SHOULD MORE THAN ONE THEORETICAL APPROACH BE USED FOR ANALYZING STUDENTS' ERRORS? THE CASE OF AREAS, VOLUMES AND INTEGRATION

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This article addresses the question: Should more than one theory be used for analyzing sources of students' errors? For this purpose, this article examines the contributions of Stavy and Tirosh's *intuitive rules theory* and Fischbein's theory regarding *the algorithmic, the intuitive and the formal components of mathematical knowledge*, for looking into students' solutions to problems dealing with definite integrals, areas and volumes. I highlight the different sources offered by these two theories for students' erroneous solutions (*e.g.*, if b - a = d - c then

$$\int_a^b f(x)dx = \int_c^d f(x)dx),$$

and consequently discuss the benefits of different theoretical frameworks to the understanding of the question "why do students err in certain ways?"

There is a wide consensus that data regarding students' ways of thinking should be used in both the designing of mathematics instruction and in its teaching (*e.g.*, NCTM, 2000; Noddings, 1992; Tirosh, 2000). For this purpose, teachers have to be familiar with students' common errors and with their possible sources (*knowing that* and *knowing why*, *cf.* Even and Tirosh, 2003). However, *why* students frequently err in specific ways is a complex question, and, thus, mathematics educators often use theoretical frameworks as connecting organizers for investigating students' tendencies to err (*e.g.*, Tall [1]; Zazkis, 1999). Typically, the analysis of the data, in studies dedicated to this issue, uses a single theoretical framework.

Should more than one theoretical framework be used for analyzing sources of students' errors for a certain topic? I shall examine the benefits of using two theories, *e.g.*, Stavy and Tirosh's (*e.g.*, 2000) and Fischbein's (*e.g.*, 1993) theoretical approaches to the analysis of students' solutions for, specifically, integration problems. Stavy and Tirosh's *intuitive rules theory* is a *task-oriented* model, claiming that students have typical (correct and incorrect) ways for solving tasks that are structured in specific ways, regardless of their content. Fischbein's *three-knowledge-components theory* [2], on the other hand, offers a *content-oriented* interpretation of students' mathematical performances, indicating ways in which students may make content-based connections between mathematical, scientific and daily issues that are not necessarily related

The following sections include:

- a brief description of the two theoretical approaches
- data about erroneous solutions to integrals
- a discussion of the data in light of the two theoretical approaches, subsequently addressing the question: Should more than one theoretical approach be used for analyzing sources of students' errors?

The two theoretical approaches

What is the intuitive rules theory?: Stavy and Tirosh (*e.g.*, 2000) formulated the intuitive rules theory, showing that students often react to scientifically unrelated but structurally similar tasks in line with three intuitive rules: *more A – more B*, *same A – same B* and *everything can be divided*. The rules were considered to be intuitive since they have the characteristics of intuitive knowledge, *i.e.*, immediacy, obviousness, self-evidence, confidence and perseverance (*cf.* Fischbein, 1987).

Here, I focus on the intuitive rule *same A – same B*. It was identified in students' reactions to comparison tasks, when they were presented with two systems that are equal in respect to one observed characteristic A ($A_1 = A_2$), and asked to compare these systems with regard to another characteristic B (where $B_1 \neq B_2$). A common incorrect response to such tasks is: "$A_1 = A_2$ therefore $B_1 = B_2$" or *same A – same B*.

Students were found to use the intuitive rule *same A – same B* in various topics. For example, they claimed that polygons with the same perimeters must have the same area, and *vice versa*: *same area – same perimeter, same perimeter – same area* (*e.g.*, Dembo, Levin and Siegler, 1997; Hoffer and Hoffer, 1992; Menon, 1998; Reinke, 1997); and in their solutions to areas-and-volumes tasks, students tended to claim *same surface area – same volume* (*e.g.*, Stavy, Tirosh and Ronen [3]).

What is Fischbein's theory?: In his seminal analysis of students' mathematical performances, Fischbein (*e.g.*, 1987; 1993) related to three components of knowledge: algorithmic, formal, and intuitive. According to Fischbein, algorithmic knowledge is the ability to activate procedures in solving problems and understand why these procedures "work". *Formal knowledge* refers to the wider perspective of the mathematical realm – what is accepted as valid and

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how to validate statements in a mathematical context – and *intuitive knowledge* is described as an immediate self-evident cognition – students are sure, feeling no need of validation. Intuitive knowledge may lead the learner to generalizations that go beyond the given.

Fischbein explained that the three components are usually inseparable, and, often, the intuitive background hinders the formal interpretation or the use of algorithms. He presented and analyzed the sources of a number of rigid algorithmic procedures, which he labeled *algorithmic models*. For example, students' tendencies to claim that $(a + b)^5 = a^5 + b^5$ or $\sin(\alpha + \beta) = \sin\alpha + \sin\beta$, were interpreted as evolving from the application of the distributive law (Fischbein, 1993; Fischbein and Barash, 1993).

Errors in integrals found in calculus-lesson segments

This section describes five lesson segments, taught by experienced mathematics teachers, in classes of 25-28 12th graders who discussed problems dealing with integrals.

Segment 1: Parameters, areas and volumes

Students were working in pairs on a problem (Figure 1); the teacher and a visiting prospective teacher, Betty, responded to requests for help.

1. Dan [4]: That's it... we finished all we had to do
2. Daffy: No ... no ... we still have the last item.

> A straight line $y = ax$ ($a > 0$) intersects the parabola $y = -x^2 + 4x$ in two points: (0, 0) and another point, P. Another straight line, perpendicular to the x-axis, passes through P.
>
> 1. Calculate – For which value of 'a' will the *area* enclosed between the perpendicular line, $y = ax$ and the x-axis be *maximal*?
> 2. Calculate – For which value of 'a' will the *volume* created by the rotation of the previous area around the x-axis be *maximal*?

Figure 1: Problem 1.

3. Dan: [sharply] It's exactly the same solution as in Item 1... The same value of 'a' will lead to the maximum area, and to the maximum of the [related] volume.
4. Daffy: [hesitant] ehh ...
5. Betty: [to Dan] You can check your solution by comparing it with the solution in the book [*i.e.*, (1) a = 3/4, (2) a = 8/5].
6. Dan: [scornfully] There might be a mistake in the book ...

Dan was confident in the correctness of his erroneous statement *same value of 'a' for maximal area – same value of 'a' for maximal volume* solution, to the point of rejecting Betty's suggestion to re-examine his solutions in light of the different solution in the textbook.

Segment 2: Adjacent areas

A problem discussed during this lesson was:

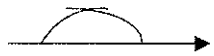
> The area limited (enclosed) between the graphs of the functions $y = \sqrt{5-x}$ and $y = \sqrt{x-1}$ and the x-axis, rotates around the x-axis.
>
> 
>
> (1) Find the marked area (2) Find the volume of the rotating figure

Figure 2: Problem 2

The solution of this problem is somewhat surprising, since in the calculation of the area:

$$S = S_1 + S_2 = \pi\int_1^3 \sqrt{(x-1)}dx + \pi\int_3^5 \sqrt{(5-x)}dx = \frac{2}{3}\sqrt{8} + \frac{2}{3}\sqrt{8} = \frac{4}{3}\sqrt{8},$$

S_1 and S_2 are equal, and in the calculation of the related volume:

$$V = V_1 + V_2 = \pi\int_1^3 (x-1)dx + \pi\int_3^5 (5-x)dx = 2\pi + 2\pi = 4\pi,$$

V_1 and V_2 are equal as well.

The students easily arrived at the area of 2 times $(2/3)\sqrt{8}$, and one student (55. Gal) wrote an expression for the volume:

$$V = \pi\int_1^3 (x-1)dx + \pi\int_3^5 (5-x)dx = \; ;$$

several students voiced their solutions for this expression:

58 Danny: I got V_1. It's 2π ... so, I actually got BOTH ... V_2 is also 2π. Sure, it's equal to V_1

60 Ron: They're [V_1 and V_2] the same ... the areas were also equal ... So, V equals two times V_1 that's two times 2π ...

61 Danny: I said it .... That's what I was saying .

Clearly, the students' "equal volumes" conclusions were based on their "equal areas" solutions. Danny and Ron confidently *deduced* that $V_1 = V_2$ because $S_1 = S_2$.

Segment 3: Areas and volumes for $f(x) = 2x$

A problem discussed in this lesson was:

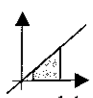
> Given is the graph of f(x) = 2x x = 0.5 and x = 3.5
>
> 
>
> 1. Find the shaded area
> 2. Calculate the volume created by rotating the shaded figure around the x-axis

Figure 3: Problem 3.

One of the students, Ron, presented a conventional correct solution on the blackboard:

$$S = \int_{0.5}^{3.5}(2x)dx = \left[2\frac{x^2}{2}\right]_{0.5}^{3.5} = \left[x^2\right]_{.5}^{3.5} = 12.25 - 0.25 = 12$$

After a short while, another student (25. Edna) was surprised to see that she reached the same, correct solution (12), although she mistakenly calculated

$$\int_2^4 \text{ instead of } \int_{0.5}^{3.5}.$$

She expected the different limits to yield different areas. The teacher asked a volunteer to check whether the areas were the same. Ann said:

27. Ann: Yeah ... What's to be done? Like . it's the same function and the same distance ehh ... from x_1 to x_2 .... so it's the same area .

Ann believed that, since it is the same function and since $x_1 - x_2$ is the same in both cases (which was a mistake), the area should also be the same. The teacher insisted that Ann show a traditional calculation. She reacted:

29. Ann: Show .. like calculate .. do the whole thing ... Yeah .. I think I can . OK ... OK ... I'll do it the long way ... [walks up to the board and writes]:

$$S = \int_2^4 (2x)dx = \left[2\frac{x^2}{2}\right]_2^4 = 16 - 4 = 12$$

You see . It's the same ... It's also 12.

Then, the class was asked to solve part 2 of the problem (Figure 3). Ron dictated his solution and the teacher wrote on the blackboard:

$$V = \pi\int_2^4 (2x)^2 dx$$

$$V = \pi\int_2^4 4x^2 dx = 4\pi\left[\frac{x^3}{3}\right]_2^4 =$$

$$\frac{4\pi}{3}\left(4^3 - 2^3\right) = \frac{4\pi}{3}(64 - 8) = \frac{224}{3}\pi$$

Suddenly the teacher realized that the original limits were between x = 0.5 and x = 3.5. She asked: "What about the second [original] volume?", and students responded:

41. Ann: It's the same areas . . it SHOULD also be the same volume . . about 75π .. we can do without the calculations . .

42. Gal: It's the same formula ... on the same function ... and in both . . the area is 12 ... It's equal . the volume .

Ann and Gal clearly and confidently expressed ideas, connecting the equal sizes of areas with the equality of related volumes, stating that "we can do without the calculations . ." Although a simple, familiar calculation could show that the volumes differed in spite of the equal areas, neither of the students bothered actually to do this calculation. They were confident in the correctness of their intuitive solutions.

Segment 4: Composite trigonometric function

Students were working on the first part of Problem 4 (see Figure 4). One student arrived at a negative area and said "I can't find what's wrong", so the teacher invited him to present his solution.

1. Find the area enclosed between f(x) = cos2(x) and the x-axis, between x = 0 and x = $\pi/2$ 2. Find the volume created by rotating this area around the x-axis.

Figure 4: Problem 4.

1. Erez: [writes] $\int_0^{\pi/2}(\cos x)^2 dx = \left[\frac{(\cos x)^3}{3}\right]_0^{\pi/2} = -\frac{2}{3}$

2. Gil: [cuts in] No ... no .. you forgot the ... to divide by the derivative of the inner function ..... can I show? [approaches the board and writes]:

$$\int(\cos x)^2 dx = \frac{(\cos x)^3}{-3\sin x}$$

3. T: [to Gil] Why is this so?

4. Gil: We did things like that ... here we divide by ($-\sin x$) .. the inner derivative.

5. T: [to Gil] What is your solution? How much is the area?

6. Gil: I did not substitute [the limits] . . You . I have to do just that .

Although the class had solved similar tasks by using the expression $\cos(2x) = 2\cos^2 x - 1$, students still tended to grasp the integral of any composite function of the type $[f(x)]^n$ where $f(x)$ is not necessarily linear

($\frac{[f(x)]^{n+1}}{n+1}$, Erez or $\frac{[f(x)]^{n+1}}{(n+1)f'(x)}$, Gil).

Erez sensed that something was wrong due to his negative result. Gil, however, did not bother to substitute the numbers, so he did not notice the zero-denominator that he would have reached.

Segment 5: Areas, volumes and intersection points

Students were asked to solve the following problem:

1. Find the area enclosed between y = (1 - x)(x - 5) and the x-axis, between x = 0 and x = 3   2. Find the volume created by rotating this area around the x-axis.

Figure 5: Problem 5.

The problem was presented with no accompanying drawing.

After several minutes of individual work in class, Eran (contribution 7) said that he got 15.75, while Shirley (contribution 8) interrupted, saying that she got 3. The teacher invited them to the blackboard:

Eran: [writes]

$$S = \int_0^3 (1 - x)(x - 5)dx =$$

$$\int_0^3 (1 - x)dx \int_0^3 (x - 5)dx =$$

$$\left[x - \frac{x^2}{2}\right]_0^3 \left[\frac{x^2}{2} - 5x\right]_0^3 =$$

$$(-1.5)\cdot(-10.5) = 15.75$$

Shirley: [writes]

$$s = \int_0^3 (1 - x)(x - 5)dx =$$
$$\int_0^3 (-5 + 6x - x^2)dx =$$
$$\left[-5x + 6\frac{x^2}{2} - \frac{x^3}{3}\right]_0^3 = 3$$

11 Eran: So … the area is the integral between the given limits . zero and three … of [points to the function] .. and I did it without opening the parentheses … integral of each [factor] between zero and three …

12 Shirley: [interrupts] But you have to simplify the expression . on this expression I calculated the integral for the area …

13 Gil: [from his place] I calculated two integrals … from zero to one and from one to three . and I did absolute values ..

14 Eran: So it's like from zero to three …

Both Eran and Shirley erroneously ignored the intersection point at x = 1. Instead of

$$\left|\int_0^1 f(x)dx\right| + \left|\int_1^3 f(x)dx\right| \text{ they calculated } \int_0^3 f(x)dx$$

and Eran also believed that

$$\int f(x) \ g(x)dx = \int f(x)dx \ \int g(x)dx$$

All in all, the lesson segments illustrate students' difficulties with definite integrals-area-volume tasks

Analysis of the data in light of the two theoretical approaches

The main aim of this article is to examine the pros and cons of implementing more than one theory for analyzing students' mathematical reasoning. To explore how the two theoretical approaches, *i e.*, the *intuitive rules theory* and *Fischbein's theory,* may contribute to our understanding of students' errors in definite integrals, areas and volumes tasks, I address the questions: What were students' errors in definite integrals tasks?; What are possible sources for students' errors? and Should more than one theoretical approach be used for analyzing students' mathematical solutions?

What were students' errors in definite integrals tasks?

A number of integration-related and area-volume related errors were evident in students' solutions. In segments 1, 2 and 3, students repeatedly expressed erroneous *same area – same volume* ideas (*e g* , Stavy, Tirosh and Ronen [3]). The novelty of the data here is that the area-volume connections are associated with integrals.

In Segment 1, there is a new variant of the belief that equal areas, when revolving around the *x*-axis, result in equal volumes. Students claimed that *the value of 'a' which yields the maximal area is equal to the value of 'a' which yields the related maximal volume.* In Segment 2, equal areas were revolving around the *x*-axis and indeed created equal volumes. However, rather than calculating and perhaps pointing to the uniqueness of the equal-volumes solution, the students spontaneously deduced the equality of the volumes from the equality of the areas. Then, Ann and Gal erroneously stated that when equal areas revolve around the x-axis, they create equal volumes

(*i.e.*, if $\int_a^b f(x)dx = \int_c^d g(x)dx$ then $\pi\int_a^b f^2(x)dx = \pi\int_c^d g^2(x)dx$).

In Segment 3, Edna absentmindedly calculated

$$\int_2^4 (2x)dx \text{ instead of } \int_{0.5}^{3.5} (2x)dx$$

and was surprised when she realized that in spite of the different limits she reached the same solution. She expected an integration of the *same function with different boundaries* to yield *different solutions*

(*i.e.*, if a ≠ b, c ≠ d then $\int_a^b f(x)dx \neq \int_c^d f(x)dx$).

Later on, in the same lesson, Ann expressed another erroneous idea, that an integration of a function between the limits a, b and between the limits c, d where b – a = d – c should yield the same solution

(*i.e.*, if b – a = d – c then $\int_a^b f(x)dx = \int_c^d f(x)dx$).

In Segment 4, two more erroneous formulas are presented:

$\int [f(x)]^n dx = \frac{[f(x)]^{n+1}}{n+1}$ (Erez) and $\int [f(x)]^n dx = \frac{[f(x)]^{n+1}}{(n+1)f'(x)}$ (Gil).

Finally, in Segment 5, Eran used a pseudo formula (Vinner, 1997):

$$\int f(x) \ g(x)dx = \int f(x)dx \ \int g(x)dx,$$

and Eran and Shirley claimed that the area enclosed by $f(x)$, the *x*-axis, x = a, and x = b, is

$$S = \int_a^b f(x)dx \quad ,$$

ignoring a significant intersection point with the *x*-axis (see also, Orton, 1983).

While it seems quite straightforward to observe "how students err" the analysis of *why* they err, and *why in this particular way*, is much more demanding. The next section offers suggestions of sources for the identified errors referring to two theoretical models simultaneously.

What are possible sources for students' errors?

This section shows how the intuitive rules theory and Fischbein's theoretical approach provide us with possible sources for students' errors. A closer look at the data from the intuitive rules theory perspective, yields that several erroneous solutions had a recurrent *same A – same B* pattern: *same* value of 'a' for max area - *same* value of 'a' for max volume (Dan, Segment 1); *same* area - *same* volume (Danny and Ron, Segment 2; Ann, Segment 3); *same* function – *same* interval - *same* area (Ann, Segment 3); and *same* formula - *same* function - *same* area - *same* volume (Gal, Segment 3)

Does this structure in the solutions imply that the students were thinking in terms of the intuitive rule *same A – same B*? As mentioned before, Stavy and Tirosh (*e g.*, 2000) identified students' tendencies to give *same A – same B* solutions to comparison tasks that present two entities that are equal in a certain respect A ($A_1 = A_2$), while asking to compare these entities with regard to another aspect B (where $B_1 \neq B_2$). Common incorrect solutions are *same A – same B*, and they typically carry characteristics of immediacy, obviousness, self-evidence, confidence and perseverance. Bearing this in mind, and re-examining the data in order to see whether the intuitive rules theory is appropriate for analyzing students' solutions in the presented lesson-sections, the students were not presented with a comparison task, but with "calculate" problems. Still, in all the cases, while explicitly dealing with calculations, the students preferred to make shortcuts and jumped to conclusions as solutions for the factors in question from an irrelevant given or from an irrelevant solution they had reached before. That is, the students, implicitly and on their own initiative, conducted comparisons, although they were neither asked to nor led to do so. This may point to the strongly preserved power of the erroneous ideas they held. The students also voiced their *same-same* solutions with no hesitations and as immediate reactions to the problems posed. The confidence they had in the correctness of their solutions and their tendency to regard their suggestions as being self-evident were expressed in their tone and in their phrasings (*e.g.*, Ann said in lesson 3, "it SHOULD also be the same volume" and "we can do without the calculations"). Most striking was Dan's firm rejection of Betty's (the prospective teacher's) suggestion to re-examine his *same-same* solution in light of the different solution printed in his book. He was so sure that he had reached the right conclusion that not only did he not agree to check it, but he assumed further that the solution in the book was incorrect.

In sum, the students might have implicitly solved self-designed comparison tasks, whose structure might have influenced the students' reasoning, eliciting answers in line with the intuitive rule *same A – same B*.

On the other hand, an examination of the data with reference to Fischbein's *three-knowledge-components theory* (*e.g.*, Fischbein, 1987; 1993), indicates that some of the previously mentioned, erroneous solutions can be interpreted by addressing students' intuitive, *content-embedded* tendencies to view areas and volumes of given figures as completely interrelated. That is, it could be that Danny and Ron (Segment 2) and Ann and Gal (Segment 3) intuitively over-generalized connections between areas and related volumes, and, thus, believed that when the areas are equal, the related volumes are equal too (see also Stavy, Tirosh and Ronen [3]).

Similarly, it could be that Dan's (Segment 1) belief that a parameter that formulates a maximal area should yield a maximal, related volume evolved from his grasp of areas and volumes being totally dependent. It is further possible that students' tendency to claim *same* function-*same* interval-*same* area (*e.g.*, 27 Ann, Segment 3), was rooted in another *intuitive, content-embedded* belief regarding connections between perimeters and areas. Possibly, the way the graph of the function and the axis surround the enclosed area reminded students of a perimeter, which they intuitively connected to the related area (see also Dembo, Levin and Siegler, 1997; Hoffer and Hoffer, 1992; Menon, 1998; Reinke, 1997; Tsamir and Mandel, 2000).

Consequently, several errors in Segments 1, 2 and 3, previously interpreted as possibly evolving from the implicit *structure of comparison tasks* and from students' use of the intuitive rule *same A – same B*, could be rooted in students' intuitive grasp of *content related* issues regarding perimeter-area-volume connections.

However, sources for all the errors identified in the data have not yet been offered. For example, what are possible sources for Erez's claim that

$$\int [f(x)]^n dx = \frac{[f(x)]^{n+1}}{n+1}$$

and for Gil's claim that

$$\int [f(x)]^n dx = \frac{[f(x)]^{n+1}}{(n+1)f'(x)} \quad \text{(Segment 5)?}$$

These cannot be interpreted by the intuitive rule *same A – same B*, but Fischbein' notion of *algorithmic models* may offer possible sources for the latter erroneous solutions. These solutions indicate students' tendencies to use previousely studied, not necessarily relevant algorithms (see also, Orton, 1983; Ferarri-Mundi, 1994). The specific mathematical conditions under which the algorithm was valid were ignored, thus the new implementation of the algorithm is erroneous, and gives rise to different algorithmic models: the *polynomial model* where students perform the integral of a composite function, while drawing on their work with polynomials

$$\left(\int [f(x)]^n dx = \frac{[f(x)]^{n+1}}{n+1} \right);$$

the *composite-on-linear model* where students treat

$$\int f[g(x)]dx \text{ as } \int f[ax+b]dx \quad \left(\int [f(x)]^n dx = \frac{[f(x)]^{n+1}}{(n+1)f'(x)} \right);$$

and the *distributive law* model

(in Segment 5: $\int f(x) \cdot g(x)dx = \int f(x)dx \int g(x)dx$,

see also Fischbein and Barash, 1993).

Clearly, both Stavy and Tirosh's and Fischbein's models contributed to the analysis of sources of students' errors in this study. A question that arises is, Should more than one model *commonly* be used to analyze students' mathematical solutions?

Should more than one model be used for analysing students' mathematical solutions?

I will first state that, in my view, using more than one theoretical framework for analyzing students' solutions is important. As shown in the analyses of students' solutions to integration-area-volume tasks, applying two models may yield three types of situation where the data can be interpreted (a) by both models (b) by only one model, or (c) by none.

In the lessons described, a number of erroneous solutions could be explained by means of each of the two models. This phenomenon may point to cases where we cannot be certain regarding the sources for students' errors. It could be that students erred merely because they used the intuitive rule *same A – same B* but it could also be that, rather than being influenced by the structure of the task, the reasons for the errors were content-oriented, *i.e.*, mal-understanding of the notions area and volume. Some students could have been influenced by both the intuitive rule and their poor familiarity with the notions at hand, and, thus, they might have become extremely certain about their erroneous solutions. Here, the use of two models provides us with multiple perspectives for possible sources of students' difficulties.

However, not all of the errors could be analyzed by both models. Several errors could be interpreted only in light of *Fischbein's algorithmic model*. That is to say, the intuitive rules theory could not suggest sources for these errors, and, without Fischbein's theory, we might have missed the theoretical interpretation of the errors, merely viewing them instead as instances of confusion. Fischbein's theoretical "lens" brings to the surface details, regarding students' difficulties, which might be ignored without them. Here, the benefits of having more than one model to interpret the data seem quite obvious

Although the two models seem beneficial in the analysis of the data, their use did not provide answers regarding the sources of all apparent errors. Some of the errors, like Edna's *different boundaries – different definite integral* solution, her mistaken substitution of 2 and 4 instead of 0 5 and 3 5, in lesson 1, and her assumption that the differences between 2 and 4 equals the difference between 0 5 and 3.5, cannot be explained by any of the two models suggested in this article. While the *different boundaries – different definite integral* solution can be regarded as being in line with a new intuitive rule, *different A – different B*, offering an extension to the intuitive rules theory. The 2 to 4 instead of 0.5 to 3 5 errors are either a result of confusion or of a general, yet unfamiliar phenomenon that needs to be interpreted by another theoretical model. Clearly, the richness of our understanding of students' ways of thinking was significantly, yet insufficiently, increased by the double-model implementation. Thus, my strong claim is that it is important to use a variety of theoretical approaches when analyzing data regarding students' mathematical reasoning

This article offers the examination of the given data in light of two cognitive frameworks In another article, Even and Schwarz (2003) showed how analyses based on a cognitive theoretical orientation and a socio-cultural orientation, led to different understandings of the same lesson The two theories explained, in different yet complementary ways, *why* students exhibited unexpected mathematical behaviors. While the cognitive analysis pointed to students' cognitive difficulties in addressing different representations, the socio-cultural analysis pointed to differences in the teacher's and the students' motives, beliefs and norms regarding school mathematics.

In conclusion, along with my strong belief that it might be beneficial to extend the use of multiple theories in the interpretations of data regarding students' mathematical solutions, I would like to indicate a point that needs some careful consideration. It is well known that theories impart interpretive and predictive powers regarding students' difficulties (*e.g.*, Guba and Lincoln, 1994; Stavy and Tirosh, 2000). It seems that the use of various theories for the analyses of research data may contribute both to the strength of the theories and the interpretations of the data However, it should be noted that theory and the interpretations of research data seem to be trapped in a vicious circle:

> research findings are interpreted in ways that confirm the theories that serve as research lenses, and correspondingly support the theories (Even and Schwartz, 2003)

In both articles, the different theories played a compatible and complementary role. What new insights can be gained by using other or additional theoretical frameworks for analyzing the data? Might we get incompatible interpretations, and, if yes, how should we go about it? Clearly, these questions call for research to accompany the wider use of theories in the analysis of data

Notes

[1] Tall D. (2005) 'A theory of mathematical growth through embodiment, symbolism and proof', plenary lecture for the *International Colloquium on Mathematical Learning from Early Childhood to Adulthood*, 5-7 July, Belgium.

[2] Fischbein addressed three components of mathematical knowledge: algorithmic, formal and intuitive However, the label *the three-knowledge-components theory* is mine.

[3] Stavy, R., Tirosh, D. and Ronen, I (1996) 'Overgeneralizations of schemes: the case of conservation', paper presented at the international seminar, *The growing mind*, Geneva, Switzerland.

[4] In transcripts, "1 Dan" indicates that Dan is the pseudonym of the student and 1 places the contribution in the sequence of the lesson; similarly, in "28 T", 'T' stands for the teachers' contribution, and '28' places that contribution in the sequence of the lesson.

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