CONTRASTING KNOWLEDGE FOR ELEMENTARY AND SECONDARY MATHEMATICS TEACHING

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Jess was a generalist elementary school teacher, with a class of 9- to 10-year old pupils in her first post-qualification appointment. She was participating in the second year of a developmental research project (Turner, 2011). In one mathematics lesson, Jess was drawing attention to the inverse relationship between multiplication and division. By way of illustration, Jess took the product 3×4, and observed that it was related to both $12 \div 3$ and $12 \div 4$. To demonstrate that this was the case, she drew the diagram shown in Figure 1.

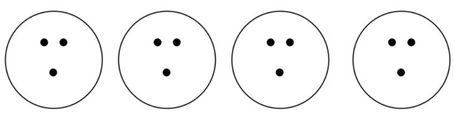

Figure 1. Jess's diagram.

This one diagram does indeed illustrate both division situations, but for $12 \div 3$ it represents a quotition or measurement structure, whereas for $12 \div 4$ the structure is partition. In the quotition structure, the total number of items (12) is specified, together with the quota, *i.e.,* the size (3) of the subsets into which they will be grouped. The "answer" (4) to the division calculation gives the resulting number of subsets. In the partition structure, the total number of items (12) is again specified, together with the number of equal subsets (4) to be made. In this structure, the "answer" is now the size of each subset. In the UK, teacher-practitioners usually refer to these two structures as "grouping" and "sharing" respectively. In a discussion with Jess following the lesson, Turner (the researcher) raised the issue of whether the dual-representation (two division structures, one diagram) was effective, especially since Jess had consistently referred to division as "sharing". Jess responded that she was aware of the two structures, which she had encountered in her pre-service teacher education program, but that she remained unsure about the distinction. She said:

> Explaining dividing in terms of grouping and sharing still gets me mixed up. It is something I need to work on myself. The aim was to explain in terms of grouping.

The same issue arose a year later, but this time Jess reported that she took great care to prepare representations of division in her lesson planning, because she was aware of ongoing confusion about the two structures in her own mind.

Knowledge for teaching mathematics

The anecdote with which this paper begins was chosen to make the point that elementary mathematics teaching presents difficulties that might not—and, probably, could not—be anticipated in advance of professional preparation, and might not be recognised even by many high school mathematics teachers. The curious thing is that the mathematical content of Jess's lesson is "easy": multiplication and division with small integers first appear relatively early in formal schooling. Jess had passed the national test in mathematics at age 16 comfortably, yet, as a teacher, struggled with an aspect of division that had neither been taught nor assessed in her previous mathematics education. The intention of this article is to highlight this pedagogical dilemma as a general phenomenon, and to argue that elementary mathematics teaching poses challenges which are qualitatively different from those that confront mathematics teachers in the secondary phase, whilst recognising that there is some kind of continuum between the two phases.

The knowledge-base that mathematics teachers bring to, and apply in, their work in the classroom has been the focus of much attention in recent years, notably in a special issue of this journal in 2009. Much of the thinking and writing in this area acknowledges its origins in the seminal work of Shulman (1986) which identified and distinguished between three kinds of teachers' knowledge of content specific to a disciplinary domain: namely subject matter knowledge (SMK), curricular knowledge (CK), and pedagogical content knowledge (PCK). For mathematics educators, pedagogical content knowledge is particularly interesting, in that it encompasses a large and increasing body of mathematical knowledge that would not be acquired in the process of learning mathematics for non-pedagogical purposes.

The approach to contrasting primary and secondary mathematics teaching in this paper is through an empirically-based, theoretical perspective (the "Knowledge Quartet") on the situations in which mathematical knowledge surfaces in teaching. This framework owes a great deal to Shulman's knowledge taxonomy, but is organised differently from it.

The Knowledge Quartet

The Knowledge Quartet (KQ) was the outcome of research in which 24 lessons taught by elementary school trainee teachers were videotaped and scrutinised. The research team identified aspects of trainees' actions in the classroom that

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could be construed as being informed by their mathematics subject matter knowledge or pedagogical content knowledge. This inductive process generated a set of 18 codes (later expanded to 20), subsequently grouped into four broad, superordinate categories or dimensions—hence the "Quartet".

A brief outline of the Knowledge Quartet should suffice here. The first dimension, *foundation*, consists of teachers' mathematics-related knowledge, beliefs and understanding, incorporating Shulman's classic taxonomy of *kinds* of knowledge without undue concern for the boundaries between them. The second dimension, *transformation*, concerns knowledge-in-action as demonstrated both in planning to teach and in the act of teaching itself. A central focus is on the representation of ideas to learners in the form of analogies, examples, explanations and demonstrations. The third dimension, *connection*, concerns ways that the teacher achieves coherence within and between lessons: it includes the sequencing of material for instruction and an awareness of the relative cognitive demands of different topics and tasks. Our final dimension, *contingency*, is witnessed in classroom events that were not planned for. In commonplace language, it is the ability to "think on one's feet". A more detailed conceptual account is given in Rowland, Huckstep and Thwaites (2005).

The underpinning foundation knowledge is rooted in the teacher's theoretical background and in their system of beliefs. Since it is the dimension explored most thoroughly in this paper, further conceptual elaboration is appropriate. Foundation knowledge concerns teachers' knowledge, understanding and ready recourse to their learning in the academy, in preparation (intentionally or otherwise) for their role in the classroom. It differs from the other three units of the Knowledge Quartet in the sense that it is about knowledge possessed, irrespective of whether it is being put to purposeful use. A key feature of this category is its propositional form (Shulman, 1986). It is what teachers learn in their personal and professional education. We take the view that the possession of such knowledge has the potential to inform teachers' pedagogical choices and strategies in a fundamental way, enabling a rational, principled approach to decision-making that rests on something other than imitation or habit. The key components of this theoretical background are: knowledge and understanding of mathematics *per se;* knowledge of significant tracts of the literature and thinking which has resulted from systematic enquiry into the teaching and learning of mathematics; and espoused beliefs about mathematics, including beliefs about why and how it is learnt.

Mathematical knowledge in teaching: the primary and secondary phases

Since the Knowledge Quartet was developed through observing and analysing elementary mathematics teaching, its relevance and application to mathematics teaching in secondary schools cannot be taken for granted. We have addressed this (implicit) question empirically in a recent project in which the Knowledge Quartet was used as a framework for the analysis of mathematics lessons taught by pre-service secondary school teachers (*e.g.,* Rowland, Jared & Thwaites, 2011). We began not knowing how well the Knowledge Quartet might fit secondary school teaching. In particular, with notable exceptions, secondary mathematics teachers are specialists with regard to their disciplinary orientation. By contrast, generalist primary teachers, who have typically specialised in the arts and humanities in their own education, often lack confidence in their own mathematical ability (*e.g.,* Hembree, 1990; Uusimaki & Nason, 2004). This absence of specialist mathematics knowledge could be expected to impose limitations on the pedagogical challenges which primary generalists are equipped to encounter. In a recent chapter, Watson and Barton (2011) emphasise the central role of "mathematical modes of enquiry" in the orientation and determination of mathematics teachers' actions in the classroom and in their pedagogical decision-making, contrasting this quality of teacher knowledge favourably with that of the teacher "who has learned a repertoire of pedagogical strategies without personal involvement" (p. 78).

The *raison d'être* of this paper is, essentially, to prise open the distinction—as far as teaching is concerned—between (merely) being mathematical and having a learned repertoire of mathematics-didactic know-how. Watson and Barton propose that the specialist-mathematician teacher is better equipped to make informed pedagogical decisions in the face of complex instructional challenges. Given that they are likely to be teaching in the secondary phase, perhaps these specialists will indeed need to drill deep into their mathematical resources. In a report of a study of nine secondary mathematics teachers, Potari and her colleagues observed that "teachers' knowledge in upper secondary or higher education has a special meaning as the mathematical knowledge becomes more multifaceted and the integration of mathematics and pedagogy is more difficult to be achieved" (Potari *et al.*, 2007, p. 1955).

Could it be, then, that teaching secondary school mathematics is in some way harder than teaching mathematics in the elementary school? The question is, as yet, rather loosely formulated, but I propose that mathematics teachers in primary and secondary school settings draw upon different (but not disjoint) mathematical knowledge resources; and that each kind of knowledge is profound in a different way. To develop and justify that claim in this article, I examine two lessons conducted with classes of pupils whose ages differ by about seven years. One class is at the beginning of compulsory schooling in England (Year 1, pupils aged 5-6), the other in lower secondary school (Year 8, pupils aged 12-13). The analytical framework is the Knowledge Quartet in both cases and the focus is on foundation knowledge in particular. I argue that whereas from the mathematical point of view, the subject matter under consideration with the Year 8 class is significantly more complex than that in the Year 1 lesson, the pedagogical content knowledge necessary to teach the latter well is invisible to the uninformed observer, and is rarely made explicit in instruction. Therefore, the work of both teachers is mathematically demanding, but in different ways, where "mathematically" is taken to encompass mathematical knowledge for teaching in the widest sense, as indicated by Shulman and made explicit by Ball, Thames and Phelps (2008). I suggest, therefore, that the "mathematical modes of enquiry" heuristic has its limitations, and is better matched to secondary school mathematics teaching than to primary school.

The pattern in the following two sections will be to give a descriptive synopsis of the lesson first (*i.e.,* to say what the lesson was about), followed by an account, necessarily selective, of the foundation knowledge relevant to teaching this lesson.

Year 1 subtraction

The teacher, Naomi, was in preservice teacher education. The learning objectives stated in her lesson plan are as follows:

> To understand subtraction as "difference". For more able pupils, to find small differences by counting on. Vocabulary—difference, how many more than, take away.

In her introduction, Naomi sets up various difference problems, initially in the context of frogs in two ponds. Her pond has four, her neighbour's has two. Magnetic frogs are lined up on a vertical board, in two neat rows. She asks the class first how many more frogs she has and then requests the difference between the numbers of frogs. Pairs of children are invited forward to place numbers of frogs (*e.g.,* 5, 4) on the board and the differences are explained and discussed. Before long, she asks how these differences could be written as a "take away sum" [1]. With assistance, a girl, Zara, writes 5-4=1. Later, Naomi shows how the difference between two numbers can be found by counting on from the smaller number.

The children are then assigned group tasks. Some of these tasks involve lining up various icons (such as cars and apples) to show the difference. Others use multilink plastic cubes, following the frogs pairing procedure. Two groups are directed to use the counting-on method to find the differences. In a whole-class discussion following these group activities, it becomes apparent that there is still widespread confusion about the meaning of difference among the children.

Foundation knowledge: subtraction

Carpenter and Moser (1983) identify four broad types of subtraction problem structure, which they call *change, combine, compare, equalise.* Two of these problem types are particularly relevant to Naomi's lesson. First, the change-separate problem is exemplified by Carpenter and Moser by: "Connie had 13 marbles. She gave 5 marbles to Jim. How many marbles does she have left?" (p. 16). The UK practitioner language for this is "subtraction as take away" (DfEE, 1999, p. 5/28).

Secondly, the compare problem type includes this example: "Connie has 13 marbles and Jim has 5 marbles. How many more marbles does Connie have than Jim?" (Carpenter & Moser, 1983, p. 16). This subtraction problem type has to do with situations in which two sets (Connie's marbles and Jim's) are considered simultaneously, involving "the comparison of two distinct, disjoint sets" (p. 15). No change or transformation is involved in these scenarios, whereas change problems involve an action on and transformation of a *single* set (Connie's marbles). Again, the National Numeracy Strategy Framework [2] (DfEE, 1999) reflects the tradition of UK practitioners in referring to the compare structure as "subtraction as difference". I return to this point below.

The National Numeracy Strategy guidance (DfEE, 1999) reflects typical Early Years education practice in recommending the introduction of subtraction, first as take-away, in Year R (pupils aged 4-5), then as comparison in Year 1. One consequence of this Early Years initiation is the almost universal use of "take away" as a synonym for subtraction (Haylock & Cockburn, 2003, p. 38). Another peculiarly British complication is that the word "difference" has come to be associated in rather a special way with the comparison structure for subtraction (Rowland, 2006). Crucially, as I remarked earlier, the National Numeracy Strategy itself refers to the compare structure as "subtraction as difference". However, at the same time, the term difference is the unique name of the outcome of *any* subtraction operation, on a par with sum, product and quotient in relation to the other three arithmetic operations. There is evidence that these complexities, and others, present obstacles to the pupils throughout Naomi's lesson, as illustrated in the following fragment from later in her introduction:

Naomi:	If I've got 15 rings, and Martin's got 20 rings, what's the difference? What's the difference between 15 and 20? [...] 15, 16, 17, 18, 19, 20. What's the difference? How many more does he have?
Child:	Five.
Naomi:	Five more. How could we write that as a take away sum? Lucy.
Lucy:	Eleven take away one.
Naomi:	No, I've got 15 and he's got 20. Right let's do some more differences first. If I've got 10 flowers and Jade's got 13 flowers, what's the difference?

It is apparent that, notwithstanding her careful introduction with the frogs, the children have not connected this new subtraction structure, comparison, with their earlier "take away" experience. In a sense, they are intended to see it as a new idea, with new procedures (principally, matching and counting up). Yet Naomi persists in asking about writing it as a "take away sum":

Naomi:	Right, I've got 10 and Jade's got 13. 10, 11, 12, 13. Jake are you doing it? What's the difference? How many have we got extra? Gavin?
Gavin:	Five.
Naomi:	No. 10, 11, 12, 13. Jim?
Jim:	Three.
Naomi:	Three. Good. Right, how can we write that as a take away sum? Jim? [...] Right, we start with the big number, 13, and take away 10, and then we know the difference. What did we say was the difference? Andrew?

While it is true that subtraction problems can be distinguished by the two semantic structures identified by Carpenter and Moser (1983), this distinction is unlikely to be found outside elementary mathematics textbooks and handbooks for elementary mathematics teachers. The educated adult is unlikely to pause to select a structure—change, separate or comparison—when presented with a subtraction word problem, or to reflect on related solution strategies, such as counting up for comparison. Yet the elementary mathematics curriculum brings the connections between these two structures to the attention of young children. It also aims to make them aware that they share a common symbolic notation $a-b=c$, and that the outcome, c, in both cases/structures is called the *difference*. In the secondary mathematics curriculum the distinction between the subtraction structures is of no consequence in the domain of natural numbers (or positive reals), especially when differences are computed using standard written algorithms. It then becomes crucial in the case of the difference of integers: 6–(-2) should hold no terrors if the difference is interpreted as comparison, whereas attempts to model as "take-away" spawn various absurd narratives about cancelling debts, extracting cold air, and the like. Even so, an opportunity sample of UK school texts [3] suggests that the semantic structures emphasised in Year 1 are not mentioned and remain implicit in representation and expositions of integer subtraction.

The subtraction structures described above were in the curriculum for the mathematics methods course which Naomi had followed at university prior to her school experience, yet somehow, it seems, the distinction had not made a strong impression on her: in effect, they remained invisible, somewhere outside the field of her awareness. For her, counting up, take-away and difference remained part of a unified and indivisible whole, as yet "unpacked" for pedagogical purposes. For the children, however, comparing Naomi's frogs with her neighbour's involved a very different action from taking 4 apples from 7 apples, say, and neither scenario entailed *quantifying* difference, as opposed to describing *what* was different. In the scenario with which this article begins, Jess knew that there are two distinct division structures, but had not yet properly understood them, or learned their names. We did not interview Naomi, but it seemed that her awareness (of subtraction in this case) was not yet awakened in the way that Jess's awareness of division was.

Year 8 graphs of linear functions [4]

The teacher, Suzie, had about 7 years' teaching experience. Following a warm-up on fractions simplification, Suzie writes the lesson aims on a board:

> Find the gradient [5] of straight lines. Use the gradient and the intercept on the y-axis to find the equation of straight lines.

Suzie asks what "gradient" means. Students' ideas include references to a road, a roof, a slide and a ski slope. Suzie develops one response—"how steep"—in terms of steep hills. She then writes on the board: "gradient = up/along" and enunciates "up over along." She rolls the whiteboard to a squared section and draws a line segment between two lattice points (4 along, 8 up). Suzie completes the triangle, using endpoints of the line segment, to show the horizontal and vertical increments. She says that the gradient is 8/4=2.

Suzie then draws another line segment alongside the first. Its gradient is 3/6. Some pupils say "2". In response, Suzie asks, "What is 3/6?" One girl asks, "Is it 1/3?" Susie says, "It is 1/2." She asks which line (segment) has bigger gradient? She says that 2 is bigger than 1/2. One pupil refers to the two completed triangles that Suzie has drawn and asks if it is about area (*i.e.,* does the first line have bigger gradient because the first triangle has greater area?). This phase lasts 15 minutes.

There is then individual/paired work in which pupils share laptop computers and use the graphing software Autograph. A worksheet asks them to draw $y=x$, $y=2x$, $y=3x$ and find the gradients (and generalise). The worksheet then shows graphs of two lines through the origin and asks for their equations. Finally, it asks for a prediction of the graph of $2x+1$, with Autograph used to check. The lesson concludes with a short whole-class discussion, during which Suzie displays a line on the large screen and asks, "What is its equation?" She finds the gradient (3) starting from (0, 1). The intercept is 1. Suzie writes $y=3x+1$, and the lesson ends.

Foundation knowledge: gradient

Some reflections of a *mathematical* kind on the nature of the gradient, a concept which occupied much of the lesson time, is prompted by the examples that Suzie drew on the whiteboard when she introduced the concept quantitatively. Her examples were of line segments, whereas gradient is an attribute of (infinite) lines. Indeed, the graphing software (Autograph) that they used later, draws lines, not line segments. Fundamental issues to be understood and considered by the teacher, therefore, include:

- the gradient of a line is found by isolating a segment of the line;
- any segment yields the same ratio (this could be tested empirically; theoretically, it relates to similar triangles).

There are also considerations of an explicitly pedagogical kind—more pedagogical content knowledge than subject matter knowledge—about the teaching and learning of the concept gradient. This is accessible in part by didactical reflections related to the mathematical observations already made:

- some segments facilitate identifying the increases in abscissa (x-coordinate) and ordinate (y-coordinate) better than others;
- the increase in abscissa should be simple (ideally 1) to facilitate calculation of the ratio (unless one uses a calculator).

There were few problems with finding the gradient of $y=mx$ because (0, 0) could be taken to be one end of a line segment, and (1, m) the other. However, $y=2x+3$ was much more problematic. So was $y=3x+1$, and it seemed that few pupils followed Suzie's demonstration at the end of the lesson.

Another difficulty had arisen earlier when Suzie drew the two line segments on the squared portion of her whiteboard (with gradients 2 and 1/2 respectively) in order to demonstrate her "up over along" proposal, intended to operationalise the talk about roads, roofs and ski-slopes. Having drawn the first line segment she said, "I've got to work out how much this line goes up and how much it goes along."

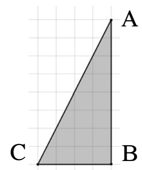

Figure 2. Gradient 2.

She gave the board-pen to a boy who had volunteered, saying, "Now, if you could just count the number of squares." The boy wrote 3½ beneath the side labelled CB in Figure 2 and then turned his attention to the hypotenuse CA. Suzie interrupted, saying, "Let me just explain, just a second … what's the three and a half here for?" The boy gestured with his left hand, starting at B and moving towards C, saying, "That's one, two, three, and that's half a square," referring to the part-square at C (the intersection of the triangle ABC and the unit square containing the point C). Suzie responded, "OK, you're looking at the area of the square there, what I actually want you to do is to look at the distance." She joined the points C and B and asked, "How many squares across is that?" Suzie erased the 3½ and wrote 4 in its place.

It is interesting to reflect that the square grid on the board facilitated the identification and completion of the "along" and "up" intervals corresponding to the line segment CA. At the same time it is reasonable to infer that the boy's misunderstanding was actually provoked by the square grid, together with Suzie's invitation to "count the number of *squares*." Yet, having identified the source of the problem ("you're looking at the area of the square there") Suzie restates the problem as "How many squares across is that?"

The iconic *Concepts in Secondary Mathematics and Science* study found "a large gap between the relatively simple reading of information from a graph and the appreciation of an algebraic relationship" (Kerslake, 1981, p. 135). In particular, the notion that proportional linear relationships hold in all segments of a line, and that lines are parallel if and only if they have the same gradient, was understood by very few pupils aged 13-15. More recently, Teuscher and Reys (2010) noted US high school students' difficulties in making connections between slope, rate of change and steepness.

Discussion: understanding concepts for teaching

A particularly pithy concept (subtraction; gradient) lies at the heart of each of these lessons and at the root of the pupils' difficulty in learning what had been explicitly stated as the objectives of each lesson. This remark is not intended as a criticism of the two teachers involved, both of whom were committed to developing their teaching and to the cause of mathematics teacher education. What I find particularly interesting is the analysis of the concepts themselves. Some of this kind of analysis is achievable by deep thought, as it were, but in some cases it needs particularly insightful observational research to prise apart, or unpack, processes and skills that inevitably become automated, and therefore trivial, to adult users of those competences. The complexity of such skills necessarily becomes invisible to the educated citizen, yet it needs to be laid bare if they set out to teach them. In his recent work on "concept study" with cross-phase teacher groups, Davis (2010)—with the concept study participants—has coined the term "substructing", in preference to Ball's word "unpacking" (Ball, Thames & Phelps, 2008). With reference to the Latin roots of the word, Davis explains that:

> To substruct is to build beneath something. In the construction industry, substruct refers to reconstructing a building without demolishing it, and—ideally, without interrupting its use. (p. 66)

This metaphor is in keeping with what elementary teachers have to do when they engage with the most elementary mathematical concepts, such as cardinality and subtraction—engaging not for personal, utilitarian reasons, but for pedagogical purposes. My proposal here is that much elementary mathematics teaching is difficult, compared with teaching in the secondary grades and beyond, because the very concepts being taught, such as subtraction, lie somewhere beneath the conscious awareness even of mathematically competent persons and their ability to analyse in pedagogically useful ways. Secondary and tertiary mathematics teaching is difficult for different reasons, where teacher knowledge is concerned. In the case of Suzie's lesson, for example, the teacher needs a good understanding of the defining characteristics of functions (*e.g.*, Even, 1990), which is advanced knowledge, in that it comes within the scope of undergraduate mathematics study. Teachers also need a thought-out, connected understanding of the different ways in which functions can be *represented* symbolically and graphically, and of how to navigate both within and between these two semiotic systems. In a recent FLM article, Zazkis and Mamolo (2011) give evidence from elementary and secondary school teaching to support the claim that advanced mathematical subject matter knowledge (AMK, acquired during undergraduate studies at colleges or universities) has the potential to enhance teachers' decision-making in response to emergent classroom scenarios. I readily agree, although I look back with mixed emotions to an early contribution of mine to the teaching of integer arithmetic (Rowland, 1982). The AMK that enabled the design of the pedagogical approach shines through the paper; so, too, does my ignorance at that time of the comparison structure of subtraction, which I clearly did not acquire in advanced study of mathematics.

Conclusion

In some ways, it is easier to continue building up the edifice of mathematics than to dig down beneath it, to establish the foundations. In the same way, engaging with the foundations of mathematical ideas that educated citizens take for granted, in order to make them accessible to young learners, poses its own distinctive challenges to teachers of students in the early grades. For more advanced mathematical topics, I suggest, the challenge for teachers lies more in the complexity of the concepts, the extent of the prerequisite concepts, and the sophistication of the semiotic systems with which they are represented in mainstream mathematical practice. Elsewhere, the case for mathematical knowledge for teaching (MKfT) being "a kind of professional knowledge of mathematics different from that demanded by other mathematically intensive occupations" has been well-made (Ball, Hill & Bass, 2005, p. 17). The thrust of this paper has been to suggest that the kind of mathematical knowledge deployed in elementary teaching is more different (from that of other professions) than that for secondary teaching, but not because it is, in some sense, easier to acquire. Using the domain-concepts and language of MKfT (Ball, Thames & Phelps, 2008), I am saying that profound knowledge of content and teaching is a particular challenge in elementary mathematics teaching—although the *representation* of abstract mathematical ideas remains crucial in the middle years. Stylianides and Stylianides (2010) have elaborated an idea of Bass (2005) to conceptualise "mathematics for teaching as a form of applied mathematics" (*passim*), discussing some pedagogy-related mathematical tasks that could help develop MKfT within teacher education programs. The exploration of conceptions of what it means for a number to be even, in their paper, comes closest to exemplifying the deep and invisible pedagogical content knowledge inherent in much elementary mathematics. Nevertheless, following Zazkis and Mamolo (2011), I suggest that secure advanced mathematical knowledge (from a course in number theory, say) should give access to useful and authentic ways of operationalising "even" for the purposes of proof, or justification, of statements about the arithmetic of parity—the field $\mathbf{Z}^2$, in effect. In terms of Shulman's taxonomy, the MKfT to be learned in initial elementary teacher preparation includes mathematics subject matter and mathematics-related pedagogy, as well as curricular knowledge. The first of these is very visible, and acquiring secure, relevant subject matter knowledge is a clear goal for prospective elementary teachers. The third, the professional "*materia medica*" (Shulman, 1986, p. 10), is all too tangible in methods courses and in the (school) workplace. However, the very existence of pedagogical content knowledge as something to be learned, and its crucial importance as a professional knowledge domain is less obvious. Drawing out the more invisible aspects of mathematics pedagogical content knowledge, some of which have featured in this paper, is necessary, but also important as a means of raising trainee teachers' awareness that not all pedagogical content knowledge can be acquired as a reflective by-product of subject matter knowledge, or even of advanced mathematical knowledege.

Notes

[1] Notwithstanding injunctions to use mathematical language correctly, it is commonplace for UK teachers, as well as their students, to refer to any kind of calculation (in this case, subtraction) as a "sum".

[2] The National Numeracy Strategy was a government initiative to improve attainment in mathematics of pupils aged 5-13 in England. The *Framework* document was superceded in 2007, but it remains influential.

[3] Porkess, R. (Ed.) (1998) *Hodder Mathematics.* London, UK: Hodder Arnold; McGuire, P. & Smith, K. (Eds.) (1998) *Oxford Mathematics GCSE.* Oxford, UK: Oxford University Press.

[4] From the ESRC-funded T-media Project 2005-07, University of Cambridge. Principal Investigator Sara Hennessy.

[5] Gradient is often referred to as slope elsewhere *e.g.,* in North America.

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