matics learning possible? Would it be so detailed as to be incomprehensible? How should we set about trying to build such a theory? (I reject the naive baconian view attributed to Begle that all we need to do is to assemble sufficient observations and then scan them for regularities.)

What defines a problem in mathematics education? Do these questions (including this one!) define problems in mathematics education?

What constitutes the answer to a problem in mathematics education?

What parameters delimit research in mathematics education?

What are the values characterising mathematics education and subscribed to by mathematics educationists. What motives (pure and impure!) drive us? (Parallel questions concerning mathematics have been studied by C.S. Fisher in "Some Social Characteristics of Mathematicians and Their Work," *American Journal of Sociology* 78(5) 1094-1118).

It has been projected that if science publications are piled up as they are published, the top of the pile will be rising faster than the speed of light by 2000 A.D. To what extent are we adding to this information pollution? Am I?

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A Teacher's Dilemma

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I remember my grandfather saying, "The man who never makes a mistake never makes anything." (Appropriately, this is a slight misquotation. The original, in a speech by Edward Phelps in 1899, was, "The man who makes no mistakes does not usually make anything.")

The first spring after I came to Lesotho I planted some peas. They failed; the harvest was just about enough for one helping. I did not learn from my mistake but tried again the second spring, with an equally abysmal result. After that I learned to plant peas in the autumn: they do not grow in the winter but they stay alive and have time to produce a crop in spring, before the heat of summer.

Last year the president of the Mathematical Association in the U.K. took up this theme and chose the title "The importance of mistakes" for his presidential address.

Mistakes are indeed important and I make no apology for publicising some mistakes made by my pupils, boys and girls in their 8th to 12th years of formal education.

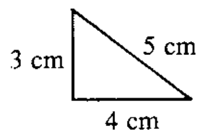
(1) $2\frac{1}{2} \times 4 = 8\frac{1}{2}$

(2) $63 \text{ mm}^2 = 6.3 \text{ cm}^2$

(3) $70 \text{ cm} = 7000 \text{ m}$

(4) $\frac{8435}{7} = 125$

(5)



area = 60 cm²

(6) (on Pythagoras' theorem)
$x = 10^2 = 100 - 8^2 = 64 = 36 = 6$

(7) $3p - (p + 5) = 3p - p + 5$

(8) (on a road going from altitude 1800 m to altitude 2500 m with an average gradient of 1 in 20)
length of road = 35 m

(9) $2^3 = 6$

(10) $\frac{0.204\not{0}}{10\not{0}} = 0.0204$

(11) radius = 2.8
$\Rightarrow$ diameter = 4.16

(12) 3.5 million = 3.5000000

(13) $\cos x = 0.5$
$\Rightarrow x = \frac{0.5}{\cos}$

(14) a wind is blowing at 12 km/h
in two minutes it blows 24 km/h

(15) $\left(\frac{1}{20} + \frac{1}{50}\right) \times \frac{1}{2} = \frac{1}{70} = \frac{1}{35}$

The trouble with most of the above statements is that they are not excusable. When you think about what they mean you realise that something is wrong—or if you do not know what they mean, why write them in the first place?

(1) Thabiso knew the meaning of $\times 4$, or he would not have got the 8 in his answer, but he did not bother to think of $2\frac{1}{2} \times 4$ as $2\frac{1}{2} + 2\frac{1}{2} + 2\frac{1}{2} + 2\frac{1}{2}$.

(2) Puleng was one of the top girls in the class; of course she could draw a square millimetre and a square centimetre if asked, and as soon as I queried the statement she realised that 63 mm^2 is less than 1 cm^2—yet she made a similar mistake on several different occasions.

(3) Mxolisi was by no means at the bottom of the class and certainly knew that 70 cm is shorter than 1 metre

(4) One year, in the top part of the school, less than half of a class worked this division correctly. Yet had I asked, "Is 84435/7 greater than or less than 1000?", the whole class would have known and been sure.

(5) This one is rather different. Rebecca probably did not appreciate the concept of area. All she knew from previous encounters with the word was that you can find the area of a rectangle by doing multiplication. She might well have been unable to correct this mistake even if she had thought about it. It is said here, "Area means multiply."

(6) What does the symbol = mean?

(7) More difficult, perhaps, yet is it not obvious that to take away $p + 5$ objects from some collection you take away p objects and then take away another 5?

(8) We can see a 2500-metre mountain from the school compound, and pupils run distances like 100 metres, so how could anyone be satisfied to submit 35 m as the length of such a road?

And so we might continue. But the point is made. Usually, with a little more thought, the mistake could have been avoided.

So why do pupils make such statements, and make them repeatedly? I will suggest two broad reasons.

First, pupils tend to be lazy. It is out of fashion to make such pejorative assertions, and I use the word "lazy" in a restricted sense, so let me elaborate. I am not referring to a reluctance to work, rather to a reluctance to think. In my experience, pupils are always on the lookout for strategies to avoid thinking. They look for methods which will produce answers without their having to attend to the situation to at hand. For me the most recent example of this came from a class during its second or third lesson on latitude and longitude. We discussed questions like, "From 10°W then go 40°E: how many degrees round?" and "Start 10°W then go 40° eastwards: what is the new longitude?" and the class wanted to make generalisations—"you add" or "you subtract." Generalisation, even generalisation from a single paradigm, is a valid and valuable mathematical activity, but it loses its validity if no attention is paid to the conditions and if the general statement goes unchecked. Another strategy is to fall back on proven written procedures. Why bother to get to grips with the 12 times table when you know how to do long multiplication? Why add numbers in horizontal format when you are used to copying out the numbers with the digits in columns? This avoidance of thinking is not, of course, peculiar to pupils in Lesotho. I have noted it in England, while Stephens has written from Australia:

> These children look at the first half of their sheets and search for clues on how to make a correct response. Often they appear to seek for a visual pattern; and having found one, they check it with another example. If the responses match, they proceed to fill in as many blank spaces as they can. They barely notice the carefully underlined sentences which are there to remind them precisely of what they are doing. When they reach a blockage and ask for help, they are willing to receive only that minimum of information which is needed to give them an answer.

And why do pupils avoid thinking? Various conjectures may be made, but surely one reason is that mathematical thinking is difficult. A mathematical expression is full of abstractions, and a good deal of mental exertion is required to examine it and assign meaning to it. To complicate matters, it is often necessary or desirable to select one viewpoint from among a number of viewpoints that are equivalent but different. For example the subtraction symbols in the expressions 1986 – 3 and 1986 – 1980 give quite different cues to most people. Further, there is a certain capriciousness about our use of symbols. Juxtaposition of symbols, for example, does not always carry the same meaning: compare 6½, 65 and $6x$. Sometimes teachers face pupils with a symbol, notably the ubiquitous x, without assigning any meaning to it at all. (Yet if *pupils* invent a new notation, or a new use for an existing notation, as in (6) and (13) above, teachers find it unacceptable!) In my commentary on the mistakes I put my notes on (6) to (8) in the form of questions, but finding the right question, if you are inexperienced, requires much thought. In summary, the interpretation of mathematical expressions involves considerable mental effort, and few pupils are so highly motivated that they will expend the effort requried.

Secondly, teachers tend to be imperious. I need to elaborate on this also. Teachers pressurize pupils. They pressurize pupils to write *something*. They get angry when homework is not completed or when a question has not been attempted. A piece of nonsense like (13) avoids a teacher's wrath, cannot earn fewer marks than the 0 awarded for a blank answer, and even demonstrates some knowledge, in this case that equations can be solved by "doing the same thing to each side." A consequence of this pressure is that pupils do not understand the nature of mathematical activity. If you are learning to speak a foreign language then you will make faster progress if you are not afraid to make mistakes. But a mathematician checks and re-checks his work before he communicates it to his contemporaries. Teachers do not do much to encourage this way of working. A shop assistant who gave the right change to only 90% of his customers would soon lose his job. But mathematics teachers are satisfied with 50% from their pupils, or even less. And in school not only is it true that wrong answers do not matter very much, right answers do no matter much either. What does it really matter whether the area of the triangle in (5) is 6 cm^2 or 60 cm^2? It might matter if the triangle were to be covered with gold leaf, but few of us are in the business of illuminating manuscripts. Most examination papers are filled with questions of little intrinsic interest, and it is tempting to spend a lot of time in class pushing as fast as possible through questions of the same trivial nature.

What has been my response to this situation?

High school pupils do not go wrong with reasonably small whole numbers. They do not write $38 + 5 = 385$, though such a mistake is of the same genre as many of the mistakes I have quoted. Pupils know the meaning of 38 well. The problems arise with numbers and expressions which they do not know so well. I have therefore tried to focus on meanings by a strategy of what I call "concretisation," in the hope that meanings will sink more deeply into pupils' minds, that symbols will become more to pupils than marks on paper or blackboard.

Very early on, all my pupils use number lines for work with whole numbers. From the beginning 99 is handled as a number in itself; the two 9's are not worked on separately. It is seen that 99 is very near to 100, even though the decimal representations of 99 and 100 are so different. When we move on to other topics a number line on the wall of the mathematics room remains as a reminder. My pupils handle cut-out shapes representing common fractions. For decimals we use multibase arithmetic blocks: $10 \times 10 \times 1$ blocks to represent units, $10 \times 1 \times 1$ blocks to represent tenths and $1 \times 1 \times 1$ blocks to represent hundredths.

When dealing with distance we go for a kilometre walk (in ten stages). For mass we have similar cartons (porridge oats cartons, actually) filled with different quantities of ballast, and pupils pick them up and feel them. For time we use old watches whose hands still turn round. For angle we use "angle indicators," whole circle protractors with cursors so that rotation can be performed as well as talked about. In geometry we try to draw interesting patterns rathers than calculate angles and lengths just for the sake of it.

Where is the dilemma? It is simply that efforts at getting pupils to associate meanings with symbols do not seem to accomplish much. Two or three years on, instead of being happy that at last I have a class in front of me that knows what it is doing, that has some feeling for the entities it is manipulating, I keep coming across mistakes like those I have exemplified. Progress seems to have been minimal, and of course I am behind with the official syllabus.

The class seems to be worse off than it would have been otherwise. When the mode of teaching is "teacher does examples, class copies method" then at least the class learns some methods (never mind that pupils get algorithms confused with one another or apply them wrongly or that they get "thrown" by anything slightly out of the ordinary). It is easy to set homework and therefore pupils spend more time on what is at least pseudo-mathematical activity. Pupils, if they copy successfully, get sums right and feel they are making progress, so class morale is good. In this way some mathematics, albeit of an inferior quality, rubs off on a majority of pupils.

By taking a more practical approach, by not setting questions to be worked by rote, by expecting pupils to think for themselves, the teacher exposes himself to the risk that pupils will not rise to the challenge. Even if they improve their understanding of selected concepts, it seems that, on returning to the trivial questions, they fail to make the connections. A distance of 120 km in a question about speed is not thought of as the approximate road distance from our town to Maseru, the capital. Unrealistic answers even turn up in questions on money; real money, which pupils understand well enough when they go to the shops, is not seen to be relevant to the mathematics of the exercise book. My approach also leads to new sources of confusion. For example, some of my pupils have mixed up 3/8 with 1/3, because of 3/8 of of a circle and 1/3 of a circle are visually similar (though I suppose a pupil who has confused 3/8 with 1/3, but later got the distinction between them sorted out, has had a richer mathematical experience than a pupil who has by-passed the comparison).

The number line work should show itself in improved mental arithmetic. We spend a good many lessons over a year with a mental arithmetic game involving matching pairs like 4×9 and $40 - 4$. But pupils tend to calculate mentally not by casting around for an appropriate method but by imagining what would be written down were a written method being used, so instead of starting at 40 and counting down 4 they will imagine 4 written underneath 40 and begin their mental calculation with $10 - 4 = 6$. The work we did with number lines has not gone deeply enough into their minds to displace the algorithmic approach learned in the primary school.

Where do I go from here? Members of the national mathematics panels, primary and secondary alike, have shown no interest in discussing these matters. They are more concerned with listing behavioural objectives, allocating topics to the weeks of the year or constructing "items," which seems to be the current term for trivial short-answer questions.

I continue with my more concrete approach because it is more interesting to me, and because I believe it ought to be better for my pupils. But those pupils do not in any noticeable way modify their attitude towards mathematics and I remain unsatisfied.

That is my dilemma.

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