Groupings in the Process of Concept Formation

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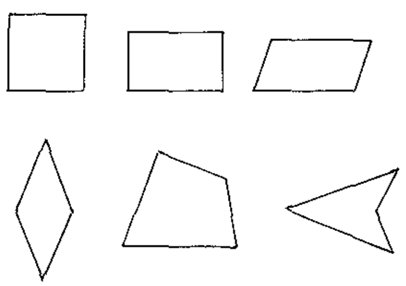
One of the tasks of school mathematics is to encourage children to engage in a variety of mental processes which characterize mathematical activity Such processes as well as their possible mutual relations must be analysed and clarified in the field of mathematics education in order to assist school mathematics in this complex and ambitious task Among the processes mentioned, organising and structuring as well as concept formation have particularly attracted our attention since they seem to be crucial in mathematical activity and since they tend to be observed simultaneously Groupings may be thought of as results of organising and structuring activities taking place in a particular conceptual context, thus developing, enriching or clarifying the given context.

In the following, we try to illustrate with the help of some examples our conception of groupings, their characteristics and possible developments, in a context of concept formation processes

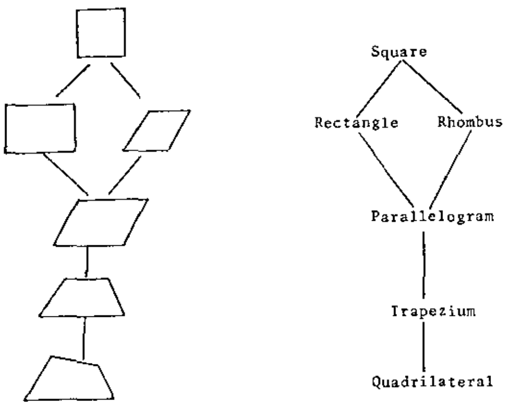
Groupings

The thought of a particular conceptual context may trigger (on the intellectual level) mental images of various kinds and intensities depending on the situation in which the thought has been stimulated and depending on previous experience which we have had in this context Thus, the thought of quadrilaterals may trigger in some persons *mental images* of various quadrilateral shapes, which may be indicated in the following way:



Images of quadrilateral chapes
Figure 1

These images reflect part of our experience in the given context by the flashing in front of our mental eyes of quadrilateral shapes which we have already encountered. Thus, depending on such experience, they may vary in number and diversity. They also vary in "mobility", by which term we like to indicate that such images might be moved, turned, transformed, and as such, related to each other This mobility stimulates relating, classifying, structuring activities, which in turn lead to the conception of *structural mental images* as shown in the following example:



Relating quadrilaterals
Figure 2

Here, generally, individual figures represent types (classes) of quadrilaterals, which themselves may be structured They represent landmarks in a map of relationships which we happened to conceive in a particular situation Different maps may derive out of different situations and maps may develop in size and detail as experience and knowledge are gathered. Such maps are the expression of the organization and integration of acquired knowledge into a system of factual and relational knowledge, which seems to reflect a much more profound appropriation of the given context.

These *structural mental images* consist of *states*, which in turn are connected by *relations*; they can be characterized by the description of the nature of these constituting elements In the preceding example, the states are either pictorial representations or names of various kinds of quadrilaterals. These states may represent concrete objects (of quadrilateral shapes) or classes of quadrilaterals or even sets of properties common to classes of quadrilaterals. Depending on the nature of their state, the relations of such structural mental images can vary accordingly. Thus, the connections between the states of the above example may mean, rather intuitively, a gradual perfecting of the "regularity" of some quadrilaterals, when read from the bottom to the top of the figure. On the other hand, they may also mean the adding of a particular property and the specification of sub-classes of quadrilaterals which have this property in common.

It is this kind of mental structural images which we conceive as *groupings*, and which reflect a state of development in the acquisition and integration of knowledge. Groupings, as already recognisable in the above example, may vary in the *level of abstraction* and in the *"richness" of their states and relations*. Furthermore, like structures in general, groupings vary in complexity. The grouping of the preceding example, which takes into account properties like the number of pairs of parallel edges, equilaterality and equiangularity, may be refined or detailed by considering additional properties such as convexity, number of line symmetries, relations between diagonals, etc. So it becomes evident that groupings develop and grow as knowledge is acquired and integrated, thus revealing a dynamic character which can adapt to further information and to changes in perspective.

As knowledge develops out of an individual interaction process between the learning subject and his environment, groupings grow as part of the learner's image or model of reality, which in turn helps him to interact more successfully and coherently with the environment. The fundamental means of interaction of the subject and his environment are *concrete (or overt) actions* performed on *real objects* of the environment. These concrete actions and real objects are the germs of what will become the relations (or operations) and the states of the grouping by a process which Piaget calls "interiorisation" and which describes a gradual detachment from reality so that states may become representations of objects, classes of objects, properties, concepts or ideas, and so that actions may develop into mental operations and more or less abstract relations. Actions, operations or relations may generate new states and new relations may germ out of comparisons of states. Thus, the formation and development of groupings is seen as a dynamic continuing process accompanying or describing the acquisition and integration of knowledge. Groupings are modified, enriched or changed depending on further information or changes of context. Temporary stability of a grouping signifies a state of equilibrium and a satisfying coherence of the individual's image of reality.

This rather global and intuitive description of our conception of groupings will be illustrated further with the help of an example which we will discuss in detail later. The conception is our understanding of Piaget's fundamental work in this context [1] and has been greatly influenced by the work of and by personal discussions with E. Wittmann [2]. Readings of both authors are necessary for further understanding of this notion. To relate this notion to concept formation processes, let us indicate some conceptions in this respect.

Concept formation

Martin [3] describes concept formation processes as stimulated by observable environmental variations. He distinguishes "two major types of environmental variability: variability *between* things and variability *within* things". Considerations of variability between things focus on differences and similarities between objects (or states in general) and "will determine whether or not an object is a member of a class". They represent discrimination activities which generate classification and differentiation criteria by the comparison of discrete objects or states. Considerations of variability *within* things focus on transformations and their effects when executed on a particular object or state. They confront us with problems like "what happens (to a given object or state) if we impose a particular transformation on this object? Will it remain a member of a given class or not"?

According to these two types of variability, Martin describes two possible kinds of content of a concept. Discrimination activities or activities which determine whether or not an object is a member of a given class are principally concerned with the "realm of objects that the concept points to or denotes. These exemplars of the concept constitute what is called the *extensive content*". This kind of content seems to focus on the boundaries of the concept without necessarily differentiating various instances within the concept. This differentiation is indicated by what is called the *intensive content* of the concept, which is developed by activities that focus on transformations or changes of objects by which these objects remain members of a given class. These activities clarify the common features connoted by all the exemplars of a concept precisely by the indication of possible variations within the concept. These common features of the exemplars develop as their invariant properties under the transformations are considered.

In the context of the development of groupings, concepts may be seen as emerging from the *structural data* of a grouping. These data are mainly properties of objects or states and their possible variations in different instances. They are the information the grouping is liable to convey or the grouping is loaded with; for this reason, they are also called the "*loadings*" of the grouping. The information, for instance, which may be transmitted or conveyed by the above example of the grouping of quadrilaterals is that such four-sided plane geometrical figures may vary according to their number of pairs of parallel edges (0, 1 and 2) and by the properties of equiangularity and equilaterality and that, according to these variations, different types of quadrilaterals (such as trapezium, parallelogram, rectangle, rhombus and square) may be distinguished. The concept of quadrilaterals unfolds in terms of a system of common and differentiating properties of its exemplars as the grouping is developed in richness and complexity. The properties common to all exemplars of the concept appear as invariant under all operations of the grouping and the differentiating properties are revealed by the same operations as their controllable and foreseeable effects. Wittmann [4] describes these common and differentiating properties as structural data "*compatible*" with the operations of the grouping. Thus, in the context of groupings, concepts emerge as structural data or loadings of groupings which are compatible with their operations and the concept formation process is strongly linked to the formation of groupings. Since, inversely, a concept may be described by a grouping of its exemplars, groupings and concepts seem to correspond in a unique way. In particular, a grouping may be considered as a dynamic description of the state of development of the corresponding concept.

This view of concept formation processes in the context of groupings seems to take into account both the extensive and the intensive content of a concept. The extensive content develops as a class of properties invariant under the operations of the grouping, while the intensive content emerges as properties not invariant but still compatible with the operations of the grouping. It shows, in particular, the complementarity of the two contents. We shall illustrate these conceptions by the discussion of the following example.

Deltahedra

In the course of a study of polyhedra, students may be attracted by several examples of regular polyhedra, the faces of which are all congruent, equilateral triangles: the tetrahedron, the octahedron and the icosahedron.[5] The conception of such a particularity may have been stimulated by the comparison of several "more or less regular" polyhedra and by the familiarity (to the students) of the equilateral triangle. Thus, they are lead to formulate a property permitting them to single out a particular class of polyhedra, the so called *deltahedra*, the faces of which are all congruent equilateral triangles. Such a property enables us to discriminate between members and non-members of the given class and can be considered as part of the extensive content of the concept of deltahedron. It provides a first indication of the boundaries of the concept and directs or limits further exploration. It does not provide us with information concerning possible variations or changes and the concept is still rather empty. This lack of "intensity" stimulates questions like: "Are there other deltahedra? What changes can we impose on known deltahedra to conceive new ones?"

Such questions normally lead to experimentations in the construction of other examples of deltahedra which depend very much on the construction material available. In the present episode, we used commercially available equilateral triangles made of plastic and which can be hinged together.[6] This experimental exploration led to a great variety of polyhedra, several of which were not deltahedra because some faces were constituted of two or more equilateral triangles and forming new non-triangular shapes. Most of the deltahedra we obtained were non-convex or concave leaving the impression that the domain of convex deltahedra might be much more restricted than the domain of non-convex deltahedra. Thus, this activity provided us, by the exploration of possible variations, on the one hand, with an element of clarification of the "extension" of deltahedra and, on the other hand, with a first differentiation criteria within the realm of deltahedra. Since it seemed that it might be easier to clarify the domain of convex deltahedra, we concentrated our efforts in this direction.

The study of such convex polyhedra may take many different courses by which various aspects of the same overall structure will emerge. The choice of the courses of study is very much influenced by already acquired knowledge and by previous experience in such activity. Since some geometrical knowledge had been available in the example of this episode, we chose to make use of this knowledge in the given context instead of simply going on in a pure exploratory way. Convex deltahedra will certainly vary in the numbers of their faces (F), their vertices (V) and their edges (E). But this variability is restricted by the Euler-formula, which is valid for all simple polyhedra:

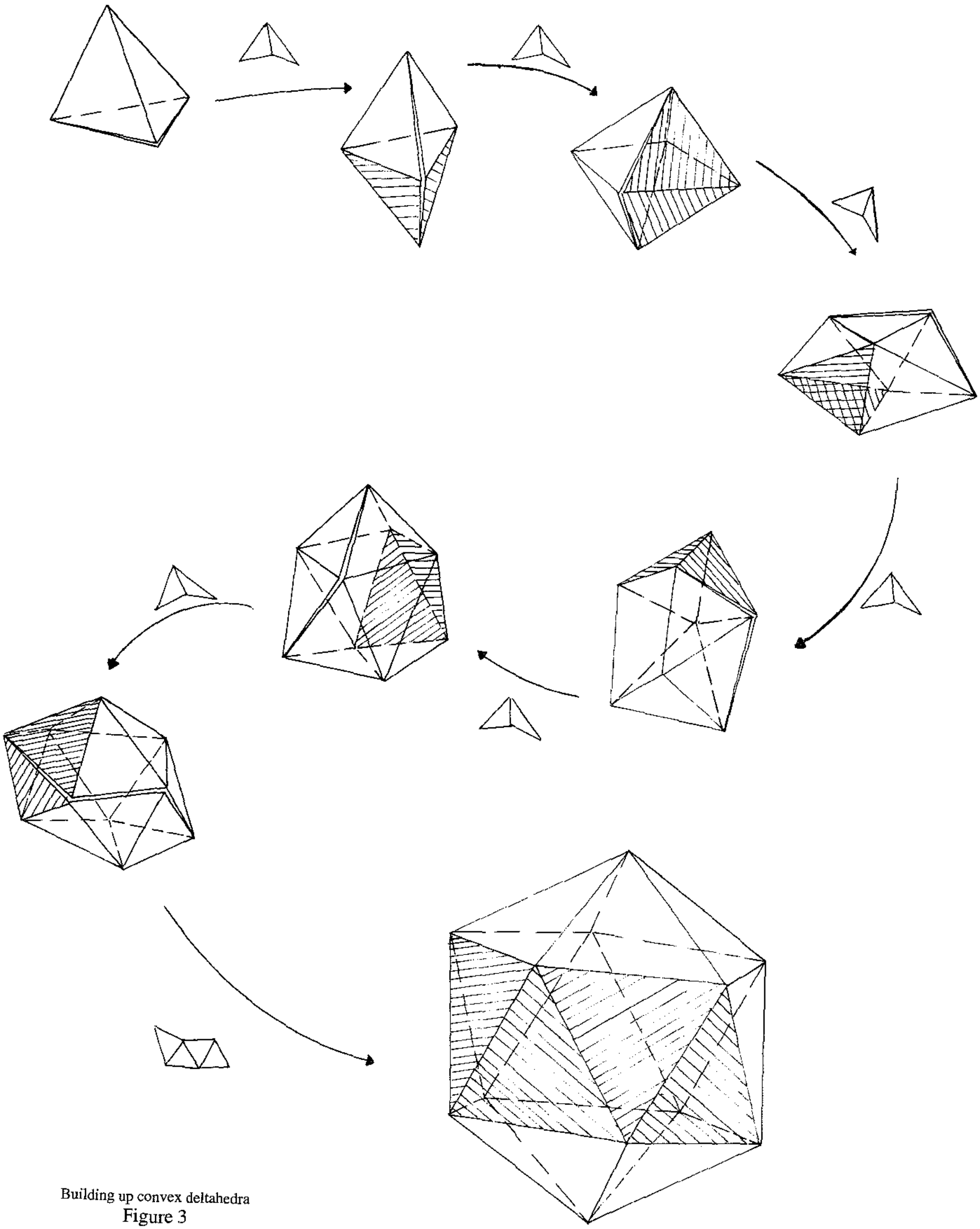
(1) $F + V = E + 2$

In the case of simple deltahedra, this relation between faces, vertices and edges can be specified even more since all faces are of triangular shape. This means that each face is surrounded by three edges and if we count three edges per face we have counted each edge exactly two times since precisely two faces form an edge. This leads us to the following relation between the numbers of faces and edges of simple deltahedra:

(2) $3F = 2E$

We deduce from this formula, that the number of faces of any simple deltahedron has to be an even number since the number of edges, obviously, has to be a natural number. After having specified the relation between faces and edges of simple deltahedra, we can look at the vertices of such polyhedra. Since we concentrate our study on convex deltahedra, the number of triangles that meet at any corner has to be at least three (to give solidity) and at most five (to retain convexity). These observations permit to determine the range of the domain of convex deltahedra in the following way:

> The "smallest" deltahedron is one whose vertices are all constituted of three triangles and therefore, are all joints of three edges. Since each edge joins exactly two vertices we have the following relation between edges and vertices for this "smallest" deltahedron:
>
> (3) $3V = 2E$
>
> The conjunction of the conditions (1), (2) and (3) describes the "smallest" deltahedron as being constituted of 4 faces, 4 vertices and 6 edges, which indicates the class of regular tetrahedra. The "biggest" deltahedron is characterised by vertices which are all formed by five triangles. Thus, we obtain for such deltahedra the following relation between edges and vertices
>
> (3') $5V = 2E$
>
> (3') in conjunction with (1) and (2) describes this "biggest" convex deltahedron as being constituted of 20 faces, 12 vertices and 30 edges, which indicates the class of regular icosahedra. Between those two extreme cases we find the convex deltahedron which is characterised by the fact that all its vertices are formed by 4 triangles. Thus, the relation between its edges and vertices is
>
> (3'') $4V = 2E$
>
> Therefore we obtain in this case the class of regular octahedra constituted of 8 faces, 6 vertices and 12 edges.



Building up convex deltahedra
Figure 3

This exploration of possible variations and restrictions of convex deltahedra in the context of previous knowledge and experience has provided us, up to now, with some more insight in the structure of deltahedra in general, and convex deltahedra in particular. On the one hand, we realized that all simple deltahedra are characterized by a rather simple relation between the numbers of their faces and edges ($3F = 3E$). Furthermore, since the number of faces has to be even, the transition from one class of deltahedra (of equal number of faces) to the next one has to be by steps of two (or a multiple of two) faces. Finally, we found that the "smallest" deltahedron, that is the one with the smallest number of faces, is the regular tetrahedron. Thus, we have enriched our knowledge of properties common to all simple deltahedra. On the other hand, and concerning convex deltahedra, we found that there is one which is the "biggest" and which is constituted of 20 faces, 12 vertices and 30 edges; it is the regular icosahedron. Thus the regular deltahedra we started with can now be seen as a minimal and a maximal example with an intermediate representation in between. In this way we are starting to conceive a possible structure within the realm of convex deltahedra.

The question which imposes itself now is: "Are there, apart from the regular octahedron, other convex deltahedra between the two extreme representatives?" The answer to this question can be given in two parts.

1. If there are other representatives, than they must be part of a certain structure: Their number of faces varies between four and twenty by steps of two. But since the numbers of faces, vertices and edges are related by the conditions (1) and (2) we can indicate a complete list of possible types of convex deltahedra

Number of faces F	vertices V	edges E	Names
4	4	6	tetra-deltahedron (regular tetrahedron)
6	5	9	hexa-deltahedron (triangular dipyramid)
8	6	12	octo-deltahedron (regular octahedron, quadrangular dipyramid)
10	7	15	deca-deltahedron (pentagonal dipyramid)
12	8	18	dodeca-deltahedron
14	9	21	tetracaideca-deltahedron
16	10	24	hexacaideca-deltahedron
18	11	27	octocaideca-deltahedron
20	12	30	icosa-deltahedron (regular icosahedron)

Possible types of convex deltahedra
Table 1

2. To find out whether or not representatives of all these types really exist, we could attempt to build up (mentally and physically) such exemplars in a systematic way starting from the already known minimal regular tetrahedron and adding gradually two triangular faces.

We actually do this by opening two adjacent edges of an existing exemplar (double lines in Figure 3) and by integrating two connected triangular faces (hachured triangles) into the exemplar to form a new deltahedron. This operation leads us in six steps from the regular tetrahedron up to the hexacaideca-deltahedron (16, 10, 24). At this point, it seems to be impossible to insert two triangular faces without creating one vertex of degree six or leaving the domain of deltahedra. This difficulty may lead us to further study the nature of the so far successful operation. Thus, we shall discover that, by adding the "spatial rhombus" composed of two triangles, we always join two vertices of degree four or less. The "distance" between those two vertices can be at most two edges. Although there still are two vertices of degree four in the constructed hexacaideca-deltahedron, these two vertices are three edges apart. This explains the impossibility of integrating two new connected triangular faces but it suggests, at the same time, considering the operation which integrates four connected new faces by which the two vertices of degree four can be joined. Such an operation finally generates the maximal deltahedron, that is the regular icosahedron.

These considerations have provided us with a structured conception of convex deltahedra. There are at most nine different types (classified by the number of faces), eight of which we were able to exemplify and to construct systematically by adding or taking away two (or a multiple of two) connected triangular faces. For the class which we have not been able to exemplify, we have gained a rather strong intuition that such a representation does not exist. Why does it not exist? Is our conception of deltahedra or convex deltahedra incomplete or inadequate? Have we overlooked a property of such geometrical figures that would explain the missing representation or should we still try to construct one? These "disturbing" questions demand further studies of the context which will subsequently lead to its deeper understanding and to a refined structuring.

So far, our investigation has led us to gradually develop a structural mental image of the studied context which is summed up in Figure 4. This grouping is loaded with all the information we have gathered so far concerning the concept of deltahedra. But, since this information is the loading of a grouping, it is co-ordinated and integrated in the context of a structure. The overall structure is constituted by relations of particularization which bring out common and differentiating properties of deltahedra. The particular context of convex deltahedra presents a structuring of the existing types by the operation of adding or taking away two (or a multiple of two) connected faces. It is a grouping within the overall grouping and it shows possible variations within the realm of convex deltahedra. Due to this structuring of information, we realize its incompleteness and become aware of possible refinements or enlargements. Thus, the grouping can be considered as a dynamic description of the state of development of the concept of deltahedra.

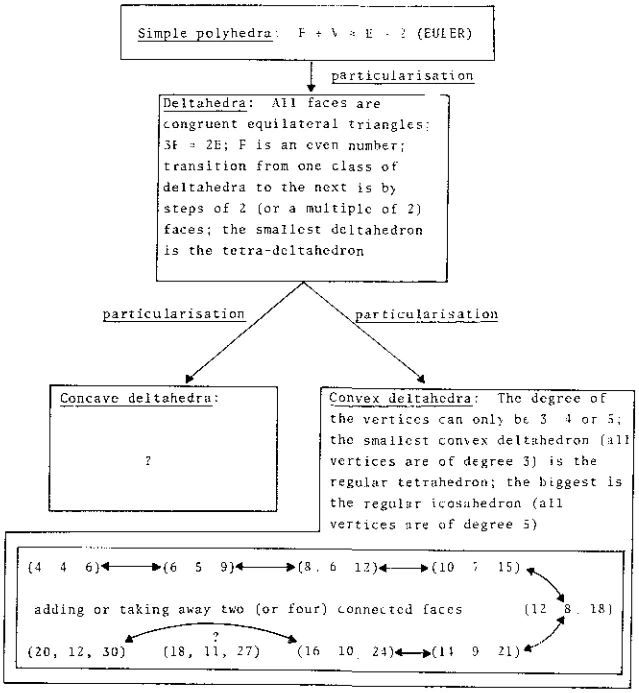


Figure 4: Partial structuring of the context of deltahedra

Partial structuring of the context of deltahedra
Figure 4

Conclusion

Concept formation processes seem to be intimately linked to the development of groupings of the conceptual context. A concept may be considered as knowledge co-ordinated and integrated by the structure of a grouping. Such loading of a grouping can be classified in common and differentiated properties of its states and thus convey both the extensive and the intensive content of the concept. Inversely, mental structural images of particular contexts or situations bear co-ordinated and integrated information relevant to the given context and, as such, describe the context conceptually. Thus, the study of groupings and their development in the process of learning may shed new light on concept formation processes which are crucial in the learning and teaching of mathematics.

Notes

[1] J. Piaget, *Psychologie de l intelligence* Paris: Armand Colin, 1967
J Piaget, *L'équilibre des structures cognitives problème central du développement* Presses universitaires de France, 1975

[2] E Wittmann, The concept of grouping in Jean Piaget's psychology: formalisation and applications *Educational Studies in Mathematics* 5 (1973): 125-146
E. Wittmann, Natural numbers and groupings *Educational Studies in Mathematics* 6 (1975): 53-75

[3] J Larry Martin, A test with selected topological properties of Piaget s hypothesis concerning the spatial representation of the young child *Journal for Research in Mathematics Education* 7, 1 (1976): 26-38

[4] E. Wittman, Piagets Begriff der Gruppierung. Unpublished paper

[5] This episode has been inspired partly by A. Holden, *Shapes, space and symmetry*. New York: Columbia University Press 1971. page 3

[6] TRI LOGIC Mag-Nif Inc , Mentor, Ohio 44060

THE READER AS INVENTOR OR DISCOVERER?

I planned to write so that the learner may always see the inner ground of the things he learns, even so that the source of the invention may appear, and therefore in such a way that the learner may understand everything as if he had invented it himself

Leibniz, quoted by Polya in *Mathematical Discovery*, volume 2

Faraday, on the other hand, shows us his unsuccessful as well as his successful experiments, and his crude ideas as well as his developed ones, and the reader, however inferior to him in inductive power, feels sympathy even more than admiration, and is tempted to believe that, if he had the opportunity, he too would be a discoverer Every student should therefore read Ampère's research as a splendid example of scientific style in the statement of discovery, but he should also study Faraday for the cultivation of the scientific spirit, by means of the action and reaction which will take place between the newly discovered facts as introduced to him by Faraday and the nascent ideas in his own mind.

Clerk Maxwell, introduction to *A treatise on electricity and magnetism*
