A Mathematics Educator Looks at Mathematical Abilities [1]

DAVID WHEELER

Introduction

The professional interest of a mathematics educator in mathematical abilities may perhaps be expressed by the following questions:

> Can the mathematical abilities of students be developed, and if so how?
>
> Can the mathematics they are taught be cast in such a form that it can be apprehended with the abilities they have, and if so how?

Both are good questions (or batteries of questions) in search of convincing answers, but it is worth remarking that they approach the central issue from different points of view The first emphasises that students do not have, or do not yet have, fully developed abilities The second suggests that students already have abilities and that one could begin the task of educating them from there.

My own interest focuses on the second group of questions, for several reasons The first concerns time: in principle, answers to the second group should be quicker to obtain and lead to more immediate actions that would benefit students (Students cannot wait for researchers to undertake and process longitudinal studies; they tend in the meanwhile to cease being students and become, in a number of cases, educational failures) The second reason derives from the pedagogical point that teaching has to take account of 'where the students are' - a point that often appears stronger in its negative than its positive form, however. And, indeed, it was first formulated as a protest against teaching practices which ignored both individual differences between students and what students in general might be expected to understand at a particular stage of development 'Starting from where the students are' can only be a *nostrum*, if the techniques for achieving it are not known or not used Even so, it may bear repeating as a reminder of an ideal.

Thirdly, I opt for the more optimistic - and I venture to say realistic - attitude that accompanies a belief that students come into the classroom with a lot going for them already Even if they come with low achievement in mathematics, they nevertheless possess a range of skills, know-hows and understandings, many of which do not seem intrinsically more difficult to learn than the skills, know-hows and understandings required by elementary mathematics. It seems to me that with this attitude, teaching them something then turns into possibility, perhaps even a probability, whereas the approach that stresses the skills and abilities they do *not* possess makes teaching into a burden and a cumulative frustration.

It is no accident that my reasons are concerned with attitudes and values as well as with facts. What teachers and students actually do is a function of these variables, and more

I report one of my own values here too. I would prefer teachers to regard the students they have to teach as the 'given' from which they must work. I notice that they almost always regard the content they are supposed to teach, the mathematics, as the 'given'. Why must we expect students to accommodate to the mathematics and resist altering the mathematics so that it is assimilable by the students? If we could really hold fast to the notion that there is a dialectical process taking place in the classroom between the poles of accommodation and assimilation, we might do justice to both aspects. But simple observation shows that the subject matter almost never gives an inch. The teacher is expected to carry the remaining load of accommodation by persuading, cajoling, threatening, 'motivating' and behaving generally in a rather peculiar, un-adult-like way.

If anyone is worried by the hint of irresponsibility in the foregoing, I can only say that I believe that mathematics is more likely to preserve its purity by being willing to compromise it; certainly that spirit and power of mathematics are not indissolubly linked with particular forms, as a glance at its history will show. If I can not go as far as Bruner (1960, p 33) with his now notorious (and, no doubt, regretted) remark to the effect that anything can be taught to anyone at any age, that is because he not only ignored that there is a hierarchy of learning, though it may be, in mathematical terms, a partial and not a total ordering, but he also does not convince me that he knows how the teaching might be done It is clear, however, that he was allowing that the 'anything' might have to be re-cast. I will return later to this point when I take up the topic of mathematisation

As an educator, I am naturally concerned with my students' abilities, but I know I must not allow my view of these abilities to distract me. In some cases, it seems to me, teachers stick labels on their students' abilities and then look at the labels rather than the students. Admittedly, this labelling tends to be of a general nature: a teacher may speak of 'a very able student' or 'a weak student'. It is hard to avoid making some such classification of students. But if it leads to the teacher prejudging what an individual student can do, or cannot do, in the classroom, and adjusting his demands accordingly, then it has probably become a distraction We know that a cycle of confirmation is easily set up:

For the Learning of Mathematics 21, 2 (July, 2001)
FLM Publishing Association, Kingston, Ontario, Canada

the 'able' student is given repeated opportunities to confirm his ability whereas the 'weak' student has so few opportunities that he can only reinforce the impression of his weakness. The labels tend to gain face validity over time. A more detailed estimate of each student's abilities, a profile of his abilities, would not necessarily liberate the student from these restrictive practices.

I am drawing attention to the fact that although all students *are* different in their abilities, it is not the teacher's job to sustain and reinforce these particular differences, but rather to allow each student the chance to advance in directions he chooses. The variety in the classroom will not diminish, but it surely should shift and alter with time.

Too much attention given to abilities may not do the students much good unless it is accompanied by a deeply-held conviction that anything anyone thinks he knows about his students' abilities is of a partial and temporary nature and has no business to be in the forefront of his mind when he is teaching. Whatever assessments and evaluations of his students' abilities a teacher is forced to make for various purposes, he will prevent himself from functioning effectively as a teacher of *all* his students if he cannot suspend judgement when he is working in the classroom.

My final introductory point - another cautionary one - is that all the work so far done on abilities is embedded in a social and cultural context (i.e. a local context and a more global, 'national' context). We cannot escape entirely from the bias of focusing on the abilities that seem appropriate to our society and our culture. How extraordinary our judgements would seem to an Eskimo or a Trobriand Islander - as strange as theirs would seem to us. We would do well to keep the relativity of our studies in mind and avoid the unthinking *hubris* of, for example, Spearman who titled his influential book *The Abilities of Man*.

The language of abilities

When the mathematics educator listens to the psychologist or the mathematician talking about mathematical abilities, he hears very little that bears on his concerns and his responsibility. Mathematicians, with a few honourable exceptions (e.g. Gattegno (1970) or Freudenthal (1973)), pitch the threshold for the possession of mathematical abilities so high that 99% of all students will never reach it. When asked about mathematical abilities, mathematicians generally describe their own, knowing that their achievement is respected and rare (Hadamard, 1945). Even when their interest turns to mathematics education - it is not entirely cynical to suggest - they are less interested in finding out how to give some of their pleasure to many more students than in ensuring a continuing supply of eager (and able) young people who will keep their kind of mathematics alive.

The damage done to mathematics education by the narrow professional interests of mathematicians is only too evident in the onset and subsequent debacle of the 'new math' movement. (That movement, of course, exposed the weaknesses everywhere in the school system and contributed to the present awareness that it does not deliver what it promises.) But we should not have expected mathematicians, who would have learned mathematics however it was taught, to know anything about the demands and constraints placed upon the ordinary classroom teacher; nor should we have expected them, who were conscious early on of the quality of their achievement, to know much about or to care much about the mathematical abilities of students not as successful as themselves. [2]

The educator gets very little more joy from the psychologists. The flurry of experimental activity that led not only to theories about the structure of human abilities, but also to ability testing, factorial studies and the excesses of the psychometric movement, has died down, but not without leaving its mark. The impact of theories of intelligence dominates the folklore of the classroom, and 'IQ' and other 'objective' tests are still widely employed for their supposed diagnostic and predictive value. The 'nature v. nurture' controversy is more commonly encountered as a pawn in ideological arguments than as a genuine scientific question. (As I have said in another place, whichever side wins the day it is bad news for the educator.)

More recent developments in cognitive psychology bring some hope, but not yet much relief. The abilities that are identified seem to be either too general or too specific to be of help to the teacher. Abilities may be inferred from such low-level items of behaviour that one is at a loss to find anything characteristically human about them. Animals can be trained to press coloured buttons and human beings who press them on demand in laboratories have somehow been persuaded to behave like animals. Even non-trivial tasks, like matching representations of three-dimensional objects (Shepard and Metzler, 1971), often fail to convince because they leave out of account all the sustained, purposeful forms of human behaviour.

Cognitive psychologists may say that they are concerned with processes, not with purposes (Dodwell, 1978). Perhaps no single sentence could more clearly illuminate the distinction between the psychologist's and the educator's points of view and indicate why the educator cannot let the psychologist have the last word on anything, not even processes. The cognitive psychologist may be primarily interested in identifying and describing processes; the educator would like to know how to induce them. The Piagetian psychologist may spend his time delineating intellectual development; the educator would like to know how to influence it. Psychologists in general may exclude purposes as metaphysical; the educator has to put them up front.

Cognitive psychology, however, does study human processes rather than human traits, and it is possible that when it develops a solid theory and some sets of convincing results it will eventually give educational folklore a welcome change of direction. Putting it very unsubtly indeed, I would say that it is healthier for educators to think of learning as located in the brain - since everyone has a brain, and the brain functions in certain universal ways - then to think of learning as a product of intelligence, an undefined quality that everyone is supposed to possess a certain amount of.

The difference between the theoretical models can lead to striking differences in the interpretations put on students' behaviours. The 'intelligence model' suggests that when a student fails to learn something, it is because he cannot, because he does not have enough of the quality labelled 'intelligence', because he is incapable. The 'brain model', on the other hand, suggests that failure to learn something is due to the student not yet knowing how to direct his mental processes, not yet being able to program his brain. Expressing the difference with the broad strokes of folk beliefs: the brain is educable, intelligence is not. It is not difficult to see which belief gives the educator more grounds for optimism.

Theories about abilities can point in both directions When Krutetskii (1976) classifies his experimental subjects as 'less capable' and 'more capable', he is leaning towards that trait-fixed/intelligence-ineducable model; yet he also talks, and not only for ideological reasons, of abilities being developed under the guidance of school instruction and indicates that students can acquire greater mastery of their mental processes. Perhaps this ambivalence is inevitable until we have developed better theories, not only of how learning takes place and how abilities are developed but also of the precise part that teaching interventions play

These criticisms and qualifications may suggest that an educator is not interested in the subject of abilities. This is not the case, although it may not head his list of useful preoccupations

The word 'ability' is highly suggestive of attributes which are very much the educator's business. First, and most obviously, it suggests 'being able to', a capability for doing something; the educator can expect a response when he invites a student with a particular ability to do something, to perform some appropriate action. Without this, where could the educator begin? Second, 'ability' suggests a personal possession of trait; it is some capacity that belongs to a person, to this student, exclusively to him (though others may show that they possess a 'similar' ability), and which the student can use whenever he chooses. It is the basis for the active contribution that the individual student can make to the teaching-learning process.

Third, 'ability' suggests stability, some relative permanence of capability. Neither the teacher nor the student could function adequately without this degree of stability, if abilities appeared and disappeared, no one knowing why or when. Fourth, 'ability' has overtones that hint at potential as well as achievement It is this sense of power not yet fully realised that gives the confidence (to both parties) that there could be a future to the teaching-learning dialogue.

The last implication may sound somewhat unscientific, since we can only know for sure that someone possesses an ability if he shows it by doing something, either spontaneously or in response to a given task 'Potential' may be an unwarranted inference. Yet it is also the case that we sometimes know intuitively that someone is not displaying all his abilities; we suppose that he can do more, but the more never materialises. Perhaps we have picked up minute behavioural signs, too small to be consciously registered; perhaps we have sensed the presence of affective inhibitors – shyness, say, or truculence or (literally) unwillingness. The inference of potential in these cases is a matter of common experience.

But the strongest argument for giving weight to 'ability as potential' is that human learning is evolutionary Not all learning evolves; some learning, especially of skills, seems to reach an optimum point and stops developing, settling into routines and patterns of behaviour. Sometimes, other kinds of learning stop, as when we say of someone that he is prejudiced, or full of preconceptions, or wilfully ignorant The learner in these cases has decided to stop learning, for whatever reasons The pejorative flavour of our terms, though, shows that voluntary cessation of learning may be common but is not admirable. And, indeed, in a clear sense, it is not natural. Human beings are learning systems, able to adapt in sophisticated ways to new circumstances.

Abilities are the signs of human adaptability to the challenges of the social and cultural environment That abilities may be allowed to atrophy no more disproves their evolutionary status than that the failure of a species to adapt to changed circumstances disproves the general validity of biological evolution.

The nature of abilities

A question that the educator may ask is whether students' ability to learn is some function of a relatively homogeneous ability, or some set of highly differentiated abilities For the purposes of a convincing theory, this distinction is important for it has many theoretical ramifications. But, in practice, it may not appear to make much difference to the teacher whether his students have a general ability which varies from one to the other in amount or in the stage of its development, or whether they possess various specific abilities to different degrees – the unevenness of their response will seem much the same in either case. The picture of children's ability which Piaget's work gives suggests the former, general character; the results of the psychometricians and the laboratory psychologists suggest the latter, highly specific interpretation

There may be little incompatibility between these two positions, since Piagetian studies generally concern preschool and grade school children, whereas data used for factor analytical studies have generally been obtained from high-school students, older students and adults It may be that the progressive differentiation of abilities overlaps the general stratum of ability as the person grows and learns (Hunt, 1961). As Jensen (1973) points out, 'Learning abilities also grow out of learning itself' (p 339).

But we have the example of exceptional talent, perhaps in music or mathematics or athletics, to show that highly differentiated abilities can appear very early in life, apparently countering the above resolution. This kind of noticeable talent does not seem to be normally distributed in the population; the distribution appears to be highly skewed. A general, relatively homogeneous ability could be expected to be distributed normally. But, as in all research situations, the answers one gets depend a great deal on the questions one asks. If general ability is assumed or presupposed, the researcher will tend to sample behaviours across a wide band of types of situation by including one or maybe two examples from each. This procedure damps down the potential effect of superior talent in any one direction by weighting each behavioural response equally. It is not surprising, then, that an approximately normal distribution of *average* responses is achieved. IQ tests, for example, exhibit this randomisation effect (Hawkins, 1977).

More anecdotally, we may take note of the fact that a teacher is quite often surprised to discover by accident, through some event taking place outside the school, some particularly strong talent in a student that nothing in the student's classroom behaviour has signalled. The opportunities provided by the classroom have in this case damped down, or eliminated, the idiosyncratic responses that would have shown the presence of an outstanding ability It may even be the case, though it is no doubt less likely, that the usual conventions of classroom behaviour mask the presence of high levels of, say, linguistic, artistic or mathematical abilities, even though these are related to the content of the curriculum. 'Normality', for all sorts of reasons, tends to be self-confirming

Any account of learning ability must be able to explain not only what students learn in the conditions of school, but what students learn by themselves before, during and after their schooldays. The phenomena of human abilities include abilities that are common across the species, differentiating it from all other species; various abilities that express themselves in the form of distinctive talents; abilities that appear to be embedded in a particular culture; abilities that appear to be associated with particular stages of human development - and perhaps more.

That these categories overlap and interact is obvious, as is the fact that the particular 'mix' of abilities in an individual cannot be explained on the basis of a simple appeal to the effects of heredity or environment Whatever matrix of possibilities may be characterised by an individual's genetic make-up, his local environment, culture and education, these forces do not determine what he will become We know that although a student can be taught by a teacher, there is no simple and transparent connection between precisely what he learns and what his teacher meant to teach him (Hawkins, 1977, p 78) There is no evidence to suggest that the individual's response to any of the chemical, behavioural or cultural impacts on his system is any more precisely prescribed.

> Heredity, education, environment, physiological constitution, are the great explanatory idols of our time - and they explain nothing. The true cause, the real motive of human behaviour, is an original project-of-being freely chosen at the moment one wrenches away from the *en-soi* to create one's own world. (Sartre [3])

Whatever the exaggerations of Sartre's protest (which would have us believe, which we cannot, that we can throw off all constraints at will), it reminds us of a component of reality that psychologists and educators have largely chosen to ignore Indeed, the biological story of human abilities can never tell us all we need to know if it systematically excludes consideration of the facts of human consciousness

Bartlett, in the conclusion to his book *Remembering*, points out that the psychologist begins to investigate the phenomena of memory long after the subject has laid the main tracks and established the general mechanisms for remembering.

> Already the long struggle which results in the specialisation of the senses has attained its main ends, already the organism with which the psychologist is concerned has discovered how to utilise the past in such phenomena as those of lowered threshold, of chain and conditional reflexes, of 'schematic' determination, and in the sequences of relatively fixed habit. But these, all necessary in their way, still cramp and confine man's activities For in them all the past operates *en masse*, and the series is of greater weight than its elements. [...]
>
> If any marked further advance is to be achieved, man must learn how to resolve the 'scheme' into elements, and how to transcend the original order of occurrence of these elements. This he does, for he learns how to utilise the constituents of his own 'schemes', instead of being determined to action by the 'schemes' themselves, functioning as unbroken units. He finds how to 'turn round on his own schemata', as I have said - a reaction literally rendered possible by consciousness and the one which gives to consciousness its pre-eminent function (1932/1967, p. 801)

The investigator of human abilities is in a similar situation The groundwork has been laid before he comes on the scene; some if not all of the abilities he finds are well-established. The duration of his contact with the subject is generally extremely short compared with the time needed to develop a particular ability, certainly too short to detect any further shifts or developments. In this situation, the investigator can easily slip into the way of regarding abilities as traits, inferring permanence as attributes of the individual that they do not intrinsically possess. (His confidence in abilities as predictors of future behaviour stems from this) And if the investigator uses tools that elicit largely habitual, schematic responses from his subject, he will obviously confirm his preconceptions

Bartlett's reminder of the role of consciousness, that it enables the individual to modify and rise above his schematic responses, suggests that the investigation of abilities might include opportunity for the subject to show what he can do when confronted with an unusual situation. Students given non-standard problems to solve frequently give evidence of abilities that their teachers would not have expected Krutetskii again sits in the middle, deriving many of his experimental problems from school mathematics, thus tending to call on schematic responses, but asking his subjects to 'think aloud', a procedure that alerts consciousness.

The mathematical abilities of students

> The justification for treating human language as a biological phenomenon comes from modern linguistics According to Chomsky and his followers, human language, irrespective of race and culture, is based on innate grammatical and syntactic structures common to all normal human beings. To a biologist, this can mean only that somehow the inner structure of language is genetically determined. That is, language and its intellectual correlates are the functional manifestations of a specific, genetically determined system of nervous connections in the cerebral cortex The enormous growth

of the human brain cortex in the astonishing short time of a few hundred thousand years may have been a correlate of the development of language, just as the expansion of a lobe of the brain of electric fishes was bound to the dependence of these animals on the detection of electric fields.

> Note that a biology of language as I envisage it here will include a biology of thinking processes such as logical structures, *a priori* ideas, artistic creation, and even ethical principles. To a very large extent, the actual contents of these areas must, of course, be of environmental origin, just as the actual language you and I, or a Chinese or a Bantu, speak is dictated not by genes but by upbringing. At the same time, a biology of language could be a truly humane science since it would address itself to qualities common to all men, not to differences between men. It may generate an applied science too, by discovering better ways to teach, to learn, and to make use of what we learn. (Luria, 1974, p. 28)

The above remarks may no longer surprise us very much, though it is worth recalling how recently they have become an acceptable challenge to science. How would they be received, I wonder, if 'mathematics' were everywhere substituted for 'language'? Perhaps one might prepare a case for the legitimacy of the substitution by attempting to spell out the reciprocal aspects of the relationship between language and mathematics

That language is a precursor of mathematics in the individual is generally taken for granted, yet the example of born-deaf children shows us that the absence of language does not entirely inhibit the development of mathematics however much it may impair it (Furth, 1966) We may notice further that when we try to articulate young children's competence in spoken language, we are more or less forced to use mathematical descriptors, which leads to the inference that at least some of young children's mental processes have a mathematical character. Indeed, it appears we may simply say that mathematics needs some language, and that language needs some mathematics The suggested substitution may not be as improper as it first appears.

Children's acquisition of the spoken language of their culture and the development of their visual perception are two examples to show that pre-school children have the ability to mathematise - that is, they have the ability to perform mental operations such as classification, transformation, substitution, reversal - and the almost universal achievement of these skills shows that this ability is common to everyone, an attribute of their common humanity The fact that we can subsequently observe great differences between people in the extent to which they exhibit a talent for mathematics is no more surprising than that people can also be differentiated in adult life by their talent for using language.

But the extraordinary complexity of the skills that have to be mastered in order to achieve functional, everyday speech from nothing, as it were, demonstrates that even the 'least gifted' have, in fact, been endowed with substantial genetic resources of linguistic and mathematical ability. This substratum of ability has been overlooked by experimental psychologists (and by Piaget) and has rarely been mined by teachers, except by accident.

That the ability to mathematise, though overlaid by complex learned systems, remains present and on call can be confirmed by observing some of the spontaneous behaviours of older children, adults and experienced mathematicians

Six-year-old children taught addition as 'counting on' spontaneously develop the strategy of counting on from the larger number, however the addition is presented to them (Resnick, 1976).

A twelve-year-old girl, enumerating the paths along the edges of a rectangular parallelopiped from one vertex to the opposite vertex, says that there must be the same number of paths that start with each distinct edge through the first vertex:

> because it is obvious that there are. There would be if it was a cube and the different shape does not make a difference (Kent, 1978, p 13)

A thirty-six-year-old man, burdened with a phenomenal memory, solves the problem of adding a metre length to a cord wrapped around the equator and finding the resulting height of the cord above the earth's surface:

> Then I saw a box near the door I made a sphere out of it [...] I added exactly one metre to the corners of the box [...] I drew the strap back along the side and came out with a measurement of 12.5 cm for each side [...] I then converted the box back to its normal shape All I'd have to do was cut off the corners to convert it into a sphere, and I'd get the same result again (Luria, 1968, pp, 102-103)

Leibniz notices that if each of a sequence of diminishing line segments in geometric progression is laid along a line, starting from a common point, the differences between the segments, read in the reverse order, form another geometric progression (Hofmann, 1974).

In these paradigmatic cases, the crucial step in the argument is a momentary surfacing of the ability to mathematise, apparently uninfluenced by learned patterns of thought or behaviour. It is at least plausible that this latent power, though not often noticed, is operating whenever people are grappling with mathematics and not merely imitating it Mathematising is brought into play whenever the individual is reconstructing mathematics, even when the results are entirely conventional in the eyes of an observer

Although mathematisation must be presumed present in all cases of 'doing' mathematics, or 'thinking' mathematically, it can be detected most easily in situations where something not obviously mathematical is being converted into something that obviously is. We may think of a young child playing with some blocks and using them to express awareness of symmetry, of an older child making shapes on a geoboard and becoming interested in the areas of the triangles he makes, of an adult observing a building under construction and asking himself questions about the design, and so on We notice that mathematisation has taken place by the signs of organisation, of form, of additional structure, given to a situation.

Consider the experience of solving a non-standard problem or mastering a new game In each case, there are moments when the whole situation, or a part of it, is suddenly seen differently; it may be difficult to recapture earlier impressions after such moments. (There is the common experience of retreading unsuccessful tracks in the pursuit of the solution to a problem and the difficulty of wiping out these attempts in order to start again *de novo*.) The perceptual differences mark new stages in the structuration of the situation.

In analysing acts of mathematisation, it is not particularly difficult to describe the structures that are successively imposed, but it seems much more difficult to get close to the 'reasons why', to the actual source of the decisions that the mathematising person takes The energy source that powers the structuring activity appears to be awareness - awareness of some characteristic feature of the situation that lends itself to being further structured: one suddenly 'notices' something that one had previously overlooked

The surge of energy that results may sometimes be strong enough to produce a 'Eureka' effect. But why 'this' awareness rather than that? 'What' triggered the awareness? These and similar questions of 'how' we recognise objects and situations is unanswerable We only know that we do and that there is evidence that everyone else does too.

A difficulty in devising a pedagogy that utilises the power to mathematise is to give a clear picture of what a teacher should do about it A pedagogy based on mathematics - especially 'ready-made mathematics', to use Freudenthal's term - shows us a teacher who is mainly an expositor, someone who displays to his students selected parts of the mathematics that is already known This is not a trivial function; it requires considerable skill and it provides a simple image of what a teacher does that, with a few elaborations, is part of the folklore.

But a pedagogy based on mathematisation must give more responsibility to the students It is they who by the exercise of their mathematising powers will cause the mathematics to be 'acted out' (Freudenthal's term again) - i.e. reinvented, rediscovered, recreated, 'brought into being'. The image of the teacher in such a situation is much less simple, more shadowy. What is he actually doing? He is not teaching mathematics, in the customary sense: but nor is he teaching the students to mathematise, since they already know how.

An analogy may he helpful here Everyone is born with an innate ability to perceive, but particular perceptions cannot be taught directly (or only with extreme difficulty and under unnaturally controlled conditions). Nevertheless, the ability to perceive is clearly educable and is influenced both by experience and by the interventions, intentional and unintentional, of a great many people. Mathematisation seems to me to be an entirely similar case The teacher, from this point of view, is a professional interventionist who brings to the students those situations which will educate their mathematising capacity. In the process, they will learn mathematics [4]

A pedagogy based on the ability to mathematise must include at least the following elements

1. The teacher/educator must be able to reorganise and recast substantial portions of the mathematics curriculum so that they can be mathematised. An extreme example which may make the point is the restructuring that needs to be done by anyone wanting to make a mathematical film The content has to be detached from its normal context, its formal expression and the method of its original invention and considered afresh to see how it might be apprehended 'from scratch' by someone viewing the film.

2. The teacher/educator must be able to select certain 'proxy' experiences which show the students how certain situations have been mathematised Solutions to problems, examples from history, on-the-spot demonstrations of his own ability to mathematise are suitable media

3. The teacher must be able to take advantage of the spontaneous events of the classroom that will occur when students are given the freedom to express their own attempts to mathematise. Indeed, this is a function that *only* the classroom teacher can perform, seizing opportunities as and when they present themselves

Teaching of this kind is a blend of knowledgeable planning and sensitive on-the-spot opportunism. It presents a different picture of what a teacher should be doing from that which is embodied in most textbooks, individualised learning schemes, and the like It requires the teacher to know something about mathematics, and more particularly about the various ways in which mathematical knowledge can be constructed

There would be no point in even mentioning it if I did not know personally many teachers who can function at this level of professional skill. Given different kinds of textbooks and resource materials, and provided with suitable support systems, there is no reason why the majority of teachers could not teach in this way [5]

It may sound to some readers as if I am trying to turn the world upside down by presenting mathematics as something that everyone can learn. They may fear that I intend school mathematics to be diluted more than it already is, or that I recommend abandoning all the traditional goals of mathematical knowledge and fluency.

I can only deny the inference. That everyone can mathematise does not *ensure* anything. All it says (though this is a great deal, I believe) is that the ability to mathematise is a personal resource that any teacher and any student can use to gain entry into mathematics, from whatever point they may start. Whether they *will* use this resource or not is up to them; whether they will like mathematics and want to pursue it or not is also up to them

When the entry into mathematics has been made, what follows will look to an outsider not unlike what he sees successful students do now Mathematics has a social face, it is recognised by its appearance as much as by its character. If students are going to show people other than their teacher that they know mathematics, they have to acquire its conventional expressions and show facility in its formalised procedures; their mathematical abilities will show up as they do now

Mathematics has a hidden life. When mathematicians talk about their activity, they occasionally give us glimpses of the processes that they use, processes that are not reported when the mathematics they have invented is written down. These processes are the mind mathematising Talking about mathematisation is not a sign of a new way of doing mathematics but a bringing into the open of the universal way of doing mathematics. The differences that talking about it may make are attitudinal and technical: someone who is aware that he can mathematise is in a different learning posture from someone who thinks that mathematics is exclusively a product of brilliant minds; the educator can use his insight into mathematising processes to develop techniques that will help the learner gain entry

This latter, technical, aspect is one of many overlooked in the nevertheless interesting article by Davis and Anderson (1979). They do not attempt to build a bridge between 'Platonic' mathematising and 'conscious mathematics', and indeed could not do so without giving the phenomenon of unconscious mathematising considerably more penetrating attention. But even if they explain the phenomenon away too cursorily, one is glad to have it on record that professional mathematicians have noticed it.

How far the ability to mathematise can fairly be regarded as the sum of anyone's distinguishable mathematical abilities is a question that has not been examined Certainly, some of the mathematical abilities that researchers have identified may be aspects of the more general ability to mathematise, but too many of the abilities defined by research are content specific in a way that the ability to mathematise certainly is not (The extreme cases of content-specific abilities, such as 'the ability to add whole numbers', are not abilities in any acceptable psychological sense but are just performance descriptions. It is necessary to distinguish between the inference of abilities from performance, which is reasonable, and the identification of abilities with performance, which is absurd.)

I conclude this attempt to reorientate the *pedagogical* study of mathematical abilities with one further observation. We sometimes use the word 'gift' to describe an ability that strikes us as exceptional. The word suggests that the possessor has been the fortunate recipient of his ability (from his parents? from his circumstances? from God?), but when we talk about a 'gifted musician', say, we are actually describing someone who has *given himself* - his energy, time, patience and enthusiasm - to the realisation of his potential. To a lesser degree, but necessarily, the possessor of an ability has given something of himself to its development. The spectrum of a person's abilities, and the estimated measures of each, do not tell us what that person was given to start with, but are the record telling what functions he has worked on, and to what extent. [6]

Individual differences

In a fundamental sense, individual differences of students are none of the educator's business The teacher should, I have suggested, consider his students as the most important of his 'givens' They come to him as they are, a multiplicity of states of being, over which he has no control, and to do his job even moderately effectively he has to accept the situation he finds

It seems strange to me that very few educators appear to see that if this act of acceptance is valid at the beginning of a teacher's contact with his students, it has equal validity at every moment of their time together To suppose otherwise is to fall, perhaps without knowing it, into the error of believing that it is a teacher's duty to change his students, or to change some of them in some respects. There are philosophies of education which consider the teacher to be essentially a moulder of unformed and inadequate lives. I reject such philosophies, and I reject any equivalent set of educational principles whenever I am alert enough to notice the similarity.

Unfortunately, teachers often invest in their students' development, and are encouraged to do so by the rewards and recognitions of the system they work in But teaching is difficult enough without burdening teachers with a responsibility that they do not have the powers to discharge. The students will learn, develop, change, if and only if they do these things for themselves.

It may be a civil rights issue to give every student the opportunity to receive a high-quality education; it is also each student's inalienable right to take it or reject it, to take some of it and reject the rest, to take some of it sometimes and to reject the rest of it the rest of the time, and so on. The teacher's responsibility ends with providing the opportunity to learn, develop, change, to all his students - a difficult enough aim, in all conscience. He need not congratulate himself or blame himself for what his students make of what he offers. (We know the customary variant of this procedure, of course.)

It will be apparent to readers that I am trying at various points to stress that the usual system of values assigned to differences and sameness might benefit by being stood on its head.

> A teacher meeting a class for the first time is confronted with a vast and unknown variety, not a group having a uniform mastery of last year's curriculum
>
> No class of students is ever homogeneous
>
> Every student present during a common lesson learns something different.
>
> The learning difficulties of students are always specific.
>
> Buttressing the variety of individual abilities possessed by students are the common abilities of humanity.
>
> Tailoring educational 'treatment' to the needs of each student implies finding pedagogical methodologies that are effective across the spectrum of abilities

I am not attempting to explain away the matter of individual differences. It is symptomatic of a mis-diagnosis somewhere that the subject has only recently appeared to grab the attention of educators and researchers. Yet who can ever have doubted that every student is different from every other student? The concern *cannot* be about the facts of human differences. It may therefore turn out that to research differences in abilities, or any other human attribute, to document and describe them, is to expend a great deal of energy in tackling the wrong educational problem

The chances are, too, that researching differences will never achieve the definitive conclusions that may be anticipated, especially if the aim is to tell teachers how to deal with them [7] I am reminded of the extraordinary difficulty that has been experienced in analysing young children's mastery of speech, or in programming computers to translate simple statements from one language to another The researchers no doubt said to themselves, "If it can be done - and children can certainly speak and linguists translate - than we should be able to analyse how it is done"

The examples perhaps indicate the great gap between successful functioning and an analytical knowledge of that functioning. I venture to make a parallel with the gulf between a teacher's ability to function in a complex world of diverse learning abilities and the analytical knowledge that would be required to program someone to function in that world Some researchers, I believe, are deluding themselves by thinking that at the end of their labours they will be able to instruct teachers how to behave.

I admit inexactnesses in the parallel, but I find enough truth in it to suggest that the problem of individual differences should properly be called the problem of functioning as a teacher with heterogeneous groups of students. Such functioning is not, and could not be, based on a complete analytical knowledge of the situation And if we want teachers who cannot yet function in this way to be able to do so, we should keep on reminding ourselves that no one *tells* a young child how to speak or a translator how to translate. Most modes in which people learn to function are acquired, not taught, and we would be better occupied in looking directly at what they do when acquiring the function, not pecking away at the function itself.

Having expressed sufficient scepticism about the more grandiose ambitions of researchers into individual differences, let me return to one positive value that research into individual differences might have Many teachers do not appreciate the wealth of the evidence available to them and are not sufficiently sensitive to the ways in which ability differences show themselves (Easley, 1980). For those teachers who are already aware that there is more under their noses than they are able to detect, research findings can alert them to the range of possibilities and the kind of signs to look for

In this sense, such research can become an invaluable observational tool to assist them in their functioning in their own situation. It can help them to interpret the manifold feedback of their students. Opening the eyes of teachers who do not know that they are missing a lot is a different problem and conventional research will not touch it.

It seems as if individual differences in the classroom constitute a problem: in everyday life, they are a source of delight. Perhaps teachers cannot be expected to take joy in the variety surrounding them, as neither they nor the students chose their mutual proximity. But it is common sense to acknowledge the differences between students and only humane to respect them

Conclusion

The tenor of this piece has been that the teacher of mathematics and the mathematics educator - by which I mean a person who has given his time to study of the practices of teaching and their theoretical bases - do not always find their concerns being addressed by those who are investigating mathematical abilities In concluding it, I want to say explicitly that the questions that belong to the field of mathematics education are *sui generis* and that it should not be hoped or expected that they will be answered by mathematicians, or by philosophers, by psychologists or sociologists, who have their own concerns

When the mathematics educator goes to, say, the psychologist for help, he has to do what the engineer does with the techniques and knowledge of the natural scientist; convert what he is given to his own purposes and test it in the field, not in the laboratory. Until it is transformed, it is, as far as he is concerned, inert and useless. While it stays in the textbook or the learned paper it is, literally, only of academic interest to him.

The mathematics educator, like the engineer, the doctor, the lawyer, is quintessentially a practitioner, in that his primary concern is to find what will work in practice - i.e in particular cases. At the same time, as a professional, he has a responsibility to find what will work not only when *he* practises it but when others practise it also But he is not - as I said earlier - under pressure to find what is true in general, in all cases everywhere. Unlike the experimental scientist, he must always return to the source of his questions in all its complexity [8] This does not make him any less a scientist, but he cannot use the laboratory techniques of the natural scientist - unless, indeed, he regards the classroom as his laboratory.

The following, although written about the medical scientist, may reinforce the point:

> One should not fling a raw fact onto paper in public, as a keeper flings a chunk of raw meat to a tiger I believe that in medicine we have a unique advantage in this respect over the purely experimental scientist, in that medicine, while becoming increasingly an experimental, has long been and must continue largely to be an observational science. In its observational aspect it deals with a supremely difficult material under conditions that make constant demands upon intuition and judgement. Nature is not interested in scientific method, and the experiments she provides for us in the guise of disease and injury we have to take as we find them; we cannot subject them to the necessary but artificial simplification that is the essence of a good experiment We are therefore forced to think, to synthesise, and to interpret our evidence to a point rarely necessary in the designed laboratory experiment. (Walshe, 1945, p. 725)

The educator, somewhat like the *bricoleur*, will take what he can and bend it to answer his needs, while his experience of the various problems he encounters sharpens his eyes to the potential usefulness of the material produced by others working on quite different problems. This back-and-forth process is not as difficult to learn as it sounds; the educator does not need to know all that is known by the people he steals from But it is hard enough, or perhaps unfamiliar enough, for few to immerse themselves in it

Anyone can work on the problem of identifying and describing mathematical abilities, and the educator may well say, 'all contributions gratefully received' But when it comes to the point of trying to apply these discoveries for the benefit of teachers and students, the mathematics educator is the only person in a position to decide what should be done. Let us hope he will steel his nerve and rise to the responsibility.

Acknowledgements

This piece, never published before, was the lead paper for a conference held in June 1980 in connection with the Mathematical Abilities Project directed by Jeremy Kilpatrick at the University of Georgia. Support for the project was provided by the National Science Foundation under Grant No SED 77-17946. The opinions, findings, and conclusions or recommendations expressed in the paper are those of the author and do not necessarily reflect the views of the National Science Foundation.

The editor is also grateful to Dianne Boyko and Dave Wagner for assistance in the preparation of this manuscript for publication

Notes

[1] [*Editor's note*] See the Acknowedgements for a brief account of the genesis of this piece The manuscript carried the dating 'Drafted June 11, 1980, Revised July 1, 1980'

[2] It will be unfortunate, however, if the interest of mathematicians in education continues indefinitely to be suspect, for mathematics education needs mathematicians Mathematics in schools, particularly now that it is subject to such intense public scrutiny and concern about falling standards in the 'basic skills', understanding of the fundamental concepts, and the like, is, for most students, limited, unadventurous and excessively boring We need mathematicians to remind us that mathematics can be exhilarating; to use their imaginations to show us how excitement can be found in the most elementary parts of the subject. But let them intervene no further than this - unless, that is, they give their time to gain the experience and acquire the sensitivities of the good classroom teacher.

[3] [*Editor's note*] Part of this quotation attributed to Sartre is very similar to p. 559 of Hazel Barnes' (1956) English translation of *L'Etre et le Néant (Being and Nothingness)* There, her English rendering of the original reads:

> Open any biography at random, and this is the kind of description which you will find more or less interspersed with accounts of external events and allusions to the great explanatory idols of our epoch - heredity, education, environment, physiological constitution.

However, although being very consistent in terms of style and terminology no subsequent paragraph in the book matches the second part of this purported quotation.

[4] Descriptions of how this can be done can be found in, for example, Gattegno (1974) and Goutard (1970).

[5] 'Could': i e 'could if they would'. I have tried to avoid talking in this piece about 'ideal' teachers and classrooms. Ideal teachers and classrooms exist only in Never Never Land But I do intend to contrast the 'real' classroom with the 'realisable' classroom, i e. the one which good teachers approximate on their good days.

[6] In this section, I have in places drawn on the text of a talk I gave on 'Mathematisation as a pedagogical tool' to the ICMI subsection of the IMU Congress, Helsinki, in August 1978 [A version of this paper is also reprinted in this issue - *editor.*]

[7] Commenting on a piece of research on the teacher's role in knowledge acquisition, Philip Jackson (1977) noted that the research 'seems yet to have caught up with common sense' (p 400)

[8] There are two orders of complexity in the classroom, one of which may interfere with the other. The one I have mainly in mind is the complexity of the teaching-learning dialogue, given a reasonably free teacher and reasonably co-operative students - the 'realisable' classroom. But the classroom is also the complex intersection of many fields, some of which act to constrain the teaching-learning dialogue - the textbooks, the curriculum, the administrative superstructure, parental and public expectations, current folklore, etc., etc The sociology of the 'real' classroom is illuminated in Lortie (1975).

References

Bartlett, F. C (1932/1967) *Remembering*, Cambridge, Cambridge University Press.

Bruner, J. (1960) *The Process of Education*, Cambridge, MA, Harvard University Press.

Davis, P. J. and Anderson, J A. (1979) 'Nonanalytic aspects of mathematics and their implications for research and education', *SIAM Review* **21**(1), 112-127.

Dodwell, P (1978) 'The scope of modern cognitive psychology', in Higginson, W (ed.), *An Image of the Whole: Knowledge and Curriculum in an Age of Fragmentation*, Kingston, ON, Faculty of Education, Queen's University, pp 97-106

Easley, J (1980) 'Alternative research metaphors and the social context of mathematics teaching and learning', *For the Learning of Mathematics* **1**(1), 32-40.

Freudenthal, H. (1973) *Mathematics as an Educational Task*, Dordrecht, D Reidel.

Furth, H (1966) *Thinking Without Language. Psychological Implications of Deafness*, New York, NY, Free Press.

Gattegno, C. (1970) *What We Owe Children: the Subordination of Teaching to Learning*, New York, NY, Outerbridge and Dienstfrey.

Gattegno, C (1974) *The Commonsense of Teaching Mathematics*, New York, NY, Educational Solutions

Goutard, M (1970) *Mathématiques sur Mesure*, Paris, Classiques Hachette

Hadamard, J (1945) *The Psychology of Invention in the Mathematical Field*, Princeton, NJ, Princeton University Press

Hawkins, D (1977) *The Science and Ethics of Equality*, The John Dewey Society Lecture No. 16, New York, NY, Basic Books.

Hofmann, J E (1974) *Leibniz in Paris: 1672-1676*, Cambridge Cambridge University Press

Hunt, J McV (1961) *Intelligence and Experience*, New York, NY, Ronald Press.

Jackson, P. (1977) 'Comments on Chapter 11 by Berliner and Rosenshine', in Anderson, R. C , Spiro, R J and Montague, W E (eds), *Schooling and the Acquisition of Knowledge*, Hillsdale, NJ, Lawrence Erlbaum, pp 397-402.

Jensen, A. R. (1973) *Educational Differences*, London, Methuen

Kent, D. (1978) 'Isobel', *Mathematics Teaching* **85**, 12-14

Krutetskii, V. A (1976) *The Psychology of Mathematical Abilities in Schoolchildren* (Teller, J. trans., Kilpatrick, J and Wirszup, I, eds), Chicago, IL, University of Chicago Press.

Lortie, D C (1975) *Schoolteacher: a Sociological Study*, Chicago, IL, Chicago University Press.

Luria, A R. (1968) *The Mind of a Mnemonist* (Solotaroff, L , trans), New York, NY, Basic Books.

Luria, S E. (1974) 'What can biologists solve? , *New York Review of Books*, February 7, 27-28.

Resnick, L B. (1976) 'Task analysis in instructional design: some cases from mathematics , in Klahr, D (ed.), *Cognition and Instruction*, Hillsboro, NJ, Lawrence Erlbaum, pp. 51-80.

Sartre, J -P. (1956) *Being and Nothingness* (Barnes, H. E , trans.), New York, NY, Philosophical Library.

Shepard, R. N. and Metzler, J (1971) 'Mental rotation of three-dimensional objects', *Science* **171**, February 19th, 701-713.

Spearman, C A. (1927) *The Abilities of Man. Their Nature and Measurement*, London, Macmillan

Walshe, F. M. R (1945) 'The integration of medicine , *British Medical Journal* **1**, May 26th, 723-727