BEYOND PROPORTIONS: CONCEPTUALISING PROPORTIONAL REASONING

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If you ask someone you know what proportional reasoning is, the answers will undoubtedly relate to the idea of the cross product or something similar to the search for a fourth proportional. Although these answers are not necessarily wrong, they might seem highly unsatisfactory and reductive. Many consider proportional reasoning to be at the heart of school mathematics, describing it, in the words of Lesh, Post and Behr (1988), as both the cornerstone of elementary mathematics and the steppingstone to secondary mathematics. In this sense, proportional reasoning must be much more than a procedure or a simple search for a missing value! The aim of this paper is to explore this 'much more', and to offer a conceptualisation of what characterises proportional reasoning and what it implies.

Since the world of proportionality is vast and diverse, a few clarifications are needed. These act only as general starting points, which are refined throughout the paper. There are several types of proportionality (*e.g.,* direct, inverse, quasi-, multiple, linear). The conceptualisation developed in this paper focuses on direct proportionality, hereafter referred to as 'proportionality'. For Lamon (2007), proportionality is a way of modelling the underlying structure of a situation. It describes an invariant relationship between quantities (which are linked and vary simultaneously). This modelling involves the concept of ratio. According to Mochon (1993), a ratio can be described as the establishment of a comparison between quantities that are related in terms of a quotient (as opposed to a difference). For example, if a classroom contains 3 boys and 4 girls, these quantities can be put in ratio and expressed in symbolic form (*e.g.,* $3/4$ or 3:4) or in words (*e.g.,* 3 to 4) [1]. A proportion, on the other hand, is often seen as an equivalence between ratios, resulting, as Comin (2002) shows, arithmetically to the relation $A/B = C/D$ (A, B, C, D being non-zero) or algebraically to a linear relation [2].

However, these 'definitions' struggle to give a clear idea of what lies behind the development of proportional reasoning. Would it refer to the development of a reasoning that combines (some of) these elements? As Lamon (2007) points out, proportional reasoning is difficult to pin down, or even to define, and reconciling the literature on the meaning to be attributed to it is a complex endeavour. Van de Walle and Lovin (2008) come to the same conclusion in their textbook, which offers a synthesis of work carried out over a number of years:

> Il est malaisé de définir le raisonnement proportionnel. Il ne s'agit pas d'un type de raisonnement qu'on est capable de tenir ou non : il s'acquiert progressivement au fil du temps. On peut le décrire notamment comme la capacité à réfléchir à des relations multiplicatives entre des quantités et à comparer de telles relations, représentées symboliquement sous forme de rapports. (p. 163) [3]

While Van de Walle and Lovin's explanations are enlightening about the difficulties, and even the limits, of a refined definition of proportional reasoning, what they propose remains at the level of general guidelines. Nevertheless, they provide fertile grounds for revealing the 'much more' at the heart of proportional reasoning. The conceptualisation developed in this paper takes as its starting point Van de Walle and Lovin's proposal, which acts as a structuring framework for analysing the potential implications of proportional reasoning. Three elements of their proposal are examined:

1. 'multiplicative relationships'
2. 'between quantities'
3. 'comparing such relationships' [4]

The analysis carried out on the basis of these elements is intended to stimulate reflection on each of them, but also to explore them in greater depth. Far from representing a review of the literature, or the formulation of a definition, this paper aims to move towards an enriched conceptualisation of what is involved in proportional reasoning.

First element: 'multiplicative relationships'

It seems obvious to make a claim for the presence of multiplicative relationships and structures when it comes to proportionality (Vergnaud, 1991). However, proportional reasoning obviously involves more than simple multiplications. As an example, consider the following problem, inspired by the work of Mai Huy, Theis and Mary (2013) and Steinthorsdottir and Sriraman (2009):

> My 2 cats eat 180 grams of food. My neighbour's 5 cats are as hungry as mine. How many grams of food do they eat?

Several multiplicative dimensions can be revealed by solving this problem from a proportional point of view. We

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might consider that my neighbour has 2.5 times as many cats as I do (2 × 2.5 = 5), so we multiply the 180 grams eaten by my cats by this 2.5 to obtain 450 grams for the neighbour's 5 cats. In purely numerical terms, a link can be established between the number of grams and the number of cats. This numerical relationship between cats and grams of food shows that 90 times as many grams of food are eaten as there are cats, since the 180 (grams) is 90 times bigger than the 2 (cats). In these various cases, this reasoning is often symbolised by the equality shown in Figure 1 [5]. There are several other ways of approaching this problem proportionally, involving multiplicative work: using a unit rate, using another basis of comparison (*e.g.,* 10 cats for 900 grams) or using properties such as 'the product of the means is equal to the product of the extremes'. These resolutions illustrate the extent of the multiplicative work that proportional reasoning can entail. However, although these proportional calculations involve multiplicative elements, what reasoning lies behind these multiplicative strategies, procedures and techniques? And, what meaning can be given to them?

Beyond the multiplicative: repetition of a relationship

From the outset, multiplication carries with it the idea of repetition, of a 'certain number of times', of multiple 'folds' ('a three-fold increase'). For example, 4 × 3 can be understood as '4 times 3' or '4 repeated 3 times'. Proportional reasoning thus involves the *repetition of a relationship.* Solving the cat problem proportionally involves preserving the relationship between food consumption of the two groups of cats, and then repeating that relationship between the 2 cats and their 180 grams of food *for* the 5 cats (and 450 grams). In other words, what happens between 2 and 180 (or 2 and 5) must happen between 5 (or 180) and the sought-after value. It is this repetition of the relationship from one case to another in a proportional situation that forms the basis of the multiplicative work underlying proportional reasoning.

A strategy frequently observed with students that describes this repetition is what is known as 'building up', which consists precisely in focusing on the relationship between two quantities and extending it to the others by means of addition (Tourniaire & Pulos, 1985). For example, a student might say: "2 cats for 180 grams, 2 more, so 4 cats for 360 grams, and 2 more gives 540 grams for 6 cats. That's too much. We have to take away 1 cat, so half the associated food which is 90 grams, which gives 5 cats and 450 grams". This can be illustrated as follows:

2 cats → 180 grams

4 cats → 360 grams

6 cats → 540 grams

(remove half, so 1 cat and 90 grams)

5 cats → 450 grams

Although expressed in additive terms, this building up strategy draws on the idea of a repeated relationship: one more time, another time more, a third time more, half a time less. This is directly aligned with what lies at the heart of multiplicative strategies: the creation of consistency through the repetitions that these multiplicative strategies impose. The repetitions associated with the multiplicative work create and express this consistency, this same rhythm, which characterises the proportional situation. So, beyond the multiplicative work, the recognition of a certain relationship, and its preservation, by applying it repeatedly throughout the situation, is a central dimension of proportional reasoning.

Beyond the multiplicative: imposing consistency

The multiplicative work also emphasises that proportional reasoning *imposes the consistency of a relationship.* Freudenthal (1973) refers to this conservation of consistency in terms of uniformity, which implies that there is no appreciable variation from one case to another in a proportional situation. Proportional reasoning thus imposes that the 5 cats of my neighbour are fed according to the same rhythm as my 2 cats, requiring that their appetite is uniform and that each of his 5 cats eats as much as each of my 2 cats. From the start, it is assumed that the relationship between my cats and the neighbour's cats is the same. In a way, it is by forcing the rhythm of food consumption of my cats on the neighbour's cats that the 450 grams are found. If the problem were not tackled proportionally, my cats' rhythm of food consumption would not necessarily be imposed on the neighbour's cats. Proportional work implies that these relationships are everywhere in the problem, and that they must be found and imposed everywhere for all the cases in the situation addressed proportionally. A link established between 2 and 180, for example, is transposed to the pair to be found, to the fourth proportional sought; and this imposition carries with it all the other links at play, such as those between the 2 and the 5 or between the 5 and the 'new' 450.

This imposition of consistency, which prescribes the rhythm to be followed, is all the more striking when considering that cats might not always have the same appetite. Indeed, Mai Huy, Theis and Mary (2013) explain that this cat problem is a quasi-proportional one, where the one solving acts 'as if' the situation is proportional. And this way of acting 'as if' is precisely aligned with this imposition of rhythm on the whole problem in order to preserve consistency; this is

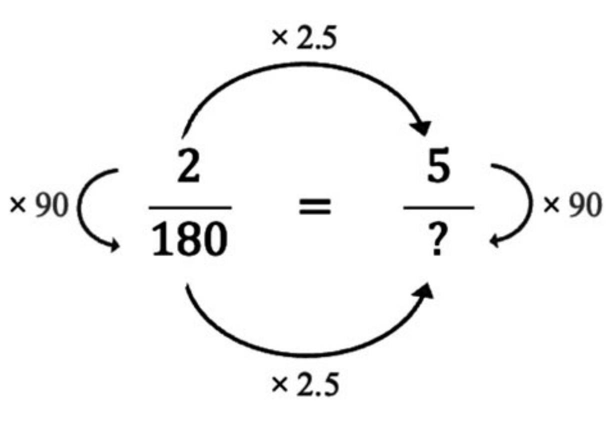

Figure 1. Usual symbolisation of a proportion.

what the phrase 'are as hungry' explicitly requires in the problem statement, despite the context of the cats, which might lead one to think that it could be otherwise. In short, beyond a multiplicative structure lies this idea of imposing consistency via the recognition and repetition of a certain rhythm, resulting in the imposition of this rhythm over the whole situation in order to determine the sought-after values.

Second element: 'between quantities'

Proportional reasoning obviously involves a significant amount of quantitative work, as illustrated by the solving of the cat problem. However, working out ratios and the various calculation strategies involve much more than simply playing with quantified attributes and numerical procedures: proportional reasoning above all involves dealing with relationships, whether quantified or not.

As Freudenthal (1983) explains, students have no difficulty accepting enlarged images without any numbers being involved. Students rapidly say that a reproduced image is inadequate if, for example, the head is too big compared to the body or if an object is too long compared to its width. What matters is the recognition and conservation of the relationships present from one image to another. And the recognition that relationships are preserved, whether or not they are quantified, can then involve qualitative dimensions (visual, geometric). For example, during an interview, Marie (aged 13) had to solve the following problem (adapted from Copur-Gencturk, Baek & Doleck, 2022):

> The Science Club has four separate rectangular plots for experiments with plans. Which rectangle(s) looks more like a square?
>
> 1 metre by 4 metres
>
> 17 metres by 20 metres
>
> 7 metres by 10 metres
>
> 27 metres by 30 metres

Marie's answer is that the 27 × 30 metre rectangle is the closest to the square. She explains that, although all the rectangles have a difference of 3 metres between their sides, this difference with dimensions of 1 and 4 metres would appear to be a lot, whereas it would be less and less visible with the other rectangles: 'almost nothing' or 'three small metres' for the 27 × 30, she would say, as shown in Figure 2. This type of reasoning highlights a certain geometric quality of the numbers involved and of the rectangles they form together; quantities that carry a certain quality [6]. In fact, Marie's strategy goes beyond quantifying the difference between the ratios that form these rectangles in order to assess their 'squareness'. Her strategy focuses on the quality of the constant 3- metre difference between the sides: the 'same' 3-metre difference is sometimes small, large, almost nothing, *etc.,* in relation to the dimensions of the rectangle under consideration. How does this emphasis on relationships come into play in proportional reasoning?

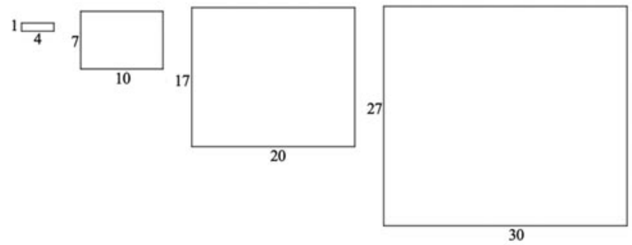

Figure 2. Marie's classification of rectangles (in squareness).

Beyond quantitative: proportional compatibility

In the same line as Freudenthal, Rouche (1998) explains that it is easy to recognise whether two objects share an appropriate ratio, regardless of any number. He talks about 'a feeling for', a qualitative relationship. He gives the example shown in Figure 3 and adds:

> On voit tout de suite que le bonhomme de la figure [3a] possède un parapluie de la bonne grandeur pour lui. Sur la figure [3b], le parapluie est trop petit pour le bonhomme, ou le bonhomme est trop grand pour le parapluie. À la figure [3c], c'est l'inverse. (p. 37) [7]

In these images, the appropriateness of the size of the umbrellas is judged in relation to the character (too big, too small, too tall, *etc.*). Resnick and Singer (1994) talk about the recognition of a relationship of compatibility, citing the story of Goldilocks and the bear family and the use of the expression 'just right' to refer to the evaluation of the size of the chair, the temperature of the porridge and the firmness of the bed. These situations require proportional reasoning based on the quality of quantities in context, namely their proportional compatibility, without requiring quantification.

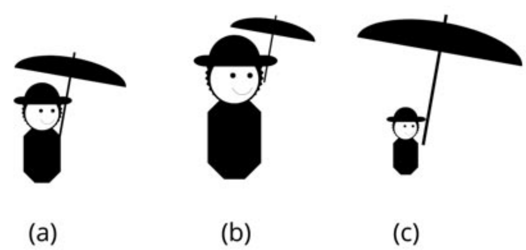

Figure 3. A comparison of characters and their umbrellas (redrawn from Rouche, 1998, p. 37).

This qualitative view of proportionality can also be found in various standard dictionaries (Robert, Oxford, Larousse, *etc.*). In addition to references to calculations, solving techniques and equality, the concept of proportionality is often presented from a qualitative standpoint, relating to the harmony between things, their equilibrium, suitability, balance, fairness, or even beauty, adequacy, rightness and goodness. Moreover, what is not proportional is often described, literally, as being 'out of proportion', discordant, frightening, excessive, inadequate, and so on.

Proportional reasoning interweaves these quantitative and qualitative dimensions, going beyond a purely calculative exercise; it is connected with the ‘feeling’ of things, of quantities, with the appreciation of their qualities and their compatibility and harmony, for ‘keeping things in proportion’ [8]. As the problem of rectangles and squares shows, proportional reasoning is not restricted to quantitative dimensions, but also calls for an appreciation of the proportional quality of the quantities involved in terms of the relationships that exist between them.

Beyond the quantitative: covariation

Another dimension that goes beyond the quantitative aspect relates to the variations and transformations involved in proportional work, better known as covariation (see Thompson & Carlson, 2017). For Oliveira (2008), a fundamental step in the development of proportional reasoning is the recognition both of the invariance of the relationship between two quantities (*e.g.,* 2 cats for 180 grams), of its repetition as mentioned above, and of their covariance in a proportional situation: the quantities related are connected and vary together. In a one-to-one interview with Marco (aged 11), he was asked to solve the following problem derived from Herron and Wheatley (1978).

> If a strip of metal 8 cm long has a mass of 28 g, what is the mass of a strip of the same type, which is 22 cm long?

Marco first explains that 8 × 3 gives 24 and 28 × 3 gives 84, so a 24 cm bar weighs 84 g. Then, because the 24 cm metal bar is too long, he has to adjust it. He divides 8 by 4 to get 2 and 28 by 4 to get 7, so 2 cm weighs 7 g. He then adjusts his 24 cm bar weighing 84 g accordingly, so that it now measures 22 cm (from 24 – 2) and weighs 77 g (from 84 – 7).

Marco’s strategy shows this simultaneous transformation by the initial multiplication by 3 of both the length (8 cm × 3 = 24 cm) and the weight (28 g × 3 = 84 g). Tripling one quantity causes the other quantity to triple too. Because this new metal bar is too big, it is *at the same time* too heavy. As the bar has to be adjusted, its length and weight have to be adjusted *and* in the same way. Removing 2 cm of metal from 24 cm, is equivalent to removing 7 g of excess weight. At no point is the length of the bar considered independently of its weight: the two quantities vary simultaneously.

This reciprocal variation is what is implied by covariation. The prefix ‘co’ emphasises that proportional analysis demands a consideration of both the quantities involved in the problem and, more importantly, the relationships between them. Everything is connected and ‘moves’ together. Although this is a typical search for the missing-value problem, Marco’s strategy focuses not only on the missing value, but on its relationship with the other quantities in the problem with which it covaries.

This is true with numbers, but the same type of adjustment is necessary in the case of a reproduced image or any non-quantified situation. It is this covariation that characterises the aforementioned acceptance of the enlargement of an image on the board, where everything that is enlarged in an image forces the rest of the image to be enlarged at the same time and in the same way: even without numbers, enlarging one part of an image enlarges the other parts of that same image at the same time. Here again, everything varies together! Conversely, a shift in these transformations from the initial drawing to the final drawing, in other words a loss of consistency in the relationships between the parts, leads to disharmony, whether or not numbers are involved. It is in this sense that proportional reasoning goes beyond purely quantitative work and centres on the relationships that exist between the quantities in the problem, through their covariation.

Third element: ‘comparing such relationships’

Proportional reasoning requires comparisons that take different forms depending on the type of context and problem at hand. An example illustrating this idea is the series of tasks developed by Noelting (1978), in which different juices composed of varying mixtures of water and orange concentrate have to be compared in order to establish which is the sweetest or the least sweet. Figure 4 shows some examples (black glasses = orange, white glasses = water).

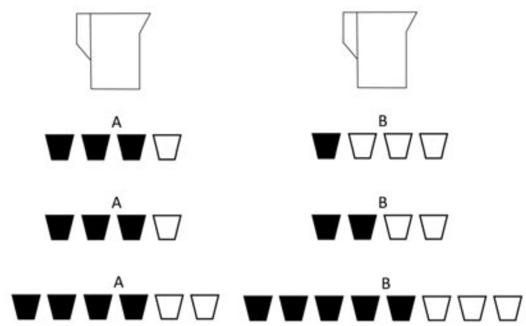

Figure 4. Examples of mixture comparison tasks (redrawn from Noelting, 1978, p. 163–168).

In task 1, there are a range of possible ways of explaining why mixture A is sweeter. For example, for the same quantity of liquid (*e.g.,* 4 cups), mixture A is sweeter because it has three orange concentrates and mixture B only one; or, mixture A is composed of more than half orange concentrate, whereas mixture B has less than half. As stated by Lesh, Post and Behr (1988), solving these tasks involves analysing the situation by establishing comparisons, which can take several forms and require some analysis of the structure and relationships between the quantities in the problem (2 times more, 4 times less, 3 times less, one more, *etc.*). How is this analysis carried out in proportional reasoning?

Beyond comparison: establishing a norm

Although this remains implicit at times, comparison in the context of proportionality involves reference to a certain norm. Rouche (1988) argues that without this norm, which provides a basis for comparison, it is not possible to offer a proportional appreciation. He gives the example of the geometric figure shown in Figure 5.

He explains that with that figure alone, it is not possible to offer a proportional appreciation, because there is no reference on which to base a comparison. However, if we add two other pentagons (Figure 6), we can say that the middle pentagon is thinner than the first and that the one on the right

Figure 5. *Single pentagon (redrawn from Rouche,1998, p. 39).*

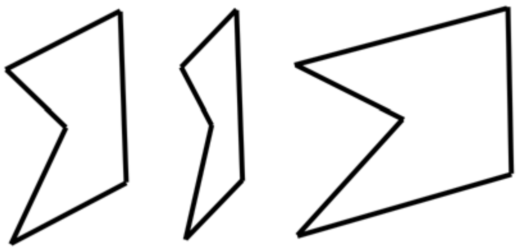

Figure 6. *Three pentagons (redrawn from Rouche, 1998, p. 39).*

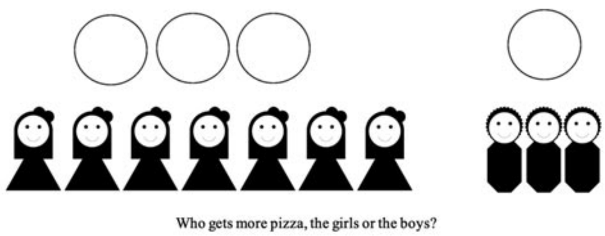

Figure 7. *Pizza comparison task (redrawn from Lamon, 1994, p. 102).*

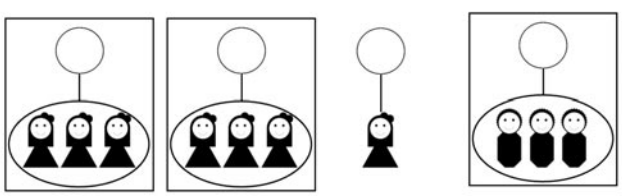

Figure 8. *Comparison relationship for the pizza task (redrawn from Lamon, 1994, p. 110).*

is wider. The initial pentagon then becomes a referent or norm for describing these new pentagons and asserting that they are not proportional by being 'thinner' or 'wider'.

The work of Lamon (1994) shows that reference to a norm not only makes it possible to make a comparison, but that, in order to approach a problem proportionally, some students actually create a norm for themselves. For example, for the task in Figure 7, Lamon shows how some students use the ratio of boys (3 boys to 1 pizza) as a norm to compare with the situation of girls (Figure 8). By using this norm of '3 boys to 1 pizza' to reinterpret the girls' situation, *i.e.* by finding it twice, they are left with one girl and one pizza 'too many'. And, as one girl for one pizza is more pizza per person than the norm of one pizza for three people, it is possible to conclude that girls have more pizza. This creation of and reference to a norm for assessing the proportionality of a situation can also be seen in the resolution of the juice mixture tasks presented above, where one of the characteristics of the two mixtures could be used as a basis for comparing how much sweeter or less sweet the proposed situation is (*e.g.,* total liquid, amount of orange concentrate, relationship between the glasses of water and the total). So, beyond comparison, the creation of and reference to a norm appears to be another key dimension of proportional reasoning.

Beyond comparison: families of ratios

The question of the norm involved in proportional reasoning highlights the idea of finding and repeating this norm in the situation to be compared. Through this repetition, other ratios are produced which preserve the proportionality of the situation. This possibility of repeating and preserving a ratio leads to the production of other ratios that follow the same relationship. The repetition of a ratio is what happens in the building up strategy for the cat problem, where the ratio of 2 cats per 180 grams of food produces the ratio of 4 cats per 360 grams, 6 cats per 540 grams, one cat per 90 grams and finally 5 cats per 450 grams. Each of these ratios produced from the 2 → 180 relationship maintains the same rhythm of cat food consumption (*i.e.,* 4 → 360; 6 → 540; 1 → 90; 5 → 450).

All these ratios produced by repeating another ratio can be seen as belonging to the same family, a family of ratios which follow the same relationship. In a way, this is the basis for making comparisons between ratios: comparing two ratios makes it possible to assess whether they follow the same relationship or whether they belong to the same family. In a way, asserting that A/B = C/D is also asserting that A/B and C/D are in the same family and that they belong to a set of ratios that follow the same relationship. In the pizza problem, the ratios '3 people to 1 pizza' and '6 people to 2 pizzas' are in the same family, but '7 people to 3 pizzas' is not. And there are an infinite number of ratios belonging to the same family as '3 people to 1 pizza'.

This idea of an infinite number of members of the same ratio family brings into play the idea of a multiple proportion, where equality is not reduced to two pairs but to an infinite number of possible pairings: A/B = C/D = E/F = G/H =…. In the case of the cat problem, the building up strategy shows and takes advantage of this infinity of ratios 2/180 = 4/360 = 6/540 = 1/90 = 5/450 = 10/900 = …. All these ratios follow the same rhythm and are in the same family. The transition from one ratio to another in the context of proportionality is supported by this idea of family, where the links connecting the members of the same family are exploited and show the fluidity that is involved in moving from one member to another. Over and above the establishment of a comparison or a search for a missing-value, proportional reasoning engages the production of ratios that are members of the same family, that is, ratios that share the same structure, and consequently have the same relationship between their quantities.

Conclusion

Drawing on the proposal by Van de Walle and Lovin (2008) has gone some way towards clarifying the 'much more' involved in proportional reasoning; components that often remain hidden under various habits in terms of calculations and processes. This deepening and broadening of the elements usually associated with proportional reasoning highlights important differences between 'proportional calculation' and 'proportional reasoning'. Both Baruk (1995) and Freudenthal (1983) are highly critical of what they call an 'arithmetisation' of proportions, where numbers and calculations take up all the space and even come to crush the possibility of proportional reasoning. Beyond numbers and calculations, proportional reasoning is conceptualised in terms of relationships between ratios (whether quantified or not), a compatibility that can be preserved, imposed and repeated, enabling them to be compared, adjusted, preserved, transformed and so on. All this goes well beyond the arithmetisation of these relationships in terms of procedures or various operations. In other words, beyond calculation, there is reasoning.

There is obviously much more to be said and analysed about proportional reasoning (for example, from the point of view of the interactions between proportional strategies and the characteristics of the problems tackled, see Adjiage, 2005; on questions of symbolism, see Proulx, Mégrourèche & Novak, 2022, or for other forms of proportionality). But already, by going beyond the multiplicative, quantitative and comparative dimensions, an in-depth conceptualisation of the meaning to be given to proportional reasoning has been made explicit and potentially provides a better understanding of its nature, and even its study in a school context.

Notes

[1] In the original, we chose to use the expression 'ratio' to avoid the polysemy associated with the expressions 'rapport' and 'taux' in French-language (academic and scientific) writing. This allowed us to make links with the idea of 'rational number', but also with the English term 'ratio'.

[2] This starting point in no way negates the different schools of thought that coexist with regard to ratios and proportions, which approach them in terms of a relationship between magnitudes (see Chambris & Visnovska, 2022; Comin, 2002), the study of rational numbers and the rational number project (see also Kieren, 1976), numbers in a purely multiplicative context (see Ben-Chaim, Keret & Ilany, 2012; Beckmann & Izsák, 2015) or an ordered pair (Freudenthal, 1983). The definition proposed by Hersant (2005) probably bridges these different schools: "La proportionnalité qualifie une relation particulière entre des grandeurs que l'on peut traduire par une relation entre les valeurs de ces grandeurs, puis une relation entre des suites numériques via les mesures de ces grandeurs" [9] (p. 6), particularly when supplemented by Simard (2012): "Deux suites de nombres qui se correspondent un à un sont proportionnelles lorsque les [ratios] de deux nombres correspondants sont égaux" [10] (p. 51).

[3] In English: Proportional reasoning is tricky to define. It is not a type of reasoning that you can or cannot do: it is acquired gradually over time. It can be described as the ability to think about multiplicative relationships between quantities and comparing such relationships, represented symbolically in the form of ratios.

[4] Another element concerns the symbolic representation of these proportional relationships. This is discussed in Proulx, Mégrourèche & Novak (2022).

[5] It is obviously possible to consider the number of grams per cat (90 g/cat) or other representations of these relationships (2/5 =180/450; 180/2 = 450/5; 5/2 = 450/180). We will not go into the distinction between internal (homogeneous) and external (heterogeneous) relationships, or their apparent complexities. This distinction is not unanimously accepted (see, for example, the debate between Vergnaud and Schwartz reported in Lesh, Post & Behr, 1988, pp. 108–109, as well as the critical analysis of Iszák & Beckmann, 2019). This homogeneous-heterogeneous distrinction was also considered outdated following developments in mathematics that made moving from a static to a dynamic vision through the notion of variable possible (Janvier, Charbonneau & René de Cotret, 1989; see also Freudenthal, 1983, Chap. 6, and Comin, 2002, note p. 47). Thompson (1994) also questioned the importance attached to this distinction, placing greater emphasis on the resolution process used and the meaning given *by the solver* to determine the nature of the relationship used.

[6] Solomon (1987) also addresses the question of proportions through the enlargement of geometric shapes, in his case rectangles, and in so doing he addresses the distinction between the quality of the shape and the quantity measured.

[7] In English: You can see straight away that the character in figure [3a] has an umbrella that is the right size for him. In Figure [3b], the umbrella is too small for the character, or the character is too big for the umbrella. In Figure [3c], it's the other way around.

[8] In the original, the expression 'toutes proportions gardées' is used, meaning to compare two things by taking their differences into account.

[9] In English: Proportionality describes a particular relationship between magnitudes that can be translated into a relationship between the values of these magnitudes, then a relationship between numerical sequences via the measurements of these magnitudes.

[10] In English: Two sequences of numbers that correspond one to another are proportional when the [ratios] of two corresponding numbers are equal.

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