are inclined to equalize the terms "intuitive interpretations", "intuitive representations" with "commonsense interpretations", "commonsense representations", etc. Instead of saying "Intuitively, space appears as three-dimensional," you may equally well say "the commonsense representation of space is three dimensional".

The fact that this equalization is partly correct has probably blocked the psychologists' interests in investigating the phenomenon of intuition.

The feeling of intuitive evidence (intuitive understanding, intuitive clarity, intuitive acceptability) seems to be so primitively, intuitively, clear to commonsense that any need to investigate what intuitive clarity means is blocked from the beginning.

This affirmation probably holds for psychologists too They frequently use terms such as "intuitive thinking", "intuitive explanations", "intuitive model", etc., without feeling the necessity to define and to explain this usage The main argument presented by psychologists, when asked why intuition is neglected by psychological research, is: the term *intuition* covers a great variety of different phenomena. It is, in fact, a commonsense notion By trying to use it as a scientific concept you will only render the problem you are investigating still more obscure

This is a faulty argument. Terms like imagination, intelligence, sentiment, character, etc., are likewise drawn from everyday language, and each covers a large variety of different meanings. Despite this, all these terms have reached a more-or-less scientific status in psychology. They have constituted the object of many investigations and have their honourable place in textbooks.

In fact, intuition has not yet found its defined place in psychology, not because it is an obscure term, but, on the contrary, because it is implicitly considered to be a primitive, self-evident term

"Intuitively, the shortest way between two points is the straight way". This statement appears absolutely clear One needs no further explanation. The term "intuitively clear" itself appears, in that context, intuitively clear. Intuition – in the sense of intuitive acceptance – appears as something so primitively obvious to commonsense that no need is felt to investigate, analyze, or interpret this term. We do not feel any need to explain what we mean by terms like "red", "between", "I feel", "I am sorry", "I see", "straight line", etc. They have, for us, a full intuitive meaning. It seems to be the same with a term like "intuitive understanding", even for most psychologists.

The main impulse toward, and the most interesting suggestions concerning the necessity for, investigating the intuitive forms of knowledge have their origins in mathematics, in science and, more specifically, in mathematics and science education.

Each effort to improve the structure of a branch of mathematics or of science is related to the effort to improve its formal, rational, and, finally, axiomatic structure. A scientist is always striving to define, to describe, to explain general features and regularities as objectively as possible. As far as possible he tries to exclude from his descriptions, interpretations and predictions, his subjective non-analyzable contribution Therefore in relation to each basic scientific interpretation he is naturally led to compare his and others' commonsense interpretations with scientific findings Each new discovery that contradicts an intuitively well-accepted truth will increase the scientist's efforts to be more careful with his interpretations. The progress of science has always been related to a permanent endeavor to liberate the views of the scientist from his own primary, intuitive representations and interpretations.

This tendency is still more obvious in the history of mathematics Efforts to formalize completely the structure of mathematics have led the mathematician to strive to detect, and, as far as possible, to eliminate all those aspects which have only an intuitive expression or justification. This is a permanent and difficult struggle which the mathematician has to conduct with his own mind. Yet the problem of intuitive interpretations and intuitive biases appears more acute in science education and mathematical education. One cannot simply say: "The intuitive interpretations and representations of the child do not bother me. I shall teach him good, rational science, or pure, axiomatized mathematics, and he will learn them as I teach them".

Every teacher knows that the pupil is not a passive receiver of information and of solving procedures. The Piagetian concept of assimilation is essential for describing and explaining the process of learning in relation to that of development. We learn, we understand, and we use information by processing and integrating it according to our own mental schemas hierarchically organized.

Mental schemas are not rigid. By lasting and laborious processes the mental schemas accommodate themselves, in the long run, to the features of real situations and become progressively more fit to manage them and to solve the problems with which we are faced.

Each period of mental development is characterized by a system of basic mental schemas which determine the capacity of the child to learn, to interpret, and to use the information he gets.

For instance, according to the Piagetian theory, an eight year old child is not able to learn thoughtfully the meaning and the use of mathematical proofs because, at that age, he has not yet mastered the required mental qualities: hypothetic-deductive thinking, full capacity to produce and prove assumptions, second order mental operations, etc.

But the Piagetian theory – which is highly resourceful in explaining the schemas and the development of analytical thinking – is rather silent with regard to the implicit, non-accountable forms of cognition and of productive thinking. An intuitive representation of a phenomenon seems to be, according to that theory, only a substitute before a complete formal understanding becomes possible. [See Beth and Piaget, 1961, pp 223-41]

Our view is that the intuitive structures are essential components of every form of active understanding and of productive thinking. They may constitute the only or the main forms of knowledge if the corresponding analytical structures are lacking (as is the case during the period of intuitive thinking, in Piagetian terminology). But the role of intuitive structures does not come to an end when operational analytical forms of thinking become possible. As we have already said, they are, in our opinion, indispensable components for every form of productive thinking. But to accept this point of view we must extend the meaning of intuition far beyond the commonsense interpretation of intuition as simply commonsense. We must deepen our understanding of its role and of its mechanisms. We must accurately define its qualities and its connections with other related psychological phenomena such as perceptual and mental representations, mental skills, personal experience, etc.

We have explored these topics in earlier papers and do not wish to dwell on them here; we give only a brief summary of some of our views below.

2. The nature of intuition

Let us first refer to the distinction between *anticipatory intuitions* and *affirmatory intuitions*.

Anticipatory intuitions have been investigated and described by psychologists in relation to problem solving. O. Selz [1922] used the term "anticipatory schemes". The Gestaltists have frequently used the term "insight" and others, like Bruner [1965, p. 55-68] and Wescott [1968], have used the term "intuitive" – or "intuitive thinking" – for describing almost the same phenomenon. The fact to which they refer is that, while striving to solve a problem one suddenly has the feeling that one has grasped the solution even before one can offer any explicit, complete justification for that solution.

However there is also a second use of the term "intuition". We speak about "intuitive understanding", "intuitive meaning", "intuitive interpretation", "intuitive models", "intuitively accepted truth". Piaget uses the adjective "intuitive", generally, in this way.

In this case, we are referring to a representation, an explanation or an interpretation directly accepted by us as something natural, self-evident, intrinsically meaningful, like a simple, given fact. If one asks a child, "Can you tell me what a straight line is?" he will try to draw a straight line or he will offer the example of a well-stretched thread. He will not feel the need to add something which could complete or clarify the notion (for instance an explanation, a definition, etc.). The concept of a "line" is intrinsically meaningful for the child. We may say that the concept of a line has, for the child, an intuitive meaning. When one asks a child, "What is the shortest way between two points?" he will answer, without any hesitation: "The shortest way between two points is a straight line". This statement appears to the child as obvious as the notion of line itself.

When we claim that the psychologists have neglected the concept of intuition we refer especially to the category of intuitions that we have termed "affirmatory intuitions". *The fact that there are representations, notions, interpretations, statements, which are accepted directly as intrinsically meaningful while others are not, has not been investigated by psychological research.*

The feeling of intuitive evidence itself (intuitive understanding, intuitive clarity, intuitive acceptability) appears as so primitively, intuitively clear to commonsense that any need to investigate what intuitive evidence means is blocked from the beginning.

Our claim is that the quality of self-evidence, which is the basic characteristic of an intuition – of an intuitively accepted truth – is itself only a pseudo-self-evident concept. Let us examine some examples.

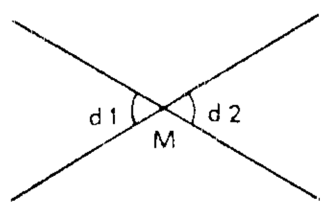

Figure 1

"Two lines intersect in point M. Are the angles d_1 and d_2 equal?" The answer is doubtless "Yes". In a previous research carried out by us this question, and the respective answer, scored very high for "self-evidence".

This is not so with statements like: "The sum of the angles of a triangle is 180°", "The three altitudes of a triangle intersect at a point", "The sum of the squares of the cathetae in a right triangle is equal to the square of the hypotenuse",etc.

These theorems are not self-evident. We are rather surprised when learning them. We are surprised by the strange possibility of such invariant properties in essentially variable geometrical structures. But we learn the proofs and we become convinced that the above statements are genuine mathematical truths. We are fully convinced that the theorem of Pythagoras or the "sum of the angles of a triangle" theorem, etc., express undoubted mathematical truths. If we know more about various geometries we will add "in the domain of Euclidian geometry" but this does not change anything with regard to our formal conviction. In order not to complicate our discussion, let us come back to the pupil who has not yet heard about non-Euclidian geometries.

The basic fact being stressed is that there are frequent situations in mathematics in which a formal conviction, derived from a formally certain proof, is NOT associated with the subtle feeling of "It must be so", "I feel it must be so".

We have, then, three different kinds of convictions. One is the formal extrinsic type of conviction indirectly imposed by a formal (sometimes a purely symbolical) argumentation. The second is the empirical inductive form of conviction derived from a multitude of practical findings which support the respective conclusion. The third is the intuitive intrinsic type of conviction, directly imposed by the structure of the situation itself. In this last case the term "cognitive belief" seems to be very appropriate. The term "belief" expresses the direct, the sympathetic, form of knowledge, the feeling of the implicit validity and reliability of the respective representations or interpretations and their extrapolative capacity. One may be convinced that a statement is true – because one completely agrees with the correctness of the proof – but, at the same time, one may be surprised, even upset, by the same statement.

Let me recall the example of the 13 year old girl who learned about the equivalence of the following two sets: the set of natural numbers and the set of positive even numbers. After learning the proof she exclaimed: "The proof is irreproachable but it seems so strange! It is very difficult to accept that there is a one-to-one correspondence between these two sets".

On the other hand, a certain statement may appear fully evident, as directly acceptable, though one knows one does not possess any proof for it. *Such a situation, in which one fully and subjectively accepts a statement without possessing a formal proof for it, creates that specific feeling of belief.* Of course, one may have a third category of situation where a proved statement appears as an intuitively acceptable truth as well. It appears intuitively clear that the diagonals in a rectangle are equal, that the shortest way between a point and a line is the perpendicular drawn from the point to the line, etc. At the same time these statements can be proved, although no proof is needed, and, as a matter of fact, seems rather superfluous.

Let us broaden the frame of this discussion with regard to the relations between these two basic ways of knowledge – the analytic and the intuitive.

Three main standpoints can be schematically described:

– The Bergsonian approach: intuition and intelligence are two distinct, even opposite ways of knowledge. Intuition offers a sympathetic knowledge of movement, of life, of spirit, of everything which implies continuity in time. Intelligence (analytical knowledge) is adapted to action, to solid bodies, to spatial discontinuities.

– The Piagetian approach: intuition and operational thinking constitute two different levels in the process of knowledge. Intuitive representations are based on global configurations while operational thinking is constituted of composable and reversible analytical structures. In this approach intuition appears as an inferior form of knowledge in comparison with operational (logical) thinking.

– The behavioral (or synthetic) approach: the intuitive and the analytical forms of knowledge are complementary and deeply interrelated. They are two facets of an unique mental productive behavior.

This third one is our standpoint. Let us make it more explicit.

What characterizes analytical (or operational or logical or symbolic) thinking is, essentially, its explicitness. The symbolic thinking of human beings takes advantage of the enormous experience accumulated by society. That experience is preserved, generalized and transmitted by linguistic means. As a consequence, the analytical and conceptual structure of logical thinking enables us to reach rigorous and safe conclusions. Symbolic behavior represents, of course, an enormous progress in the development of knowledge.

But at the level of practical current behavior (either mental or external) this highly reliable cognitive instrument presents an essential drawback: its lack of immediacy, of direct effectiveness. In practical situations we need global, rapid evaluations which can promptly elicit effective adapted reactions.

Consequently we submit that the essential function of (intellectual) intuition is to be the homologue of perception at the symbolic level, having the same task as perception: to prepare and to guide action (mental or external).

Intuition is simultaneously a derived form of cognition – as thinking is – and a programme for action – as perception is. Intuition and perception have essential common features and for this reason the term *intuitive knowledge* is sometimes used to designate both categories. Both are global, direct, effective forms of cognition.

The difference between intuition – as a specific form of knowledge – and perception is that intuition does not directly reflect objects or events with all their concrete qualities. Intuition is mostly a form of *interpretation*, a *solution* to a problem, i.e. *a derived form* of knowledge, like symbolic knowledge. On the other hand the difference between intuition and analytical thinking is that intuition is *not* analytical, not discursive, but rather a compact form of knowledge like perception, and, again like perception, it does not require extrinsic justification. With perception we have the feeling of being plunged directly into the world of material objects; perception appears to be reality itself rather than mere appearance With intuition we have the same feeling – of being *in* the object and not a mere interpreter of it

Being a derived form of knowledge, like analytical thinking, intuition is able to organize information, to synthesize previously acquired experiences, to select efficient attitudes, to generalize verified reactions, to guess, by extrapolation, beyond the facts at hand The greatest part of the whole process is unconscious and the product is a crystallized form of knowledge which, like perception, appears to be self-evident, internally structured and ready to guide action

In its anticipatory form, intuition offers a global perspective of a possible way of solving a problem and, thus, inspires and directs the steps of seeking and building the solution. In its conclusive form the role of intuition is to condense – again in a global compact manner – an analytical solution previously obtained. In this form, too, the role of intuition is to prepare action That final concentrated interpretation is destined to make the solution directly useful in an active, productive, thinking process

An intuitive conviction is extrapolative, very much like an *a priori* conviction. At the same time, it includes the feeling of being factually meaningful – as an inductively supported conviction does

What distinguishes the intuitive form of conviction from both the formally and the inductively based, is its *intrinsic character*: no direct need is felt to prove formally or factually the respective interpretation or representation

With regard to the *origin* of intuitions, let me outline only a few remarks. Very frequently, intuitions are considered as mere mental skills. I agree that intuitive forms of acceptance, interpretation or solution are based – at least partially – on certain mental skills. Moreover, I consider that, generally, intuitions are deeply rooted in our previous, practical and mental, experience But *intuitions are not mere skills* The formula of solving a quadratic equation may become, through practice, a skill without resulting in the structure of an intuition. An intuition is a *form of knowledge*. It has the role of a programme of action – but it is a cognition, a representation: drawing a line and *the intuitive form of understanding what the concept of a line means* – are not the same thing. Knowing and using the distribution law: $a\,(b + c) = a\,b + a\,c$ and intuitively understanding the same law are not, of course, the same. In order to eliminate typical errors, such as $\sin(A + B) = \sin A + \sin B$ it is not sufficient to practise the corresponding trigonometric formula. The simpler and widely used distribution law will always endanger the memory of the correct trigonometric formula (if it is assimilated only as a skill). In this case, too, a structural intuitive understanding of the meaning of each of the formulae is necessary.

An intuition cannot be built by mere verbal explorations nor by blindly practising solving procedures. An intuition can be elaborated only in the frame of practical situations as a result of the personal involvement of the learner in solving genuine problems raised by these practical situations The same happens with the child in his early years when a large variety of practical genuine problematic situations shape its basic intuitive representations of space and time, of objectual identifications, of quantity and chance estimations, etc.

The same must happen if new intuitions are to be built in the process of the intellectual education of the pupil, in his school years. Every problem used must imply some request for a global, predictive, extrapolative interpretation having a behavioral meaning

For instance, in order to create new correct probabilistic intuitions the learner must be actively involved in a process of performing change experiments, of guessing outcomes and evaluating chances, of confronting individual and mass results with *a priori* calculated predictions, etc. New correct and powerful probabilistic intuitions cannot be produced by merely practising probabilistic formulae. The same holds for geometry and for every branch of mathematics.

3. The role of intuition in mathematical activity

The main practical problem posed by this paper is: *what should be the attitude of the mathematics educationist with regard to the relation between intuitive and analytical ways of thinking?*

Since mathematics is in essence abstract, since it is essentially logical, since a mathematical truth is equivalent to a pure formally proved statement, why do we need to bother with intuitive representations and interpretations? We must strive to get rid of them as soon and as completely as possible.

My point – which of course, is not original – is that such a complete "purgation" is not possible and not desirable. Obviously we refer to the psychological processes of understanding and creating mathematics and not to the final product, which must be "as pure as possible". The qualities of the product must not be confounded with those of the process of producing it

As Morris Kline [1970] excellently put it:

> There is not much doubt that the difficulties the great mathematicians encountered are precisely the stumbling blocks that the students experience and that no attempt to smoothe these difficulties with logical verbiage will succeed If it took mathematicians 1000 years from the time that first class mathematics appeared to arrive at the concept of negative numbers, and it did, and if it took another 1000 years for mathematicians to accept negative numbers, as it did, we may be sure that students will have difficulties with negative numbers Moreover, the students will have to master these difficulties in about the same way that the mathematicians did, by gradually accustoming themselves to the new concepts, by working with them and by taking advantage of all the intuitive support that the teacher can master. [p. 270]

It is impossible to explain the ways of thinking in the history of sciences and in the ontogenesis of intelligence without taking into account the lines of force created by these cognitive beliefs.

The formal logical structures, the various mental schemas, up to the operational ones, are only instruments, only virtualities The fact of knowing the four basic arithmetical operations does not enable the child to solve arithmetical problems. Of course, for solving a standard question the child resorts automatically to standard solutions. But such a question is no longer a problem The fact of knowing the English vocabulary, grammar, various syntactical structures, does not itself enable one to write beautiful poems. The Piagetian theory is a fine description of intellectual instruments But if we want to know *how* and *why* and *how successfully* the thinking activity proceeds, following a certain way rather than another, *we must take into account those internal forces created by the existing cognitive implicit beliefs of the person concerned.*

A pure conceptual structure has not, in itself, the orienting and inspiring power requested by a productive mental process. The processes of discovering, organizing and using information are largely influenced by the personal, sympathetic attachment characteristic of intuitions, by the kind of "tacit knowledge" so beautifully described by Michael Polanyi [1969]. Although we sometimes question their validity, pre-conceived tacit interpretations are very frequently taken for granted in the history of science and in the course of individual development Their influence on the ways of posing and solving problems may be considerable.

Comparing the process of a "tacit inference" with the concept of perception, Polanyi writes,

> We have noted and analysed how perception is performed by straining our attention towards a problematic center, while relying on hidden clues which are eventually embodied in the appearance of the object, recognized by perception. This, I suggest, is also how the pursuit of science proceeds; this is the unaccountable element which enters science at its source and vitally participates throughout even in its final result. In science this element has been called intuition The purpose of this paper is to indicate that the structure of scientific intuition is the same as that of perception. Intuition, thus defined, is not more mysterious than perception – but not less mysterious, either Thus defined it is as fallible as perception, and as surprisingly tending to be true. [Polanyi, 1969, p 118]

Coming back to the elementary level of pupils learning science or mathematics, we may simply say this: *When considering the possible impact of a body of information or of a pattern of mental procedures on the dynamics of productive thinking, we must take into account the kind and the strength of the credibility attached to them by the learner.*

We naturally tend to interpret a piece of information in such a way as to enable us to believe in it. *To believe in certain representations or interpretations means that you feel that they appear to you as being behaviorally meaningful.* This of course, depends on our previous, general and specific experience with respect to the given facts.

If for various reasons we do not succeed in reaching such a personal commitment, we tend to forget the fact concerned, or else its impact on the productive ways of thinking tends to be minor If we do succeed in realizing such a personal commitment, it is the specific, credible form – which may be correct or not – that will be remembered later, that will be active and that will direct our ways of thinking

For instance, people consider, generally, that the force of friction depends on the magnitude of the surfaces in contact, though they have learned the contrary (with regard to common, everyday situations).

The personal interpretation may be a simple fiction But it has been proved in the history of science that we may believe in concepts or interpretations which someday will appear as mental constructs without any objective correspondent (like, for instance, the "impetus", the "phlogiston" the "ether", etc.). The famous postulate of Euclid has been considered intuitively – and erroneously – for about 2000 years as expressing an absolute truth. It represents a basic component of Euclidian geometry, which itself was considered by Kant as apodictically certain as a whole (i e as composed of *a priori* synthetic statements).

The Aristotelian concept of infinity – which considers infinity only as a potentiality – dominated mathematics until the Cantorian era and has amply influenced its developments As a matter of fact, this concept expresses our basic, natural, intuitive interpretation of infinity. It represents a compromise between the finitist nature of our logical schemas and infinity itself A *potentially* infinite object (for instance, a line which can be extended indefinitely) has a behavioral meaning A potentially infinite operation has a behavioral meaning as well (for instance, dividing indefinitely a line segment)

An actual infinity has no behavioral meaning and therefore it is not congruent with an intuitive, insightful interpretation.

In a previous research, we asked high school pupils the following question: "Given a segment $A\ B = 1\ m$. Let us add to $A\ B$ a segment $B\ C = 1/2m$. Let us continue in the same way, adding segments of $1/4m$, $1/8m$, etc (A) Will the process of adding segments come to an end? (B) What will be the sum of the segments $A\ B + B\ C + C\ D$ . . . etc?

To the first question 84% of the subjects answered that the process will not come to an end, and this answer scored very high on our scale of intuitiveness. To the second question we get the following distribution of answers: S = 2 (correct): 5.6% and a very low level of intuitiveness; S= (erroneous): 51.4% and a very high level of intuitiveness. S approaches 2: 16 8%, and a high level of intuitiveness. The sum of an infinity of segments is naturally considered as being equal to infinity. The correct answer is given by only a very small number of subjects simply because it is counter-intuitive. Some of the subjects found a compromise, intuitively acceptable, and this is: the sum *tends* to 2, which corresponds to the interpretation of infinity as a potentiality

Returning to mathematics education, let us consider some more examples. The first refers to what may be termed as *active understanding*.

Dr. Alan Bell, of the Shell Center for Mathematical Education in Nottingham, once gave me some examples with regard to his recent researches There were a set of problems to be solved like: 3 + 7 = 22 The subjects had no idea about algebraic operations For them such a question was a real problem. There were various categories of questions according to the degree of difficulty I asked Dr. Bell why he did not simply teach the children the corresponding algebraic rules of computation? Sooner or later they will learn such rules. Was it not a waste of time to train the children in this way? Teaching the rules does not mean teaching them blindly: they will get the rational, formal proof for each rule. (For instance in an equation of the form $a + b = c$, b can be moved to the right side of the equation by changing its sign from + to – This must be based on the known principle that "As a result of extracting two equal quantities from two equal quantities, we obtain again equal quantities".

Dr. Bell's answer was, "I want to given them a basis of belief" I like this expression very much and would be ready to use it instead of "intuition" or "intuitive acceptance". Despite this, I prefer "intuitive acceptance". The reason for this is that the term "belief" is too strongly connected with religious matters and may suggest the notion of an unprovable credo. The formal inversion in the above example ("+ b" is transformed to "– b") may be formally learned and formally accepted. *But in order to reach "a basis of belief" you must live the process.* It must be the effect of some behavioral, personal (mental or external) involvement. This is what I mean by getting an intuition, an intuitive understanding of a mathematical truth

Let us consider the problem: "John has a sum of money He wants to buy a book but the money he possesses is not sufficient Mother gives him 35 pennies So he has the 67 pennies he needs. What was the sum he had at the beginning"?

If John invents, by himself, the solution he will get the "basis of belief" for replacing the apparent addition with the practically useful – and of course – correct subtraction. He has to imagine the whole situation in order to understand behaviorally, sympathetically, that he has to subtract and not to add, as it might appear from the verbal formulation of the problem In my opinion, if the formal rule can be learned on such a previously created "basis of belief", the whole acquisition will gain enormously in efficiency and durability.

Let us also recall the concept of proportion The concrete operational child gets an intuitive understanding of proportional dependencies before reaching a formalized version of them. In our opinion, even at the formal generalized level of understanding, at the formal operational stage, there is need for a complementary intuitive understanding of the entire structure. The synthesis between negation and reciprocity, the inverse effects determined in a fraction by multiplications (or divisions) operated on the numerator and respectively on the denominator, the notions of direct and inverse proportionality, etc , must acquire for the child a behavioral – global, synthetic – meaning. Such an intuitive meaning has the same basic structure as that of a behavioral understanding, for instance, of the rules which govern the dynamical equilibrium of a balance.

Briefly, in our opinion, even at the level of purely formal concepts, productive thinking needs such forms of intuitive sympathetic representations The problem for the pedagogy of mathematics is not to exclude or to avoid them – which is not possible – but, rather, to refine them and to bring them under a complete conceptual control.

What I want to emphasize is that such an insightful understanding (based on integrative, manipulative, sympathetic forms of mental activity) is possible and necessary at each level of mathematical thinking even at the level of axiomatic construction. Let me add two quotations The first is from David Hilbert:

> Who does not always use, along with the double inequality $a > b > c$, the picture of three points following one another on a straight line as the geometrical picture of the idea "between"? Who does not make use of drawings of segments and rectangles enclosed in one another when it is required to prove with perfect rigour a difficult theorem on the continuity of functions of the existence of points of condensation? Who could dispense with the figure of the triangle, the circle with its center or with the cross of the three perpendicular axes? Or would give up the representation of the vector field or the picture of a family of curves or surfaces with its envelope which plays so important a part in differential geometry, in the theory of differential equations, in the foundations of the calculus of variation and in other purely mathematical sciences? [cf Reid, 1970, p 79]

And now a quotation from Paul Cohen:

> The most natural way to give an independence proof is to establish a model with the required properties This is not the only way to proceed since one can attempt to deal directly and analyze the structure of proofs. However such an approach to set theoretic questions is unnatural since all our intuition come from our belief in the natural, almost physical model of the mathematical universe [Cohen, 1966, p 107]

As Felix Klein says: "The investigator in mathematics, as in every other science, does not work in this rigorous deductive fashion. On the contrary, he makes essential use of his imagination and proceeds inductively aided by heuristic expedients" [F. Klein, *cf.* M Kline, 1970, p. 274].

My view is that the proof itself – with its step-by-step analytical explicit structure – can and must reach the level and the form of an internally coherent synthetic grasp And this is an intiution. Let us quote Poincaré [1913]: "In the edifices built up by our masters of what use is to admire the work of the mason if we cannot comprehend the plan of the architect? Now pure logic cannot give us the appreciation of the total effect; this we must ask of the intiution " [p. 217]

Let us recall Wertheimer's [1945] famous example of the area of a parallelogram.

A child who has learned the correct solution by a step-by-step proof may miss the genuine understanding of it If the parallelogram is presented as in Figure 2 he may not be able to identify the correct formula.

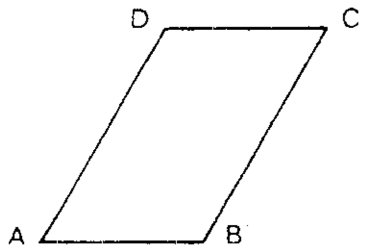

Figure 2

On the other hand, a child who has grasped that the parallelogram can be transformed into a rectangle (for instance, by using scissors) seems to acquire a genuine understanding of the equivalence between the two areas (the parallelogram and the rectangle having the same altitude and the same base as the parallelogram).

In fact, there is no contradiction between grasping the behaviorally meaningful solution "taking from the left and adding to the right" and following the steps of the logic of the analytical proof. When reaching or learning a formal proof the pupil must not abandon the structural way of understanding. Each step of the proof can be understood as a necessary part of a whole (the general idea of the proof) the meaning of each part being determined by its relations with the whole. For instance, in the example of the parallelogram: "We have then to prove that the triangles *D A E* and *C B F* are equal, etc" In other terms, introducing structurality in a proof means simply grasping the proof as a meaningful coherent whole, *hierarchically organized*.

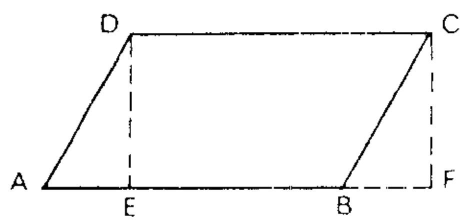

Figure 3

Let us again quote Wertheimer himself: "Someone objects – Why do you speak of grasping inner structure, inner requirements, implying that a grasp of the structural feature in your example makes the thing reasonable? What of non-Euclidian situations? What if we choose other axioms for our geometry? What is reasonable in one system may be wrong in the other. Only as long as one presupposes the naive, old fashioned belief in the unique validity of the Euclidian axioms does what you say seem reasonable".

Wertheimer continues: "The objection is blind: it does not touch the issue. Non-Euclidian geometry has structural features of its own, to which again considerations of reasonableness apply within the new, enlarged framework [p. 36]

The main point, in my opinion, is that the logical form of necessity which characterizes the strictly deductive concatenation of a mathematical proof can be joined by an *internal structural form of necessity* which is characteristic of an intuitive acceptance. Finally, both can blend in an unique synthetical form of mathematical understanding.

Sometimes there is a natural coincidence as, for instance, in the case of the theorem, "Two opposite angles, determined by two intersected lines are equal". You can directly "see" that the two opposite angles have the same complementary angle. You can "see" and you can also calculate.

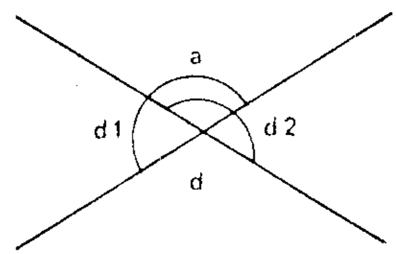

Figure 4

$\measuredangle d_1 + \measuredangle a = 180°$
$\measuredangle d_2 + \measuredangle a = 180°$
And therefore $\measuredangle d_1 = \measuredangle d_2$

But very often the formal and the structural types of necessity do not coincide naturally. My point is that such a fusion can always be attained – by using adequate didactic procedures – and that it is always desirable.

4. The concept of mathematical proof

There are, in principle, two basic ways of proof. If we are considering factual realities the proof consists mainly in producing or observing facts which will confirm the claim expressed in the respective statement. Our conviction of the validity of the statement will grow stronger as we become able to produce more facts which will fit the statement. With reference to mathematics the way of proving is different: the statement we consider must be the logical, necessary conclusion of some other previously accepted statements.

Such a proof is not based on a multitude of very well-controlled practical results. The validity of the theorem is guaranteed only by the validity of the formal logical inferences used. If one affirms: "The proof of the theorem completely satisfies me", one cannot any more claim that new facts fitting the statement will increase the conviction of its validity. The universality of the truth expressed by the theorem is guaranteed by the universal validity of the logical rules used in the proof.

Of course, one may claim that the proof is not satisfactory. Maybe some mistake has been made, maybe some implicit affirmations have been taken mistakenly for granted. In this case further checks are justified.

On the other hand, since we are acting in mathematics only with properties and operations completely and explicitly defined, we are also able to determine, from the very beginning, the exact level of generality to which we refer, no matter which particular example we are using. A drawn triangle is always a completely determined particular triangle but, despite this, we are able to neglect the size of its sides and angles and to consider only the triangularity.

The level of generality of the theorem is then explicitly defined by the theorem itself and the proof refers exactly and clearly to that level of generality. Therefore one cannot logically affirm: "I am sure that the proof of some mathematical statement is complete, is irreproachable" and to claim, *at the same time*, that the analysis of further examples will strengthen confidence in the general validity of the theorem. These two affirmations are logically contradictory.

How shall we qualify a pupil's answer representing such a situation? We can simply say that this pupil does not understand what a formal (mathematical) proof means. And this is true. But, in my opinion, such a situation has much deeper implications.

Let us, briefly, describe a research which may cast some light on that problem. Our main question was: how far are high school pupils – with an advanced training in mathematics – aware of the profound distinction between an empirical proof and a formal (logical, mathematical) proof? About four hundred pupils enrolled in three Tel Aviv high schools (grades 10, 11, 12) were questioned. They were presented with a mathematical theorem, the complete proof was given, and afterwards they had to answer various questions referring to the validity of the theorem. For half of the pupils the theorem used represented a numerical relation, and for the other half a well known geometrical statement.

We shall refer, here, only to the numerical example. The subjects were given the expression: $E = n^3 - n$ and were asked to find if this expression is divisible by 6, first for numbers (positive integers) from 1 to 10 and, in continuation, for integers greater than 10. After these practical checks, the pupils were presented with the following questions: "Dan claims that the expression $E = n^3 - n$ is divisible by every n and he gives the following proof:

$$n^3 - n = n(n^2 - 1) = n(n+1)(n-1) = (n-1)n(n+1)$$
i.e. $n^3 - n = (n-1)\,n\,(n+1)$

Thus we have obtained the product of three consecutive numbers ($n-1$, n, $n+1$). Among three consecutive numbers there is *always*, at least, one number divisible by two and there is *always* one number divisible by three. Therefore the product is divisible by 2 and by 3. The product of three consecutive numbers will then be divisible by $2 \cdot 3 = 6$.

The subjects were then asked: (a) Is Dan right when affirming that the expression $E = n^3 - n$ is divisible by 6 for every n?; (b) Is the proof given by Dan fully correct?; (c) Is the generality of Dan's affirmation determined by the proof he gave?

Let us first consider the results of these three last questions: (a) 68.5% claimed that the expression $E = n^3 - n$ is always divisible by 6; (b) 81.5% claimed that they checked the proof and have found that it is correct in every detail; (c) 60.0%: the generality of the theorem is guaranteed by the proof.

Logically these three affirmations must be considered as equivalent: to agree with the correctness of the proof; to agree with the theorem; to agree that the generality of the theorem follows from the (accepted) proof.

Instead, we find here a revealing cleavage in the pupils' answers, leading to an interesting hierarchy: 81.5% affirm that the proof is fully correct; 68.5% agree with the theorem; for only 60% of the subjects the generality of the proof is justified by the theorem. It was found that only 41% of the subjects answered all of these three questions correctly.

In continuation, the following questions were asked: "Moshe claims that he has checked the number $n = 2357$ and has found that $n^3 - n = 2357^3 - 2357 = 105\,514\,223$ and that this number is not divisible by 6. What is your opinion on the matter?"

Only 32% of the subjects consider that "it must be a mistake" (or "it is impossible"). It is also interesting to mention other categories of answers: "The theorem is true only for some classes of numbers" – 18%. "Moshe is correct: the number is not divisible by 6" – 6.5%. "This result refutes Dan's statement" – 15%. The others did not answer or gave irrelevant explanations.

The next question was: – "If you consider that Dan's proof is correct (referring to the theorem: $n^3 - n$ is divisible by 6 for every n, please answer the following question: "Do you consider that additional checks (with other numbers) are necessary in order to increase your confidence in the theorem?" Only 24.5% of the entire population accepted the correctness of the proof and *at the same time* answered that additional checks are not necessary.

Let us quote some categories of answers referring to the fact that an additional control is necessary: (a) "We have to check numbers in order to prove the theorem" (6.0%) (b) "An exception is always possible" (4.5%) (c) "The results depend on the category to which the number belongs" (3.5%). (d) "If we would continue to check, we would get more precise results" (4.5%), etc. "It would be wise to continue to check other numbers" (after accepting the theorem) 5.5%, etc. Many did not answer.

Finally we found that, of the entire population investigated, there were only 14.5% who were completely consistent in their correct answers and interpretations. What explains the fact that, while 41% of the subjects are consistent in accepting the validity of the theorem and its proof, only 14.5% of the subjects are consistent *right to the end* (i.e. do not feel the need for further empirical checks)?

A very simple explanation would be: Only 14.5% of the students really understand what a mathematical proof means. I fully agree, but this is only a first-step explanation.

Let us consider again the theorem and its proof. The statement is extremely simple: a certain expression is divisible by 6 for every n. What is the proof saying? The proof refers to the almost evident fact that among three consecutive numbers one, at least, is divisible by two and one is divisible by three. Why, then, do only a third of the pupils who fully accepted that the theorem was proved also consider that more checks would be useful? Logically these properties are equivalent.

For a mathematician such a discrepancy must appear rather surprising, but it can be explained, in our opinion, in the following way: *The two basic ways of proving – the empirical and the logical – are not symmetrical, they do not have the same weight in our practical activity.* For current adaptive behavior the main way of producing general representations and interpretations is that of empirical induction: as we succeed in accumulating more facts which confirm a certain general view, we become more convinced of its validity. The degree of belief depends, naturally, on the quantity and, of course, on the variety of confirmatory facts we succeed to accumulate.

The difference between the layman and the scientist in this matter is that the scientist is much more cautious in his generalisations and predictions. He is always ready to check new types of situations and he always tries to refine his techniques of analysis. But these improvements only continue the same main process of empirical knowledge. Certainly this is not yet the way the young child proceeds because he does not yet possess a clear notion of proof. He confounds his desires and his imaginative representations with real facts. But in the course of time the child, the "operational child" in Piaget's terminology, learns that a prediction is sometimes refuted. He learns, stepwise, to relate the validity of his assumptions to the amount of practical confirmations he can produce. His daily accumulated experience and the way science is taught in school help him in the same direction.

The ways of thinking of the formal operational child evolve in this direction. He is able to isolate the components of a given situation, to formulate assumptions related to these components, to search systematically for facts which will confirm or refute his assumptions. The whole process continues the main way which characterizes empirical adaptive knowledge. We can find its roots in the biological process of the elaboration and extinction of conditioned reflexes.

The concept of formal proof is completely outside the main stream of behavior. A formal proof offers an absolute guarantee to a mathematical statement. Even a single practical check is superfluous. This way of thinking, knowing and proving, basically contradicts the practical adaptive way of knowing which is permanently in search of additional confirmation. In principle, the formal structure of the adolescent's thinking possesses all the basic ingredients necessary for coping with both formal and empirical situations. Despite this, the current ways of trying and evaluating are mainly adapted to empirical contents. In solving a problem the mathematician proceeds, at the beginning, in the same way as the "empirical" scientist. He analyzes the given situation, he tries to identify some general properties, some invariant relations or dependencies, etc. But at a certain moment this search process stops and a new situation appears: the mathematician has found a complete proof for his solution or theorem. Such a proof is the absolute guarantee of the universal validity of the theorem. *He believes in that validity.* This is a new situation in relation to natural mental behavior. Naturally, intuitively, we continue to believe in the usefulness of enlarging our field of research, of accumulating more confirmation. To think means to experiment mentally. Mental experience is the duplicate of the practical trial-and-error goal-oriented process. Therefore the ideal, the perfect proof, has no meaning for the natural empirical way of thinking. In order to really understand what a mathematical proof means the learner's mind must undergo a fundamental modification.

Of course he can learn proofs and he can learn the general notion of a proof. But our research has shown that this is not enough. A profound modification is required. A new completely non-natural "basis of belief" is necessary, which is different from the way in which an empirical "basis of belief" is formed. *The concept of formal, noninductive, nonintuitive, non-empirical proof can become an effective instrument for the reasoning process if, and only if, it gets itself the qualities required by adaptive empirical behavior!*

In other terms: *formal ways of thinking and proving can liberate themselves from the constraints of empirical knowledge if they become able to include in themselves those qualities which confer on the empirical search its specific productivity.* We refer to the global, synthetic, intuitive forms of guessing and interpreting.

It is not enough for the pupil to learn formally what a complete, formal proof means in order to be ready to take complete advantage of that knowledge (in a mathematical reasoning activity). A new "basis of belief", a new intuitive approach, must be elaborated which will enable the pupil not only to understand a formal proof but also *to believe* (fully, sympathetically, intuitively) in the *a priori* universality of the theorem guaranteed by the respective proof. As in every form of thinking, we need, in addition to the conceptual, logical schemas, that capacity for sympathetic, direct, global acceptance which is expressed in an intuitive approach. After learning a formal proof we have to reach not only a formal conviction – but also the internal direct agreement which tells us: "Oh yes, it is obvious that the described property *must* be present in every object which belongs to the given category". *The feeling of the universal necessity of a certain property is not reducible to a pure conceptual format. It is a feeling of agreement, a basis of belief, an intuition – but which is congruent with the corresponding formal acceptance.*

Let us consider again the theorem: "The sum of the angles of a triangle is equal to two right angles".

Let us now draw a segment AB and the perpendiculars MA and NB to the segment. The angles MAB and NBA are right angles. We can "create" a triangle by "inclining" MA and NB. So, it can be seen that the angle APB "accumulates" what is "lost" by the angles MAB and NBA when "inclining" MA and NB. (Figure 5)

Of course this is not mathematical language. It is rather a story about lines and angles, but a story which can catch the spirit, which can impose itself as *intrinsically* true. The same story can be translated into the form of a mathematical proof. Consequently the formal necessity and the intrinsic necessity will coincide.

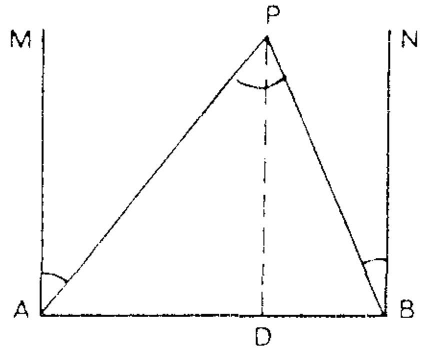

Figure 5

I exposed that procedure to a group of master's students. One of the students presented the following technique which, he said, is more srikingly intuitive and easier to understand. You cut a triangle out of a piece of paper. Let *ABC* be that triangle. Then you fold the triangles *ADE, DBG* and *EHC* so as to make the angles 1, 2 and 3 fit as in Figure 6. You can *see* that their sum equals two right angles.

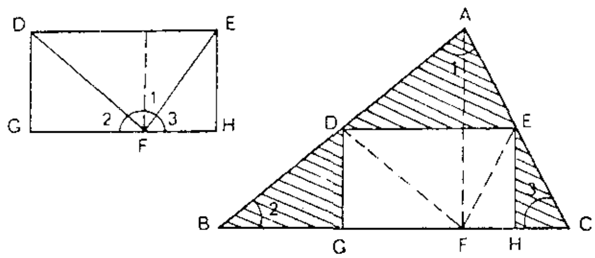

Figure 6

I do not agree with the above technique for the following reason. To "grasp intuitively" does not simply mean to "see". In the example with the sum of the angles of a triangle what you have to grasp is not merely that, in a particular case, by joining the angles *pratically* you will get two right angles. The problem is to grasp intuitively *why* that constant effect is *necessarily* conserved, imposed, in the variable conditions of a non-determined triangle. Intuitively it must be a problem of compensation. Therefore the matter is not of *showing* practically that in a particular example the angles fit as the theorem predicted. What we have "to see" is that, in *variable conditions, by way of compensation*, the sum *must* be conserved. And this, I think is better suggested by my procedure because (a) a triangle appears as a particular case in a changeable situation; (b) the compensation leading to a constant result can directly be grasped; (c) we are behaviorally involved not in merely collecting angles but, rather, in *a process of transformation* which leaves constant the sum of the angles; (d) this representation can be translated directly into a formal proof. The formal proof and the intuitive interpretation are perfectly congruent.

We have to consider three levels of intuitive acceptance.

A first level refers to the fact expressed by the statement itself. For instance, in the previous numerical example, the pupils seem to have a clear understanding of what the statement means ($n^3 - n$ is divisible by 6 for every n). But they may or may not agree, intuitively, with the affirmation expressed by the statement (i.e. they may or may not consider it as intrinsically obvious).

A second level refers to the structure of the proof. A pupil may intuitively grasp the meaning of a theorem but he may not be able to grasp intuitively the structure of the respective proof (though he is able to memorize and to understand formally its steps). For instance he may intuitively accept the following statement: "If a line l_1 not contained in plane P is parallel to a line l_2 contained in plane P then l_1 is parallel to the plane P". In contrast he may have some difficulty in grasping intuitively the sense of the proof (though he may be able to reproduce its steps).

The third level refers to the fact of understanding the universal validity of the statement as guaranteed and imposed by the validity of the proof.

As we have seen in our numerical example, most of the pupils understood the theorem. They understood the proof and seemed to accept its validity (based on the fact that the product of three consecutive natural numbers must be divisible by 2×3). And yet many of the same pupils expressed the view that more practical empirical checks would increase their confidence.

Formally, there is no difference between accepting the correctness of a mathematical proof and accepting the universality of the statement as guaranteed by that proof. The fact that, for the pupil, there is a difference between accepting a proof and accepting the universality of the statement proved by it, demonstrates that an additional element must be taken into consideration.

That additional element is constituted by *the need for a complementary intuitive acceptance of the absolute predictive capacity of a statement which has been formally proved*. This is only a particular case of a more general hypothesis discussed in the previous pages, namely: a mathematical truth can become really effective for productive mathematical activity if, together with a formal understanding of the respective truth, we can produce that kind of synthetic, sympathetic, direct acceptability of its validity. The same holds for a concept, a statement, a proof, and for the basic principles of generalizing, deducing and proving in mathematics. Consequently the pupil's mind has to synthesize two contradictory requirements into one mental structure. On the one hand it is necessary for him to keep full conceptual control of the facts under consideration. On the other hand, mathematical reasoning must preserve the qualities of mobility and empiricity which are the *sine-qua-non* conditions for productive mental activity. It may be supposed that the child – who admits the correctness of

Continued on page 24