MAKING MATHEMATICS TALK: TOWARD A RESEARCH PROGRAMME ON SCHOOL MATHEMATICS

JÉRÔME PROULX

This article is about, of, and for school mathematics, addressing the nature of the (meta-)mathematical interactions I have had with David Pimm over the years. I have always felt that David has a very open mind as to what mathematics are and could be, especially when it comes to school mathematics. Whenever I present him with students' work (be it with school children, university students or school practitioners), and the mathematical meanings I draw from it, David is always listening. He does not listen by being silent, because he intensely discusses these mathematics with me (and, yes, they are plural! [1]). David listens by letting the mathematics "do the talking", giving it its "chance" to speak, without imposing on it pre-established ideas. He remains open to the mathematical possibilities-pluralities and what can be gained from them. As my title alludes to, I have at times referred to this attitude as "making mathematics talk", which seems to me to be as much about making mathematics *talk*, as it is about *making* mathematics. And, in these "making mathematics talk" moments with David, school mathematics is constantly at the centre of attention, being explored, scrutinized, reflected upon, suggested, imagined, developed, *etc*. In the following, I synthetically organize and report on a variety of these moments with David since I met him during my PhD work. These years of working with David contributed and gave shape, implicitly and explicitly at times, to the generation of a research programme on school mathematics. In this article, I put forward three axes of this research programme [2].

Making mathematics talk: Initial steps and a first axis for thinking about mathematics education

One of the earliest discussions that I recollect having with David occurred when I was designing an in-service session for secondary mathematics teachers about operations on fractions during my PhD research (Proulx, 2007a). As I was discussing my preparation of the tasks I had thought of offering the teachers, David told me about an observation made when visiting classrooms in China. He had seen an 11-year old using the following procedure to divide fractions:

$$\frac{26}{20} \div \frac{2}{5} = \frac{26 \div 2}{20 \div 5} = \frac{13}{4}$$

My initial reaction was to doubt the correctness of the procedure. But then I became fascinated with the mathematical reasoning underlying it, which forced me to integrate it in my research design. Through an investigation of this algorithm the participating teachers and I unpacked it as a multiplicative structure. This led us to establish a number of connections with and reflections about other algorithms for operating on fractions presented in schools (+, −, and ×; reports on this investigation appear in Proulx, 2007b).

Another mathematical event from my PhD research that I remember David and I discussed concerned an interaction between teachers about the usual orientations of prisms, where some were said to be "standing up" and others to be "lying down", as Figure 1 shows. This positioning of the prisms unleashed a series of arguments between the teachers and I about identifying the base(s) of this prism, and whether they "changed" depending on the prism's orientation (reports and analyses of my interactions with teachers can be found in Proulx, 2008a, pp. 345–349). Discussing these events with David made us raise significant questions about the nature of the bases of prisms: what they stand for, their use and their meaning in mathematics. These mathematical reflections on bases of prisms led us to reflect on their importance and meaning in terms of the role they play to create what could be termed "solids of translation", where bases are instrumental in the creation of these prisms. Figures 2a and 2b represent the translation of different bases to create both prisms.

What was pleasing and compelling about exploring these mathematics was that it was about *school* mathematics. It was fascinating to see how inquiring into what could be seen as elementary mathematical notions (*e.g.,* fractions, prisms) could enable us to go beyond these notions, to extrapolate them, to establish unthought links, and even to question their usual meaning. Exploring the mathematics of students' and teachers' work and investigating school mathematics ideas became a core aspect of my interactions with David (most of

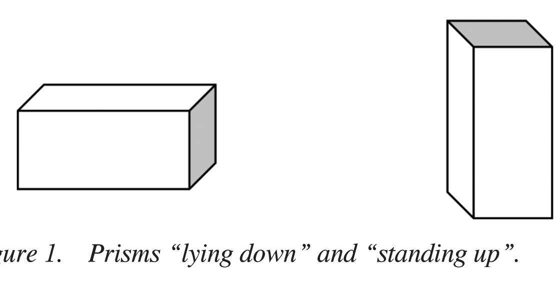

Figure 1. Prisms "lying down" and "standing up".

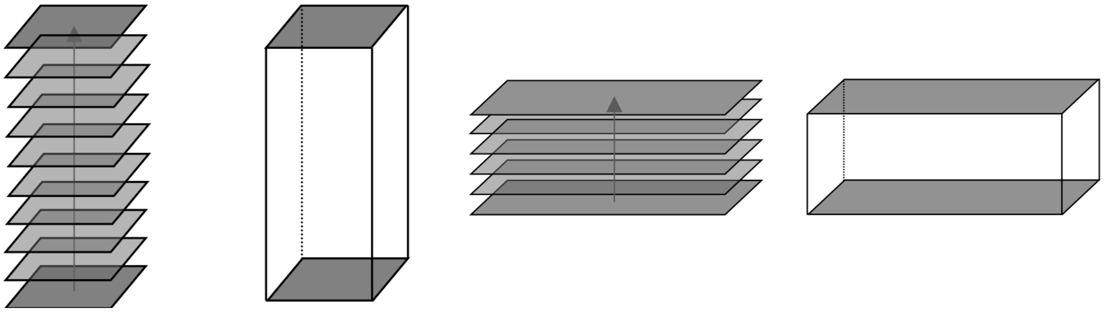

Figure 2. Prisms of translation: (a) a prism "standing up", (b) a prism "lying down".

them happening in room 933 on the 9th floor of the University of Alberta Education South building). They were moments where David showed me how to let the mathematics talk; how lots could be gained from being attentive to what was lying underneath seemingly simple strategies, comments, events and notions like the ones presented above. These moments rhymed well with the following quotation from Stephen Brown, to which I was introduced by David:

> One incident with one child, seen in all its richness, frequently has more to convey to us than a thousand replications of an experiment conducted with hundreds of children. Our preoccupation with replicability and generalisability frequently dulls our senses to what we may see in the unique unanticipated event that has never occurred before and may never happen again. That event can, however, act as a peephole through which we get a better glimpse at a world that surrounds us but that we may never have seen in quite that way before. (Brown, 1981, p. 11)

I suppose David introduced me to Brown's work and ideas because of this interest of mine in exploring and reflecting in depth about the mathematics of students (be they students from elementary school or in university, or even (future) teachers, superintendents, policy makers, and so on).

In a similar way, he also introduced me to David Henderson's reflections on mathematics as an ever-growing discipline and on how students' work was a constant source of inspiration for him (see David's biographical recollection of Henderson's work in Pimm, 2019). This second quotation is taken from one of Henderson's FLM articles about his experiences teaching a junior/senior level geometry course at Cornell university:

> What I have discovered is that through this process not only have the students learned, but also I have learned much about geometry from them. At first I was surprised—How could I, an expert in geometry, learn from students? But this learning has continued for twenty years and I now expect its occurrence. In fact, as I expect it more and more and learn to listen more effectively to them, I find that a larger portion of the students in the class are showing me something about geometry that I have never seen before. (Henderson, 1996, p. 46)

The title of Henderson's article—"I learn mathematics from my students"—still catches the eye today, coming as it does from a mathematician. Henderson explains how he continually comes across inspiring mathematics from his students that pushed his thinking, even if many of them were not necessarily high-achievers in mathematics.

This immersion in some of David's intellectual influences, like Brown and Henderson, acted as a fertile ground that encouraged me, as a PhD student, to see relevance in these constant explorations of the mathematics underlying students' work. And, along these lines, to give prominence to school mathematics issues in my research in mathematics education. As these initial examples show, this mathematical focus was conceived of as instrumental to inspire and stimulate ideas *for* mathematics education, as captured through the following quotation taken from an FLM communication I wrote following my PhD:

> De ces fouilles et travaux mathématiques émergent des idées inspirantes et novatrices pour l'enseignement et l'apprentissage des mathématiques […] Lorsque nous explorons les concepts mathématiques en détail et en profondeur […] nous réalisons certaines subtilités présentes à l'intérieur d'un concept, nous réfléchissons à des façons possibles d'approcher ces concepts avec les élèves, nous pensons à de nouveaux problèmes et questions qui amèneraient les élèves à explorer certains aspects, *etc*. Ce type de travail mathématique a le potentiel de faire évoluer notre compréhension de ces mathématiques, de leur apprentissage et de leur enseignement et peut donc inspirer et générer de nouvelles pratiques et approches d'enseignement et de recherche. (Proulx, 2008b, p. 28)

Making mathematics talk: Continuing steps and a second axis for (re-)thinking school mathematics

At first, this school mathematics focus was mostly conceived as a way to stimulate ideas for my work in mathematics education, like in the design of in-service sessions or classroom experiences and to think of ways of working on these notions with students and teachers. However, as time went on, more was starting to happen in these mathematical investigations, and it somehow went beyond thinking of mathematics education issues. These investigations became also about mathematics itself, about revisiting already known school mathematics ideas as a way to deepen them, to make them appear under a different light, to dig further into them. This happened when, for example, I told David about the work I had done with teachers by exploring various equivalences of volume between solids through Cavalieri's principle (see Figures 3a, 3b, 3c).

David was interested in these ideas, which reminded him of aspects of the history of mathematics in relation to infinitesimal calculus, as discussed in Gray (1987). Encouraging me to follow this route further, he suggested writing an article for FLM about possible extrapolations of Cavalieri's principle for understanding elementary volume, as well as area, under a new lens. Whereas establishing equivalence of area between figures or volume between solids is a well-known idea (*e.g.,* when figures or solids share same dimensions or have similar formulas), David and I explored this idea further through conceiving of *families* of area-equivalent figures and of volume-equivalent solids. What could be termed a "mathematical theory of families" enabled us to establish a multitude of transformations from one figure to another, or one solid to another, within a family of equivalences (Figure 4a for parallelograms, Figure 4b for trapezoids, and Figure 4c for triangles, offer a glimpse at the notion of families of equivalences developed in the article; see Proulx & Pimm, 2008, for more details and other transformations [3]).

This work was not only inspiring mathematics education ideas (*e.g.,* for developing in-service sessions or classroom activities), but led me to re-think these mathematics, to see them differently, to work at them under a different lens. The theory of families offered an additional view for conceiving of solids and of figures under an equivalency principle, where

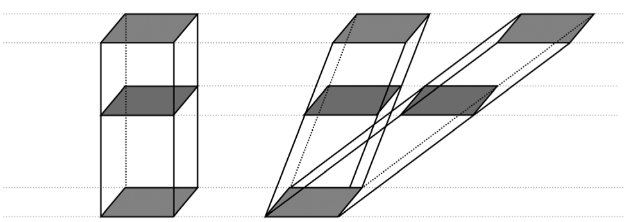

Figure 3a. Equivalencies of volumes of prisms through Cavalieri's principle.

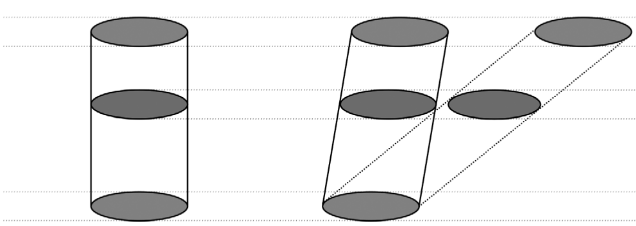

Figure 3b. Equivalencies of volumes of cylinders through Cavalieri's principle.

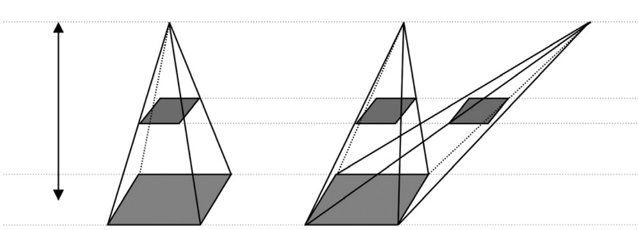

Figure 3c. Equivalencies of volumes of pyramids through Cavalieri's principle.

they could all be linked together as well as possibly reduced to a simpler member of the family sharing the same properties (rectangle, right-angle triangle, right-angle trapezoid, *etc.*) [4].

In a similar vein, with my PhD colleague Mary Beisiegel, who also interacted regularly with David, we found in the already-mentioned work of Brown (1981) a particularly interesting view of what division could imply. Brown explains that the definition of Euclidean division respects some conventions— or conditions—where the remainder r is defined as being situated between 0 and the divisor. Thus, $18 \div 4$ gives $4r2$ and not $3r6$, even if both results are conceptually adequate in relation to the division process. Brown himself became interested in the possibility of the remainder being a variable for a division following what Sharon, a child he worked with, offered as answers to some division problem. Hence, $18 \div 4$ could give $4r2$, $3r6$, $2r10$, $5r^{-}2$, *etc.,* as conceptually adequate answers, provoking mathematical reflections at the same time about the significance of division's conventions and about definitions of divisions. Based on Brown's work, Mary and I decided to explore further this idea by trying to understand its meaning in relation to negative numbers. For example, we raised and explored questions of the sort: What would be the value for $^{-}18 \div 4$? Is it $^{-}5r2$, $^{-}4r^{-}2$, $6r6$, or something else? What about $18 \div {}^{-}4$? What about $^{-}18 \div {}^{-}4$? To which answers does the convention point for each of these divisions? What do various hand-held calculators point to? Is there a need to develop a different convention for dividing with negative numbers? How could we go about defining it? These questions unleashed a series of mathematical investigation about the division of negative numbers in which algorithms, usual definitions and the use of calculators were brought into play (see Proulx & Beisiegel, 2009).

The work on families of equivalences and on division of negative numbers acted as a trigger for me. I felt, when

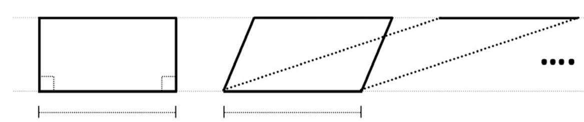

Figure 4a. Parallelograms of the same area-equivalent family.

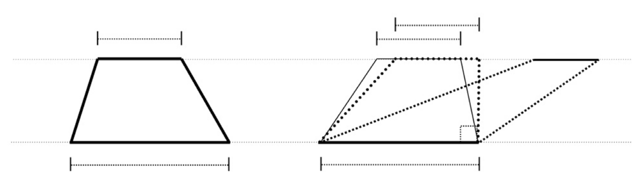

Figure 4b. Trapezoids of the same area-equivalent family.

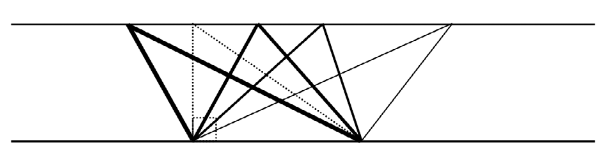

Figure 4c. Triangles of the same area-equivalent family.

doing this work, that I was going a step further in my explorations of school mathematics. It felt as if I was not only thinking about mathematics education, but was also exploring mathematical ideas, pushing them, producing ways of conceiving them, and raising new questions to already known problems. As an illustration of this sentiment, here is how Mary and I ended the article on divisions with negative numbers:

> These issues raise for us the significance of working on the exploration of mathematical concepts as a genuine activity of mathematics educators. Albeit this is not research *per se* in its traditional sense, yet these explorations have something to offer to our understanding of the very concepts that we work on with students in classrooms. We see it [as] important to delve deeply into mathematical concepts and ideas, to understand the concepts, to make sense of what is happening, to gain a stronger footing in our own understanding of seemingly simple ideas. These sorts of mathematical developments of school mathematics appear here as initiatives driven to enhance our understandings of mathematics, a clear intention of all work being done in mathematics education. (Proulx & Beisiegel, 2009, p. 421)

Hence, not only was I letting the mathematics talk through exploring it, but I felt that I was entering in the "making" part of it. To re-use Watson's (2008) formulation, I was exploring questions that had an intellectual purpose, and not only a pedagogic or didactic one.

This second dimension associated with exploring school mathematics motivated me to undertake this work in a more systematic way. I give a detailed example of this below, to illustrate the sort of mathematical insights that could be gained through these explorations.

An example about operations on functions

In a study, Grade 11 students were asked to mentally solve graphical tasks on operations on functions (Proulx, 2015). A typical task was to have two functions represented in the same Cartesian plane; the students had to add or subtract them without making use of pencil-and-paper or any material aids. Numerous strategies were produced by students. For example, for the following task (Figure 5), students had to recover the function g that has been added to function f to give the resulting function $f + g$. In order to decipher what function g was, some students expressed that "each function was parallel to the other", that g had to be a constant function "for the curve to be translated down", and that this constant function was "negative for bringing the curve lower" as a result of its addition. These ideas initiated a geometrical understanding of the graphical representations, through the set of image-lengths of each function. This strategy brings forth an interesting mathematical tool, that is, the parallelism of curves in order to understand translations in a graphical environment. This view of operations in terms of parallelism of curves addresses the curve as a whole, translated through being added to a constant function. This leads to understanding the effect of adding a constant value, through the constant function, to another function. In this instance, *e.g.*, $f(x) = x - 2$ to which $g(x) = +7$ is added produces a parallel function $(f + g)(x) = x + 5$. Considering the function that is translated as a whole, and it being parallel to the other, leads to reflection on what happens algebraically when these two functions are added [5].

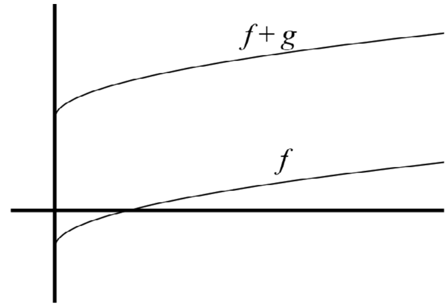

Figure 5. Task requiring the students to find the function g that was added to function f.

When functions f, g or $f + g$ are conceived in terms of their set of image-lengths, the addition of f to g through $x - 2$ and $+7$ goes beyond mere algebraic manipulations resulting in $x - 2 + 7 = x + 5$. In the case of $f(x) = x - 2$, the $x - 2$ is no longer a simple algebraic expression that has a numerical value, but represents the entire set of image-lengths of f, as shown in Figure 6.

The same is to be conceived for function $g(x) = +7$, where the $+7$ represents the entire set of image-lengths of g (Figure 7). Thus, to the set of image-lengths of f is added the set of image-lengths of g, that is $+7$. Figure 8a shows the graphical superposition of the set of image-lengths of f and g. Figure 8b illustrates the result of the addition of each set of image-lengths, resulting in $f + g$.

Considering functions as wholes and in terms of image-lengths leads to the realization that when working with functions, expressions like $x - 2$, $+ 7$, and $x + 5$ are no longer simple algebraic expressions: they carry with them the entire set of image-lengths of their associated functions, here for f, g and $f + g$. This underlines the "function" dimension of these expressions. This line of thought also leads to the

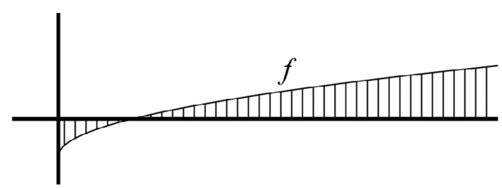

Figure 6. The entire set of image-lengths of function f.

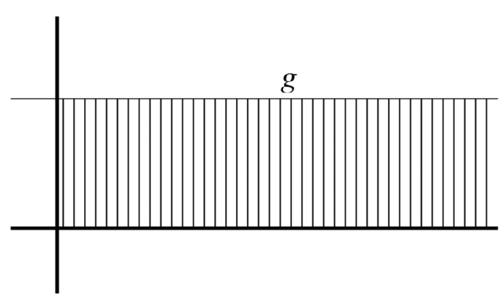

Figure 7. The entire set of image-lengths of function g.

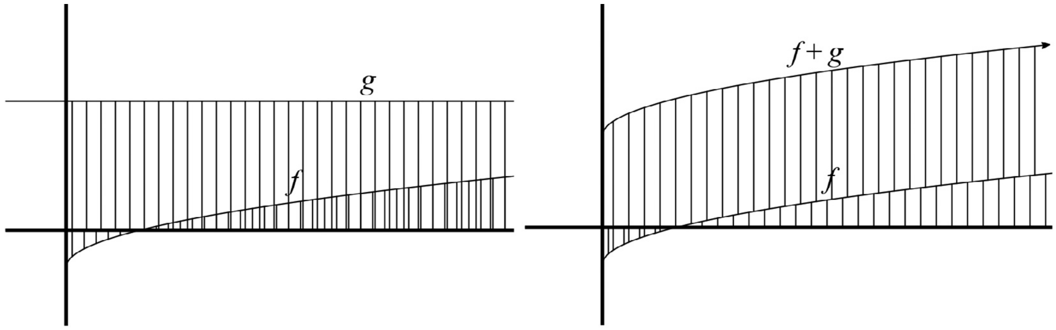

Figure 8. f and g sets of image-lengths: (a) superposition, (b) addition.

consideration of other issues relative to functions, like the composition of functions or finding zeros of functions.

The composition of functions can be conceived under an algebraic lens, where the composition of $f \circ h$ for $f(x) = x - 2$ and $h(x) = x - 3$ gives the impression of being reducible to algebraic manipulations. The composition $f \circ h$ becomes $f \circ h\,(x) = (x - 3) - 2$, where $(x - 3)$ is substituted algebraically for x in the expression $x - 2$. However, when functions are considered under their set of image-lengths, the composition of functions $f \circ h$ implies that $(x - 3)$ carries with it all the set of image-lengths of the function $h(x) = x - 3$. Thus, inserting $x - 3$ for x in $f(x) = x - 2$ means inserting the entire set of image-lengths of h, that is $x - 3$, in f, where the "set of values in x for f" becomes the "set of image-lengths of h". In other words, in the composition $f \circ h$, the values of x of the function $f(x) = x - 2$ are the set of image-lengths $x - 3$ of h. There is thus an important difference between conceiving a composition of functions through algebraic manipulations and conceiving it in terms of sets of image-lengths.

Finding the zeros of functions under an algebraic lens implies looking for the values of x that nullify the algebraic equation, that is, that give 0. For example, finding algebraically the zeros of the function $i(x) = 2(x - 4)^2 - 3$ requires that $i(x) = 0 = 2(x - 4)^2 - 3$, where algebraic manipulations lead to two values:

$$x = \sqrt{\frac{3}{2}} + 4 \text{ and } x = -\sqrt{\frac{3}{2}} + 4$$

When considering the function through an image-length conception, finding the zeros of the function $i(x) = 2(x - 4)^2 - 3 = 0$ calls for looking for image-lengths of $i(x)$ measuring 0. Finding the zeros of the function requires looking in the entire set of image-lengths of function $i(x) = 2(x - 4)^2 - 3$, that is $[2(x - 4)^2 - 3]$, for the ones that have a length of 0. In Figure 9, these values are the ones where the function crosses the x-axis, hence their length is 0. Therefore, $0 = 2(x - 4)^2 - 3$ can imply a geometrical sense in terms of image-lengths. The 0 is here not a numerical value, but a measure of image-length.

This view of functions, in terms of image-lengths, can give rise to new ways of working with functions. Other themes could be mentioned as well, like the use of parameters, of extrema, of intersection points, *etc.,* where an image-length approach might suggest particular ways of conceiving these notions when working with functions.

Towards a research programme on school mathematics

The proposition for developing a research programme on school mathematics can seem odd, since it could be said that the mathematics in the school curriculum is already "developed" and well-known; we already know these mathematics well, that we already know how to divide, calculate a mean, solve equations, and so forth. In other words, one could argue that we already understand the inner workings of these notions, that nothing new can come out of these investigations, and that what could come out of them are simply part of the already written (school) mathematical text, as Brousseau (1998) would say.

However, as the above examples illustrate [6], there are lots of mathematics to continue exploring within the mathematics of the school curriculum. In other words, there is a lot of mathematics "deep inside" the mathematics worked on in schools, still lots to unearth, to uncover, to dig into, to reflect upon, to question. Doing so aims as much to raise mathematics education issues as to delve into school mathematics itself.

Another one of David's intellectual influences, Richard Skemp, alludes to this when he draws out the relations between and intricacies within the mathematical concepts that are represented by the symbol system. For example, from the symbol 572 can be induced three specific numbers "5", "7" and "2", related to three specific powers of ten, and again to three operations of multiplication by these powers of ten (*e.g.,* 5×10^2), and finally to the addition of these products (Skemp, 1987, pp. 179–180). For Skemp, this represents how it is possible to unearth concepts hidden behind

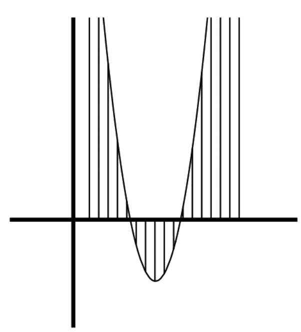

Figure 9. Finding zeros of functions with the image-lengths.

the "structure" of the numerals (here, 572), opening up an immense realm of insightful mathematical concepts:

> Once one begins this kind of analysis, it becomes evident there is a huge and almost unexplored field—enough for several doctoral theses. (p. 180)

The proposition for developing a research programme on school mathematics is about this unexplored field. When discussing and exploring these mathematics with David, it became obvious that the mathematics within school mathematics could be developed further and re-thought, along both already mentioned axes. A first research axis concerns investigating school mathematics for unearthing reflections for mathematics education. A second axis concerns the direct exploration of school mathematics itself, revisiting and rethinking school mathematics directly.

This second axis might represent more than Skemp's notion of drawing out the relations behind and within mathematical concepts. Reviewing, deepening, reformulating school mathematical ideas, proposing alternate ways of understanding, raising new questions about these mathematics, *etc.;* these are also ways of developing additional notions and concepts relative to school mathematics. In short, going deeper in school mathematics can also be conceived of as developing new mathematical ideas and objects within this school mathematics. This is what the work of raising questions about division of negative numbers was about, as was the development of families of figures and solids, and the reflections on functions' image-lengths and legitimate algebraic transformations. Research in school mathematics is not *only* about uncovering already-known aspects within a concept or unpacking it for mathematics education purposes; it can also be seen as producing mathematical ideas and objects within school mathematics. Through this, school mathematics becomes less of a finished product or a self-contained field, and more as one in which there is a world of mathematical possibilities still to understand, investigate, uncover, develop and push further. Hence, school mathematics is to be conceived of, like any field of mathematics, as a domain where there is still place to think creatively and differently—and not only repeat known ideas.

But this should not be misinterpreted as a call for research on mathematics as mathematicians do, namely by proving theorems and creating new conjectures in order to have the field of mathematics go forward. These two research axes on school mathematics offer a different view of developing mathematics, one that focuses not on moving the field forward, but on digging deeply into already conjectured, proved and formalized notions in order to review, deepen and reformulate school mathematical ideas, to propose alternative ways of understanding these ideas, to raise new questions and even to develop additional notions and concepts relative to school mathematics. In other words, whereas research in pure mathematics is often conceived of as moving the field forward, research on school mathematics aims at taking an already known piece of mathematics and enlarging or deepening it. Figure 10 gives an image of what is intended by "researching the mathematics of school mathematics".

Researching the mathematics of school mathematics is first about these two axes of (1) unearthing reflections for mathematics education, and of (2) developing new mathematical understandings. That said, these two axes also implicitly raise issues about novelty. In these explorations into school mathematics, some new ideas could be produced: new theories, new concepts, new methods, *etc.* This would represent a third axis of the research programme, about (3) producing new school mathematics, new ways of doing, new discoveries or inventions. To better understand this third axis, two examples from work we are conducting in my research team are here briefly presented.

- *Discovering new operations:* Some students' strategies for establishing the mean of a statistical distribution implied transforming the data in each distribution (*e.g.,* from 41, 42, 44, 45 to 1, 2, 4, 5). This strategy led us to explore a variety of operations that could be done to transform a statistical distribution (+, −, ×,÷), as is usually done in advanced statistics, and to evaluate its effect on measures of central tendency and of dispersion for the new distribution obtained. Then, we considered conducting these operations between distributions (*i.e.,* adding two or three distributions together) and analysing the properties of the resulting distribution and how they could be predicted from the previous distributions operated on.

- *Studying the behaviour of a mathematical object.* One student strategy for solving a system of linear

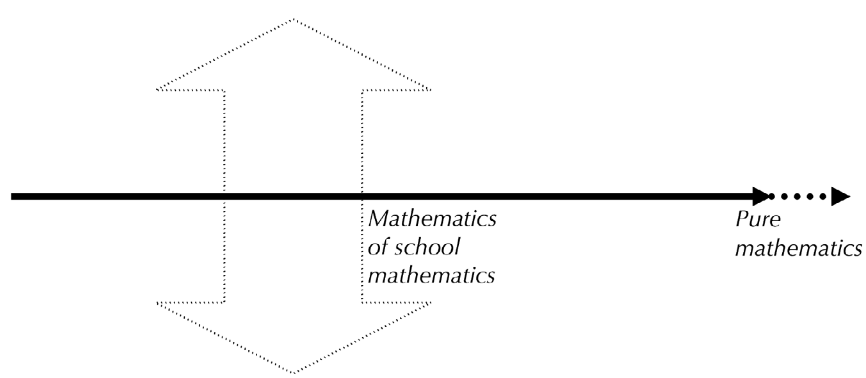

Figure 10. Tracing differences between researching the mathematics of school mathematics and pure mathematics research.

> equations centred on how the y-intercept of one equation could have an impact on the solution (point of intersection) of the system. This led us to inquire into the effect of the variation of the y-intercept in one equation of the system to be able to predict the impact of this transformation on the intersection point (see Proulx, 2020).

These examples, one on elementary statistics and one on elementary analytic geometry, represent the creation of new ideas, new constructs and new knowledge about school mathematics. They are both about producing new mathematics. Surprising as it may be, this third axis embodies well how school mathematics can be conceived as a genuine field of scientific inquiry that grows.

Concluding remarks

Research in school mathematics, along the three axes proposed here, forces a re-looking into mathematical matters we believe we know well. It pushes us in possibly uncharted territory; what David and I once called the revisiting and renewing of our (school) mathematics.

David's ways of seeing mathematics has always aligned with the fundamental intentions underpinning my research: to open up what mathematics could be, to transform what it can be, to recast it as an endeavour of meaning-making, to make accessible the fact that mathematics is not pre-decided and is something that unfolds in action. And, that mathematical ideas—even the supposedly simplest ones—are not all pre-written up somewhere, but are worth inquiring into more and making grow.

David often mentioned that doing this sort of work with school mathematics is "humanising the face of mathematics". I do not think I always understood what he really meant. But, I obviously liked the sound of it! It is as if the sometimes-alternative mathematics that I was telling him about were also offering a message about mathematics itself, and about school mathematics in particular. Above all, the development of this research programme is also a way of reminding us that, as we mentioned in Maheux *et al.* (2019), the mathematical landscape does not solely belong to mathematicians, and that anybody who does mathematics could take part in it, participate in its development, and offer inspirational ideas to it. In so doing, mathematics stops being seen as an epistemological absolute and becomes a collective epistemological endeavour. I take that to be one of David's most inspiring lessons.

Notes

[1] Nathalie Sinclair, a collaborator in David's *festschrift* monograph, has used the neologism "mathematicses" in order to draw attention to its potential plurality: "In French, one sees both the plural and singular forms used. The plural form, "les mathématiques" may refer to its multiple branches (geometry, algebra, probability, analysis, *etc.*) and/or to its multiple sources (Mayan, Chinese, Mesopotamian, *etc.*). While the plural term is more common, the singular term has also been used. For example, in the 1940s, the Bourbaki chose as a book title, *Éléments de mathématique.* This choice may seem ironic, since Gödel had proven in 1931 that a complete and consistent grounding for mathematics is impossible. However, rather than connoting any foundational assumptions, the singular term may have been chosen to denote a unified mathematical language or way of reasoning. With the option available, the French have a choice; in English the default to the plural term may obscure the aesthetico-political dynamic at play." (Sinclair, in press).

[2] What I report here is obviously my way of making sense of the (meta-) mathematical interactions I have had with David. Although personal, I like to think (1) that David will recognize himself in them and, in return, (2) that you will recognize David in them.

[3] It is of interest that Nathalie Sinclair and Dave Hewitt, collaborators on this monograph, offered interesting suggestions and comments to earlier versions of this article, many of which were incorporated, with explicit thanks.

[4] This mathematical theory of families was also extended to other concepts like algebraic equation solving and statistical distributions, in Proulx (2021).

[5] Special thanks goes to Charlotte Mégrourèche and Rox-Anne L'Italien-Bruneau for contributing drawings and ideas to this analysis.

[6] Other examples have been published elsewhere, some of them for which David has greatly contributed. See, for example, the four columns on algebraic equation solving published in the *Ontario Mathematics Gazette* in volume 59 (Proulx, 2020–21).

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