The Seen, the Spoken and the Written: a Semiotic Approach to the Problem of Objectification of Mathematical Knowledge [1]

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In a letter written in December, 1881, addressed to his brother Theo, the painter Vincent van Gogh comments:

> Theo, what a great thing tone and colour are. And those who fail to learn to have feelings for them will remain far removed from real life M[auve] has taught me to see so many things that I used not to see and one day I shall try to tell you what he has told me, as there may well be one or two things you do not see properly either. [2]

In this short passage, van Gogh provides us with a clear example of how complex it is to see. To see an object or a certain state of affairs properly, it is not merely enough to stand in front of it and look. In the quoted passage, the *same* tones and colors that van Gogh used to see suddenly appeared to him in a new light. We do not know exactly how Mauve taught him to see what he was not able to see before. Perhaps, both of them were standing in front of a painting and Mauve, holding his brush, pointed to some colours while explaining their contrast and tone differences. Pointing and words may have made visible, for the first time to Vincent van Gogh, something new - something that had escaped him until then

Mauve's lesson is an example of a process of *objectification*: that is, etymologically speaking, a process aimed at bringing something in front of someone's attention or view. Mauve's signs (e g the words and gestures that he might have used to make his point about colours and tones) are part of what I have termed elsewhere (Radford, 2003) *semiotic means of objectification* - e.g objects, artifacts, linguistic devices and signs that are intentionally used by individuals in social processes of meaning production, in order to achieve a stable form of awareness, to make apparent their intentions and to carry out their actions.

In learning mathematics, our contemporary students face a situation that is not so very different from van Gogh's. Students also need to learn to see beyond crude perception and to find something that was unnoticed previously - e g. the center of gravity of a triangle, an irrational number such as $\sqrt{2}$ or the general term of a pattern. But to make the mathematical objects somehow apparent even the most accurate ostensive gesture would not be good enough, for, as Gottlob Frege observed, mathematical objects are neither palpable nor directly perceivable

How then are we going to deal with objects that *cannot* be directly perceived, such as numbers and other mathematical objects? Frege answered: with symbols (1892/1977, p. 144). But Frege's insightful answer leads immediately to a trickier, more difficult question In what sense do mathematical objects become known through signs? Can we equate the conceptual objects with the symbols representing them? Frege categorically answered, *no,* contending that we have to avoid the danger of confusing the objects with their symbols

> So it is necessary not to overrate words and symbols [...] by mistaking them for the actual things of which they are at most the (more or less accurate) representations. [...] When I write down 1 + 2 = 3, I am putting forward a proposition about the numbers 1, 2 and 3, but it is not those symbols that I am talking about. (Frege 1895/1970, p. 482) [3]

It is within this context of the distinction between conceptual objects and their representations that, I want to suggest, the idea of semiotic means of objectification may be of interest - from a psychological as well as an epistemological viewpoint. The idea of semiotic means of objectification relies on the anthropological principle that the representation of knowledge needs to be studied in the broader context of the cultural processes of knowledge production and their technological forms of mediation (Radford, 1998a, 1998b). If it is true that symbols play an important role in our process of producing or becoming acquainted with mathematical objects, to restrict such a process to symbols is much too limiting. [4] There are other means that play a fundamental role in dealing and objectifying mathematical objects, means such as artifacts (e.g rulers, calculators, computers) and linguistic devices (like metaphors and metonymies)

Following Frege's insight, this article is an attempt to investigate the role of semiotic means of objectification in mathematics. Nonetheless, my interest is not underpinned by a philosophical concern What led me to think about semiotic means of objectification first originated in a grade 8 classroom, when the students were asked to generalize a simple pattern and to find the number of circles in figure n, prior to learning the use of letters to build algebraic expressions. (In order not to confuse references to figures used within this article, I use the symbol # before the number when referring to a particular figure number from the classroom mathematical task This was, of course, not actually said or written by the students or teacher.)

After a moment of puzzlement, one of the students looked carefully at the first figures of the pattern as they lay on the table and, in a most conspicuous tone, exclaimed: "There is no figure n!" [5] In a certain sense, the student was right. There was no figure n on the table The teacher then displayed a series of strategies while articulating in different ways several means to forge a sense for what was a new algebraic expression in the students' ontogenetic development.

I have divided the rest of this article into three sections. [6] Section 1 contains some theoretical points concerning

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the idea of semiotic means of objectification. As will become apparent, although inspired by some Fregean writings, the conceptual elaboration of the idea of semiotic means of objectification betrays Frege's thought on several counts. In Section 2, by means of an analysis of classroom activity, I pay attention to the interface between the spoken and the seen: that is, between orality and perception. I then discuss the role of a category of linguistic terms whose function is called *deixis* (meaning in Greek 'display', 'reference'): that is, linguistic terms (e.g. *this, that*) related to actions of showing or pointing out something and that, as it turns out, constitute a key element in mathematical discursive meaning production processes.

Drawing from Bühler's (1934/1979) work, I introduce the notion of *objectifying deictics* as an attempt to account for the way students become aware and come to talk about non-palpable and unperceivable mathematical objects. Section 3 draws from Duval's recent work and examines the *semiotic contraction* arising when what has been said, seen and acted upon is transformed into a symbolic written form. I conclude with a comparison of René Magritte's famous painting *The Annunciation* and the algebraic formula offered by the students. I claim that, instead of expressing a sober and timeless mathematical, relational content, the students' formula can be seen as a kind of vivid painting made up of signs whose content expresses their mathematical experience as lived in the semiotic system of speech, perception and movement.

The classroom analysis and the theoretical discussion shed some light on a more general research question that I am addressing, which can be formulated as follows: how do semiotic systems interact in the objectification of mathematical knowledge? [7]

1. Two theoretical points about semiotic means of objectification

(a) The technology of semiotic activity

As mentioned in the introduction, Frege claimed that the objects of mathematics can become known through symbols only. But he reduced the technology of the semiotic activity to *written* symbols. This reduction cannot be adequately grasped without noticing a deep epistemological conviction at the heart of Frege's view. He, as well as other mathematicians, believed that our knowledge of mathematics can be derived from a few principles that do not require any *sensible experience*: that what was required was a good symbolic language.

This epistemological conviction led him to envision a symbolic system in which, in contrast to natural language, thoughts could be expressed almost directly, without ambiguities, interference or deformation. His symbolic system – *Der Begriffsschrift* – elaborated at the end of the nineteenth century was intended to fulfill such a function. [8] Of course, Frege was dealing with the early twentieth-century problem of the foundation of mathematics. But we cannot say that the epistemological assumption dismissing the role of intuition and having an anti-Kantian flavor did not go without repercussions in the field of teaching, even now. [9]

The concept of semiotic means of objectification – in contrast to the Fregean reduction of the technology of semiotic activity to the written – relies on the idea that to study the production of knowledge requires paying attention to the use of several objectifying means, such as words, gestures, graphics and artifacts. In turn, this idea is built upon the view that a scientific language or discourse cannot become a closed system. It has to remain open and to incorporate new experiences that transcend the border of the system of the language and of the written. As Gonseth remarked many years ago:

> any attempt to base a theory of knowledge (and specially a theory of scientific knowledge) on a discourse closed in itself, that is, on a discourse into which the faculty of being open to experience is not integrated as an always returning exigency, can only miss more or less broadly its goal. (1958, p. 293; *my translation from the French*)

(b) The Fregean concept of sense: the gap between object and sign

Words and symbols, Frege said (in a previous quotation), are more or less accurate representations of their objects. In other words, there is always a *gap* between object and sign. Formalists, like Heine, hold an opposite viewpoint by taking mathematical objects as the actual signs. Hence, in answering the question "What is a number?", Heine said:

> I define from the standpoint of the pure formalist and *call certain tangible signs numbers*. Thus the existence of these numbers is not in question." (Heine, quoted in Frege, 1892/1977, p. 183).

Frege refuted formalists like Heine who, he feared, would reduce mathematics to a play of signs. He insisted that both the sign and object must be definitely distinguished. In distinguishing between sign and object and acknowledging the unavoidable gap between them, Frege was led to introduce his famous distinction between *sign, sense* and *reference* (that is, *object*). In his article '*Über Sinn und Bedeutung*', Frege observed: [10]

> It is natural, now, to think of there being connected with a sign (name, combination of words, letter), besides that to which the sign refers, which may be called the reference ['object'] of the sign, also what I should like to call the *sense* of the sign, wherein the mode of presentation is contained. (1892/1977, p. 57)

Hence, the Fregean concept of *sense* appears to be related to the particular facet of the object being intellectually grasped. One of his best-known examples is the pair 'morning star' and 'evening star'. These expressions (or signs) have the same object (Venus), but their sense is different. They are different because the expressions refer to different *aspects* of the related object – in this case, aspects concerning the time of the day when the object is perceived.

What Frege is saying to us is that sense and knowledge are deeply interrelated. He is also saying that what discloses knowledge is our becoming conscious of unexpected links between two or more different senses of the same object and that at the heart of the epistemological disclosure lies the semiotic variety through which sense is expressed. [11] But what is to be done in the case of mathematical objects where, in contrast to the case of Venus, the time of the observation is of no interest and objects are, as Frege himself pointed out, beyond the scope of perception anyway?

Frege, Weiner (1999) notes, did not give a clear and strict definition of 'sense'; he tried to clarify it using analogies. To convey the distinction between *object*, the subjective *idea* that individuals may have of the object, and the *sense*, Frege wrote:

> The following analogy will perhaps clarify these relationships. Somebody observes the Moon through a telescope. I compare the Moon itself to the reference ['object']; it is the object of the observation, mediated by the real image projected by the object glass in the interior of the telescope, and by the retinal image of the observer. The former I compare to the sense, the latter is like the idea or experience. (Frege, 1892/1977, p. 60)

In this example, the retinal image plays the role of the subjective individual's *idea*. It is opposed to the *object* itself (the Moon). Sense lies between both. It lies in the exterior of the individual and of the object. Frege's philosophy of language parts company at this precise point from subjective accounts that see sense as a private intention. There is another relevant point to be noted. Sense is embodied or expressed *in* the material telescope. While one of the senses associated with Venus, namely 'the morning star', was expressed in natural language, here 'sense' is expressed through a cultural artifact. The telescope becomes a part of the technology of knowledge representation.

In my terminology, the telescope is a means of objectification. But this means of objectification is used neither in an arbitrary manner nor in a disinterested way. Artifacts, like signs, are used in accordance with cultural modes of scientific and mathematical investigation:

> [in] a historically formed and historically developing system of 'objective notions' [...] including the grammatical and syntactical structures of speech and language and the logical norms of reasoning. (Ilyenkov, 1977, p. 77)

This remark suggests that the role that Frege attributed to sense as a mediator between the knowing individuals and the object of knowledge needs to be examined within a larger horizon. This horizon requires situating human action in its cultural context and taking into account that which makes the mathematical object relevant and possible.

The mediating role of Frege's concept of sense needs also to be examined as part of a horizon which distinguishes the cultural modes of signification on the one hand and the technologies of semiotic activity used in the objectification of knowledge on the other. Although, in the conventional sense of the word, speech is neither considered as part of our technological environment nor as part of our modes of production, speech nevertheless is part of the modes of production of knowledge. This is why I included it along with material artifacts in the technology of semiotic activity. [12]

Figure 1 provides a diagrammatic account of the objectification of mathematical knowledge. Notice that there is no line linking the individuals to the objects of knowledge directly. The absence of a direct link expresses the essential idea of semiotic mediation: between subject and object lie language, technology and, more generally, the conceptual and material cultural institutions embracing the life of the individuals.

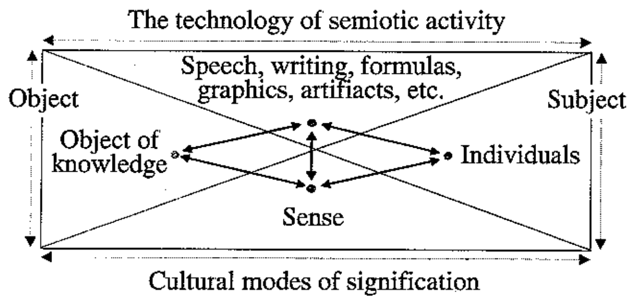

Figure 1: A diagrammatic account of the objectification of knowledge

Bearing this in mind, in the next section I investigate, through an interpretative case study, how a specific conceptual algebraic object is made apparent by the students. I examine the corresponding process of mathematical knowledge objectification as it occurs in the interface between the spoken and the seen or, more generally, of speech, perception and gestures. In taking perception as a relevant element in the objectification of knowledge, I part from Frege's analytical position.

2. Deixis

I want now to turn to a classroom episode that, like the one mentioned in the introduction, was aimed at providing grade 8 novice students with an occasion to start building their first algebraic symbolic expressions. [13] In the first subsection below, an excerpt of the students' activity will be analyzed. [14] In the second subsection, the concept of objectifying deictics is introduced in reference to the students' activity.

Figure *n* as a schema

The students were working on the pattern shown below:

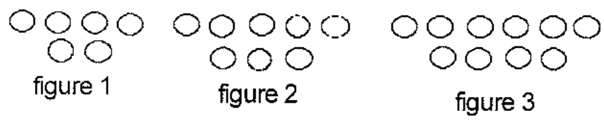

Figure 2: The first three terms of the sequence

The following excerpt comes from one of the three-student groups I monitored as part of a longitudinal research program. The excerpt begins as the students begin tackling the task of finding an expression for the number of circles in figure *n*. Right after this, the students easily completed the task of finding the number of circles in figure #11.

1 Francis: We just have to pick one [figure]; we just have to choose one of that (pointing to the figures of the pattern) . *n* equals ...

2 Dan: *(Talking to Francis)* Equal to how many at the top and then how many at the bottom *(Francis is inaudible)* ... It's pretty hard. We'll say figure #100

3. Francis: 102 then 100 . .

4 Dan: *(Talking to Francis)* How did you figure that?

5. Francis: You always add 2 on top.

6. Dan: *(Talking to Francis)* At the bottom it's 1.

7. Sylvie: *(Talking also to Francis)* Yes, but look at figure 4 *(showing figure #4 with her hand on the page)*. It's 5.

8. Dan: *(Talking to Francis)* You add 1 at the bottom too Then you add 2 on top.

9. Sylvie: Look! *(Pointing to figure #4)* It's 4, you have 7 on top.

10. Dan: We'll say *n* equals figure 100 It could be 100 ... you add 1 that would be 101 .

11. Francis: *(Understanding what Dan and Sylvie said, adds)* Then 103. *(Re-reading the question)* How many circles in total will figure *n* have? Yes If you add 103 + 101, it is going to equal 204.

In the first turn of this passage, by stating that one just has to pick one of the figures, Francis indicates a path to deal with mathematical generality. Through an indexical gesture, he points to the figures on the table and states that one of them can be picked up. After acknowledging the difficulty of the task, Dan chooses a figure that is not *materially* there To deal with figure *n*, the students choose figure #100 [15]

In turn 2 of the dialogue, to materialize figure #100 somehow, Dan talks about *top* and *bottom* rows. Through these terms, the students give shape to a figure that they have not materially seen - and will not ever see But in using this new way of seeing, permitted by linguistic objectification, Francis makes a mistake (turn 3). Focusing on the relationship between the rows, he rightly stresses the fact that the upper row has two more circles than the lower row. However, he does so without taking into account the relationship between the number of circles in the bottom row and the number of the figure.

Dan's question in turn 4 is not superfluous. Since figure #100 cannot be seen in the same way as figure #1, #2 or #3 can, Dan is not sure about what Francis is seeing. When the answer is given (turn 5), he realizes that there is something wrong with the bottom row (turn 6 hints at the missing circle) But still, this is too general. Hence, Sylvie calls her group-mates' attention to figure #4 - a figure that they materialized using bingo chips and drew on the activity sheet and that all of them could *see*. Her utterance "It's 5" (turn 7) supplies the discussion with the concrete dimension that can conclusively elucidate the argument.

This passage is followed by Dan's key verbalization of the structure of the rows of the figures - a description anchored in the number of the figure (see turn 8). This line contains the very kernel of the generalizing action - and will have repercussions in the building of the mathematical symbolic formula, as we will see later. In turn 9, displaying a visible figure, Sylvie relies again on perception At this point, Francis realizes his mistake and corrects it in turn 11.

Thus, so far, the students have objectified figure *n* in the form of a *process* that allows them to find the number of circles in any particular figure (regardless how 'big' the figure may be). In the next subsection, I will investigate the tools of the technology of semiotic activity that the students used to craft figure *n*.

Deixis at phantasma and objectifying deictics

The students' objectification of figure *n* has the form of a *schema* (in a Kantian sense): that is, a general procedure expressed in accordance to the way in which the object (in this case, figure *n*) becomes apprehended Furthermore, the sense associated with the way in which figure *n* is apprehended is *perceptual* Indeed, the schema appears as a chain joining the linguistic and the non-linguistic domains, the spoken and the seen [16]

How is the schema crafted? How do the students give shape to it with the technology of words? In the previous excerpt of student dialogue, not all the words played the same role The crucial words *bottom row* and *top row* function as means to show or make apparent a certain contextual structure - they are *deictics*

The category of linguistic terms referred to as deictics is comprised of words such as *this* or *that*, i e words whose primary function is to point to something in the visual field of the speakers As Hanks (1992) says, the basic communicative function of deictics is:

> to individuate or single out objects of reference or address in terms of their relation to the current interactive context in which the utterance occurs. (p 47)

Bühler, who carried out a systematic investigation of deictics in his book *Sprachtheorie*, published in 1934 and that became a classic of twentieth-century linguistics, cogently argued that every time that the word 'here' appears in a verbal exchange it is because something is no longer perceptually evident: thus, words have to supplement gestures and sight. He corrected and expanded a list of deictics previously identified by Brugmann (among them personal pronouns - e.g. *I, you* - and spatial adverbs like *there, here*)

More important for this discussion is Bühler's distinction among certain kinds of deixis, based on the mode of reference between the indicative or deictic word and the referred object. I discuss two of them here, which will help to understand the students' objectification of mathematical knowledge. The first occurs where deictic words are used to refer to objects in the perceptual field of the speakers. This form of deixis is called *demonstratio ad oculos* The classic example is the word 'that' to point to something in our surrounding. Francis in turn 1 gives a concrete example when he states, "we just have to choose one of *that*".

Another kind of deixis occurs when pointing is made to something absent:

> in the domain of [...] *memories* and of the constructive *imagination* (p. 21; *emphasis in original*) [17]

This type of deixis Bühler called *deixis at phantasma*. A typical example occurs when somebody uses indicative words to describe a path to follow in a city (e.g. from *there*, you walk two blocks down the street [18]).

Regardless of the kind of deixis, the referent to which a deictic term refers is seized by the contextual circumstances (see Nyckees, 1998, p. 242 ff ; Frei, 1944). As a result, the contextual nature of deictic terms renders their referents context-bound. Although this is an ordinary phenomenon in natural language, it raises a problem in mathematical discourse - a discourse whose objects are supposedly context-free. [19] It is for this reason some logicians have paid attention to deictics and why Russell claimed that they do not determine concepts (Russell, 1976; see also Quine, 1960).

Deixis is of interest here for, as could be seen in the previous analysis of student dialogue, it supports a powerful referential mechanism. Terms such as *top row, bottom row* in the students' discussion about figure #100 are *indicating* something. But this *something* is not perceptually visible. In this sense, they function to a certain extent as a form of *deixis at phantasma*. But not exactly, for figure #100 has *not* been seen by the students. In constructive imagination, by using deictics the students go beyond mere reproduction but in a different manner from the one Bühler emphasized.

In our case, it is not a question of variations of previous settings already seen and that the novelist or the speaker transforms in constructive imagination (see Bühler, 1979, p. 141). Rather, the point is to make apparent something *new*. The process that makes apparent something new I would like to term *objectifying deixis*. The indicative linguistic terms used to accomplish this process I would like to call *objectifying deictics*. Objectifying deictics allow one somehow to shape the essence or what mediaeval philosophers re-interpreting Aristotle referred to as the *quidditas* of an object (its 'whatness'). The *quidditas*, as I conceive of it here, is not something that presents itself to view, but something that is crafted out of words, signs, gestures and artifacts.

Objectifying deictics on which schemata are based make it possible to *talk* about something in *general*. But the generality thus attained is still situated or perspectival. In Bühler's terminology, it still pre-supposes an *origo* or *origin*: that is, a spatio-temporal centre making it possible to distinguish between e.g. *there* from *here*. To be handled formally, that is, according to their form and not to their content, polynomials and other symbolic expressions must go beyond the *origo*. To do so will necessitate that the students abandon the perspectival view with which objectifying linguistic deixis provides them. What happens when the written becomes the centre of the technology of semiotic activity in the objectification of knowledge? This question is addressed in the next section.

3. The semiotic contraction

Letters as nouns

The teacher came to see the students' work. He remarked that the students did not produce a symbolic expression and encouraged them to find one.

1. Teacher: There you have done it for a particular case. You have done it for 100. Eh ... for figure 100 ... But if we wanted one for any figure ... *n*?

2. Dan: You add 1 on top and 1 on the bottom too, you always add 2. You add 2 to the previous formula ... Yes, you always add 2. Look! *(Counting the circles on figure #1, he says)* 1, 2, 3, 4, 5, 6.

3. Sylvie: Yeah *(Continuing Dan's utterance and pointing to figure #1 and figure #2, she says)* 6, 8

4. Dan: Yes, what we would do is column 1, it would be like *a*, then column 2 would be like ... the top column would be *n*, then the bottom column would be like *b*. Then you do $n + b + 2$

In turns 2 and 3, the students mention the relationship linking a term and the previous one: that is, in symbols: $u_{n+1} = u_n+2$

While in turn 2 Dan offers a verbal general statement, in turn 3 Sylvie provides a concrete example, an example open to perceptual inspection. [20] Turn 4 contains the students' formula. Of course, from the perspective of a fluent algebra user, Dan's formula was not the expected one. Nevertheless, it does not imply that, for the students, the formula was meaningless. [21]

In fact, the students' formula has a hybrid function. On one hand, it functions as a numerical procedure; on the other, it appears as a narrative - a *symbolic narrative* (Radford, 2002) recounting, with symbols, a previously linguistically objectified schema. The symbolic narrative is telling us that to go from the figure at which you are now to the next, you have to add the number of circles in the top row to those in the bottom row, and, after that, you still have to add two circles. The students changed the formula because they were confronted with a situation that they interpreted as an incompatibility between its two functions, the procedural one and the narrative one. I will return to this point in the final section.

At present, my interest is to understand the exigencies that the *semiotic contraction* imposes upon the students. To do so, I first address the question of the students' mode of designation of objects through algebraic symbolism. To what do the students' symbols refer? They refer to the top and the bottom rows of the linguistic schema objectifying figure *n*. Furthermore, in the referencing act, they function as nouns. [22] One of the problems associated with the introduction of algebra is certainly to understand that letters are not merely substitutes for *nouns*. Now, if the designation of objects is not based on nouns, how then is reference possible at all? The lack of proper nouns - or *penuria nominem*, as Duval (in press) termed it - will require the use of a different kind of designation for the objects of discourse.

Algebraic reference

In the students' investigation of the general term of a sequence, two main strategies can be identified. The first one focuses on the relationship between some terms of the sequence – usually a relation between consecutive terms [23] Here, perception and language play a central role. The relation between e.g. two consecutive terms can be *seen* and expressed in natural language, even if it is not in a perfect way (there may be some ambiguities in the terms) In the second strategy, a direct expression for u_n is sought. Here, perception is much less helpful The production of a symbolic expression for u_n requires that a *point of reference* be chosen. This point of reference is related to the *unperceivable* position of the term in the sequence. Failure to take into account the point of reference leads the students to produce symbolic expressions like the previous one in which the letter n and b remain without link to the position of the term [24]

To overcome this problem, a new kind of designation of objects of discourse has to be employed One of the characteristics of the new kind of designation is that the 'nouns' for the objects of discourse are designated according to *internal* relations between them. Thus, the relationship between the number of the figure and the number of circles on the bottom and the top rows puts aside all the spatial characteristics of linguistic objectifying deictics, focuses on their internal relations, and results in the expressions n, $n + 1$ and $n + 2$ respectively. [25]

In algebraic language the mode of designation of the objects of discourse is, as Duval (in press) noticed, *functional*. By focusing attention on the functional features among objects, reference is enriched. This new kind of designation of objects compensates in this way for the problems arising from the lack of proper names. But it would probably be better to say that the compensation so achieved by functional designation does not exactly 'repair' the lack of proper names.

It is a specialization along a new line of investigation: a line that grew up with the development of symbolic languages in the sixteenth century and resulted in a non-*transparent* (Duval, 1999, p. 9) use of signs. Indeed, the development of specialized symbolic languages brought forward highly conventional procedures for the formation of mathematical expressions requiring a complex understanding as witnessed in a letter that Peano sent to Frege in February 1894 (hence only a few years after Vincent van Gogh sent his to Theo) Failing to understand Frege's *Begriffsschrift* or conceptual notation (see [8]), in his correspondence, Peano gently asks this question:

> How do you write the following propositions? [To shorten my argument, I will quote but one of them here, the simplest one.] There is a whole positive number x which satisfies the equation $2 + 3x = 5$ (given in Frege, 1980, p. 110)

On the last page of Peano's letter (p. 111), Frege wrote:

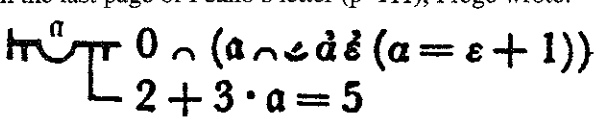

Figure 3: An example of Frege's Begriffsschrift

Peano's letter is interesting in that it shows the role of natural language in the signs-making-sense enterprise when dealing with new specialized symbolic systems. Seen from the perspective of the genesis of mathematical symbolic languages, the use of natural language appears as a support But it does not guarantee success. Indeed, in addition to the problem of the functional designation of objects required in algebraic reference, the semiotic contraction brings another concomitant exigency: the loss of the perspectival view that natural language affords

The loss of the perspectival view

As noted before, deictics in general and objectifying deictics in particular provide the speaker with a window to see the world. They presuppose a spatial and temporal system revolving around an *origo*. Thus, in using the top and bottom row as objectifying deictics, the students emphasize an *origo* and achieve an 'orientation in space' (Talmy, 1983). The students' formula $(n + b) + 2$ is written from this perspectival view, preventing them from operating with letters [26]

The use of brackets in the students' first symbolic formulas gives a clear idea of the extent of the sequential organization of the numerical action in time and in space. These actions (related to the way that the non-palpable and unseen figure n is *perceived* by the students) become symbolized and, so to speak, frozen by letters in a more complex symbolic system. [27] The formula expresses the embodiment of the students' mathematical experience The deletion of brackets, the ability to move letters from one place to another in the formula requires a re-interpretation of the previous embodied experience – in fact, it requires a disembodiment of experience and sense, as well as the loss of the perspectival view

4. Concluding remarks: the students' formula as a painting

In 1930, the Belgian artist René Magritte produced a painting inspired by a religious theme common in Renaissance Italy: it is the passage in which the Virgin Mary is visited by the Angel Gabriel who informs her that she will give birth to God's son. Instead of an angel and a pious woman, what we see are large bells hanging in a corrugated metallic wall We see also a piece of paper showing some kind of mechanical geometric cuts standing between the wall and two mysterious balustrades that seem to witness the scene (see Figure 4 overleaf).

The painting belongs to an art intellectual movement that stressed the predominance of language over the visual and offered new ways of ordering and perceiving the world of objects: for instance, unreal encounters presented with neat verisimilitude or encounters taken from the cultural stock of knowledge but presented without such verisimilitude. Magritte's *The Annunciation* is an example of the latter It is a painting that tells a classical story: a story from tradition but recounted in a fantastic way – to the extent that we have to be told the title in order to attempt to make sense of its content. And yet ...

In a similar manner, the students' formula tells a story. It is a classical story of traditional problems with pattern generalization that the students have previously encountered in

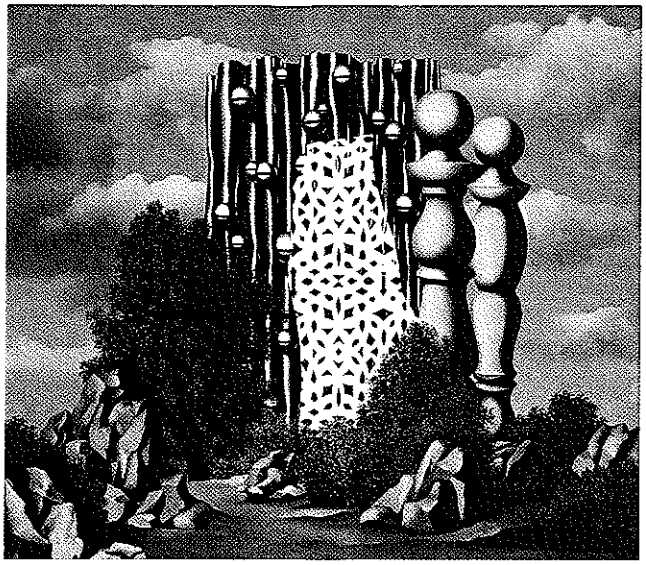

Figure 4: René Magritte, 'L'Annonciation' (1930) © Estate of René Magritte / ADAGP (Paris) / SODRAC (Montréal) 2002 and © Tate, London, 2002

school arithmetic, but one that is now recounted in a fantastic way. As in Magritte's painting, it is told using signs. [28] In both, Magritte's *The Annunciation* and the students' formula, the signs neither have an intuitive relationship to the objects which they represent, nor do they show signs of human presence. Balustrades, bells, mechanical geometric cuts, letters, brackets and the symbol for addition are at most vestiges of it

Even if the formula appears in a linearized manner, as required by the algebraic symbolism, the story (a symbolic version of a previously linguistically objectified schema) is written in a form that suggests a *narrative style* Indeed, the signs have been spatially distributed to fit and retell an organization of previous events and actions through time. The students' formula in section 3 is saying that to go from the figure at which you are now to the next, you have to add the number of circles in the top row to the number of those in the bottom row and, after that, you still have to add two more circles.

But how can we be sure that the formula is correct? The students asked themselves this question. Section 3 described how in addition to the narrative function, the students' formula also has a procedural function. Both functions are linked. The latter enacts the story told by the narrative. To inspect the adequacy of the link, the students worked on figure #9 of the pattern Their formula required that they start on figure #8 Figure #8 was never continued with the bingo chips that we provided the students with during their mathematical activity, nor was a drawing of figure #8 made by the students.

They never *saw* figure #8 They did not need it Language, with its rich arsenal of objectifying deictics, allowed them to see beyond the situated perceptual field. Yet, the way to write the number of circles in each row of figure #8 keeps the spatial distribution of the rows (See Figure 5 below: eleven circles are indicated on top of nine circles, divided by a small horizontal line)

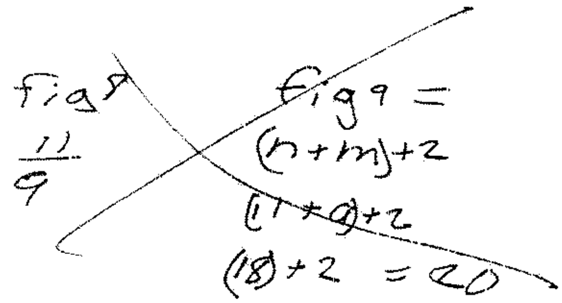

Figure 5: A narrative enactment of the students' formula

Figure #9 on the activity sheet is placed to the right of figure #8, thereby emphasizing the movement from one figure to the next In the second line, we find the formula (the letter *b* has now been changed to *m*) The substitution of values comes next The calculations are then carried out, following the chronology of the mathematical events as emphasized by the use of brackets. But the students made a mistake in the addition!

At this point, they reached an incompatibility between the two functions of their formula - the procedural and the narrative one. How simple it would have been for them to recheck the addition and repair the mistake The problem is that in these cases, if something goes wrong, the reason is attributed not to something very well known (such as a procedural calculation here), but to the new object under scrutiny (figure *n*). The students decided to give up the formula. They crossed it out and continued trying to find another one.

However, the semiotic point that I want to stress is the following: the distribution of signs in the formula is such that what the students are doing is expressing *their* experience *with the nascent mathematical object*. The students have not detached themselves from the emergent object. On the surface, without being there, deictics have left their imprint on the organization of the students' signs. As a result, the mode of designation of letters is still indicative. The letters have an *indexical meaning* only (Radford, 2000b)

If in *The Annunciation* Magritte achieved a *coup de maître* in making the traces of human presence vanish, the students' formula still demands that you can *see* (in a conceptual sense, though) the top and bottom row of the figure at which *you* are *now*, in order to *go* to the next. In doing so, the students paint (so to speak) their own mathematical experience and act much more as painters than as algebraists. The students' formula, I want to suggest, can be seen as a kind of vivid painting made up of signs whose content expresses their cultural, embodied mathematical experience as it happened in the general system of speech, perception and movement. In terms of my diagrammatic account of the objectification of knowledge (see Figure 1), we can say that the students' formula, expressed with the technology of written symbols, has not reached the level of impersonal sense. To go further, its sense has to be disembodied The question now is: *how*? [29]

Frege once thought of his conceptual notation as a tool, a kind of mechanical hand achieving a solid 'grip' on the ideas and leaving aside the imprecision and the defects of natural language. [30] But Frege was well aware of the fact that his conceptual notation, like any other symbolic system, could not avoid recourse to natural language, for primitive terms cannot be defined. So:

> on the introduction of a name for something logically simple [.... there is nothing that can be done] but to lead the reader or hearer, by means of hints, to understand the words as is intended. (Frege, 1892/1960, p 43)

It may well be that the disembodied sense required in algebraic language lies precisely in the ability to construct a new *origo*, in Bühler's terminology: that is, an origin whence to see things and grip ideas. But in contrast to traditional *origos*, the new one needs to be non-static. Perhaps it needs to be flexible to accommodate different views at the same time, so that various semiotic means of objectification can be integrated in the act of knowing.

There are reasons to believe in a strong epistemological connection between the historical emergence of algebraic symbolic languages in the work of Piero della Francesca, Bombelli and other mathematicians who predated Viète and the general culture of the Renaissance, one which allowed, among other things, a new way of writing (particularly as a result of the invention of printing and the concomitant invention of the reader) and the creation of different perspectival techniques (e.g. in stage design with movable stages for theatre performances). All this points, indeed, to a world that radically changed its mode of knowledge objectification by integrating new technologies of semiotic activity. But, at this point, I can only offer this as a conjecture.

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Notes

[1] This article is one result of a research program funded by the Social Sciences and Humanities Research Council of Canada

[2] Van Gogh (1996, p 114)

[3] Much in the same vein, Georges Glaeser in his Ph D seminar at the *IREM de Strasbourg* used to tell us that confusing the object with its representations would amount to confusing God with the old man painted in the Sistine Chapel in Michaelangelo's famous scene. Another similar argument against formalism can be found in Duval (1995). For a case against empiricism, see Steinbring (2001)

[4] Frege's position needs to be understood within the cultural history of Western thought and its legendary passion for the written - something that has often surprised other cultures. As an indigenous North American remarked:

> The white man writes everything down in a book so that it will not be forgotten; but our ancestors married the animals, learned all their ways, and passed on the knowledge from one generation to another. (Jenness, quoted in Lévi-Strauss, 1966, p. 37)

On this subject, see also Goody (2000).

[5] A description of the classroom lesson and its analysis can be found in Radford (1999)

[6] In order to keep this article within a reasonable length, I have foregone discussion about the nature of mathematical objects. The terminology that I use, however, may give the impression that I am adopting a kind of Platonist or Realist account of mathematical objects. This is not the case. Suffice it to say that I conceive of mathematical objects as conceptual entities produced in the course of cognitive reflections of the external world *in the forms of the individual's activity (i.e. cognitive praxes)*. Given that activity finds its ontological support and its sense in culture, the realm of potential mathematical objects appears subsumed in the possibilities offered by the semiotic systems mediating and objectifying the individuals' activities and the cultural modes of scientific and mathematical investigation. (A first elaboration of this idea can be found in Radford, 2000a, with further development in Furinghetti and Radford, 2002.)

[7] The idea of semiotic system that I am conveying includes classical systems of representation - e.g. natural language, algebraic formulas, two or three-dimensional systems of representation. In other terms, this is what Duval (2001) calls *discursive* and *non-discursive* registers - but it also includes more general systems, such as gestures (which have an intuitive meaning and to a certain extent a fuzzy syntax) and artifacts, like calculators and rulers, which are not signs but have a functional meaning (Radford, 2003).

[8] Frege's neologistic noun *Die Begriffsschrift* comes from *Begriff* 'concept' and *Schrift*, 'writing', so that *Begriffsschrift* means an 'ideography' or a 'conceptual notation'

[9] As an example, there is the heavy importance still given to the written in contrast to the oral, the figural or artifact-based activities in school mathematics. This commitment to the written over the figural and the spoken is well illustrated by an example that Bosch and Chevallard (1999) discuss in an article that deals with some of the issues that I am addressing here. Their example - a word-problem in which a certain number has to be found - is solved using three different methods that the authors themselves elaborated (pp 97-98). In the first one, the oral component is dominant. In the second, a diagram is the center of the oral discourse. In the third, there is neither diagram nor speech: all happens within the realm of the algebraic symbolism. They argue our evaluation of the three solutions in terms of the opposition 'reasoning versus calculation' depends on our stance towards symbols, figures and the oral.

[10] Among Fregean scholars, *Sinn* is usually translated as 'sense'. *Bedeutung*, in contrast, is variously translated as 'object', 'reference', 'designation' or 'meaning'. I will use the word 'object' here and reserve the term 'meaning' for other more general purposes

[11] As Otte (in press) has written:

> the problem of knowledge [.] is to be seen in the question of [.] how objects and signs are connected. This question can only be answered from a genetic point of view by explaining how laws or representations arise and how meanings evolve

See also Duval (1995, 1998)

[12] See also Goody (2000, p. 132).

[13] The rationale for the activities and the methodology can be found in Radford (2000a)

[14] In the students' dialogue, I have omitted (when necessary) students' repetitions, in order to facilitate the reading and to capture the essential dialogic descriptions of the problem discussed by the students and the teacher. Intonations have been ignored and gestures accompanying utterances have been underlined

[15] This strategy looks similar to the one I discussed in a previous article, where another group of students (working on a similar pattern) chose figure #12. There are important differences though. In the reported analysis, figure 12 was taken in a *metaphorical* sense (see Radford, 2000a, p. 247). Here, figure #100 is taken in a *generic* sense. Although a metaphorical sense includes some sort of generic sense, the reciprocal is not true. The mode of presentation of the general through the particular is not the same. In the metaphorical sense, one comes to terms with mathematical generality by talking about the general *through* the particular. This requires that the particular not be *seen* as it really *is*. To accomplish this, the students use linguistic strategies such as: "Let's say that .. " or "as if .. ", etc. In the generic sense, figure #100 is not a surrogate. Figure #100 can be *replaced* by any figure. Metaphorical sense is not based on substitutability as generic sense is. It is based on seeing something as it *is not*, it is based, as Aristotle said, on *transferring* something to something else (see Aristotle's *Poetics*, XXI 4). Discussions about metaphors in mathematics education can be found in e.g. Sfard (1994), Presmeg (1998) and Núñez (2000).

[16] The schema, however, is not an image. In his *Critique of Pure Reason*, Kant (1781/1996) writes:

> if I put five dots after one another . . ., then this result is an image of the number five. Suppose, on the other hand, that I only think a number as such, which might then be five or a hundred. Then my thought is more the presentation of a method for presenting - in accordance with a certain concept - a multitude (e.g., a thousand) in an image, than this image itself. [. .] Now, this *presentation* of a universal *procedure* of the imagination for providing a concept with its image I call the schema for the concept. (p. 213; *my emphasis*)

Eco (1999) has suggested that, in the Kantian concept of schema, there is a kind of 'iconic' relation between the fact and the proposition expressing the schema. The Kantian schema is, Eco claims, "a proposition that has the same form as the fact it represents" (p. 82).

[17] I am quoting from excerpts of Bühler's *Sprachtheorie* (1934), translated into English in Jarvella and Klein (1982). A complete translation into Spanish was made by Julián Marías; see Bühler (1934/1979). A book devoted to Bühler's work is Innis (1982)

[18] See for instance the interesting work of Klein (1983)

[19] Rowland (2000) has shown the ubiquitousness of deixis in school mathematical discourse.

[20] The adverb 'always' (appearing here in Dan's statement) is crucial to conveying the generalizing mathematical sense. I discuss it as a part of the generative function of language in Radford (2000a), with further analysis and development in Radford (2003).

[21] That the students found their formula meaningful is emphasized by the fact that it was not refuted by Sylvie and Francis.

[22] We have noticed in the course of our research that in more complex patterns, for example, to count the matches that figures are made up with, the students tend to objectify the general in the form of a schema. As seen in the first subsection of section 2, the schema is based on objectifying deictics such as 'roof', 'walls' and 'floor'. During the symbolization, they are inclined to introduce a different letter – one for the matches on the roof, one for the matches on the walls, etc. See also the classroom example discussed in Laborde, Puig and Nunes (1996).

[23] This case is shown in turn 4 of the last excerpt. See also Arzarello (1991); Mason (1996); MacGregor and Stacey (1995)

[24] This is the reason why I called this *the positional problem* (Radford, 2000a).

[25] Interestingly, in turns 8-11 of the first excerpt (in the first subsection of section 2), the students showed that they were aware of the numerical relation between the position of the figure and the number of circles on each row. But the point of reference was not pronounced. It remained implicit.

[26] In Radford (2003), I discuss at length the case of a group of students from another grade 8 class none of whom could identify $2n + 1$ with $(n + 1) + n$. This identification would have required the suspension of the perspectival view whence the symbolic expressions received their sense.

[27] The 'linearization' of mathematical experience, as imposed by writing, has also been addressed by Arzarello (2000) and Arzarello, Bartolini Bussi and Robutti (2002)

[28] Instead of rocks, balustrades, etc., the story is told using letters, brackets and the sign '+'.

[29] The semiotic phenomenon that I am trying to exhibit through the comparison between a students' formula and a painting cannot be limited, however, to *the* particular formula $(n + b) + 2$. It is a frequent situation arising when students start building their first algebraic expressions with letters. Expressions such as '$(n - 1) = a$' (Radford, 2000a), '$(n + n) + 1$' (Radford, 2003), the examples discussed in Radford (2002) and all those expressions that accomplish a translation into algebraic symbolism or, more precisely, a *conversion*, without being capable of *treatment* in Duval's sense (for instance, a translation into $(n + n) + 1$ without being able to transform the latter into $(2n + 1)$, has the same traits as the painting. They tell a story, but the signs remain fixed, as do the balustrades in *The Annunciation*.

[30] See Frege (1971, p. 66).

References

Arzarello, F. (1991) 'Pre-algebraic problem solving', in Ponte, J., Matos, J., Matos, J. and Fernandes, D. (eds), *Mathematical Problem Solving and New Information Technologies* (NATO ASI Series F, Vol. 89), Berlin, Springer-Verlag, pp. 155-166

Arzarello, F. (2000) 'Inside and outside: spaces, times and language in proof production', in *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education*. Hiroshima, Japan, Hiroshima University, 1, pp. 23-38.

Arzarello, F., Bartolini Bussi, M. and Robutti, O. (2002) 'Time(s) in the didactics of mathematics: a methodological challenge', in English, L. (ed.), *Handbook of International Research in Mathematics Education*, Mahwah, NJ: Lawrence Erlbaum, pp. 525-552.

Bosch, M. and Chevallard, Y. (1999) 'La sensibilité de l'activité mathèmatique aux ostensifs', *Recherches en Didactique des Mathèmatiques* **19**(1), 77-124.

Bühler, K. (1934/1979) *Sprachtheorie (Teoría del lenguaje)*, trans. by Marías, J., Madrid, Alianza Editorial.

Duval, R. (1995) *Sémiosis et Pensée Humaine: Registres Sémiotiques et Apprentissages Intellectuels*. Bern, Peter Lang.

Duval, R. (1998) 'Signe et objet, I et II', *Annales de Didactique et de Sciences Cognitives*, IREM de Strasbourg, **6**, pp. 139-196

Duval, R. (1999) 'Conversion et articulation des représentations analogiques', Séminaires de recherche de l'Institut universitaire de formation de maîtres du Nord-Pas de Calais, Lille, France

Duval, R. (2001) 'The cognitive analysis of problems of comprehension in the learning of mathematics', Paper presented at the Semiotics Discussion Group of the 25th Conference of the International Group for the Psychology of Mathematics Education, Utrecht, Freudenthal Institute, The Netherlands.

Duval, R. (in press) 'Des signes au langage et du langage à l'algèbre', *Actes du séminaire Franco-Italien sur l'enseignement de l'algèbre (SFIDA)*, 1-4, Nice, IREM de Nice

Eco, U. (1999) *Kant and the Platypus: Essays on Language and Cognition* (trans. McEasen, A.), London, Harcourt, Inc.

Frege, G. (1892/1977) 'Uber Sinn und Bedeutung', in *Translations from Philosophical Writings of Gottlob Frege*, Geach, P. and Black, M. (eds), Oxford, Basil Blackwell, pp. 56-78

Frege, G. (1895/1970) 'The whole number', *Mind* **79** (#316), 481-486. (First published as: 'Le nombre entier', *Revue de métaphysique et de Morale* **3**, 73-77.)

Frege, G. (1971) *Écrits Logiques et Philosophiques*, Paris, Éditions du Seuil.

Frege, G. (1980) *Philosophical and Mathematical Correspondence* (Gabriel, G., Hermes, H., Kambartel, F., Thiel, C. and Veraart, A. (eds), abridged from the German edition by McGuinness, B. and trans. by Kaal, H.), Chicago, IL, The University of Chicago Press.

Frei, H. (1944) 'Systèmes de déictiques', *Acta Linguistica* **4**(2), 111-129

Furinghetti, F. and Radford, L. (2002) 'Historical conceptual developments and the teaching of mathematics: from philogenesis and ontogenesis theory to classroom practice', in English, L. (ed.), *Handbook of International Research in Mathematics Education*. Hillsdale, NJ, Lawrence Erlbaum, pp. 525-552

Gonseth, F. (1958) 'Le problème du langage et l'ouverture à l'expérience', *Dialectica* **12**(3/4), 288-295

Goody, J. (2000) *The Power of the Written Tradition*. Washington, DC, Smithsonian Institution Press

Hanks, W. (1992) 'The indexical ground of deictic reference', in Duranti, A. and Goodwin, C. (eds), *Rethinking Context*, Cambridge, Cambridge University Press, pp. 43-76

Ilyenkov, E. (1977) 'The concept of the ideal', in Kuleshov, V. (ed.), *Philosophy in the USSR: Problems of Dialectical Materialism* (trans. Daglish, R.), Moscow, Progress Publishers, pp. 71-99.

Innis, R. (1982) *Karl Bühler: Semiotic Foundations of Language Theory*, New York, NY, Plenum.

Jarvella, R. and Klein, W. (1982) *Speech, Place and Action: Studies in Deixis and Related Topics*, Chichester, John Wiley

Kant, I. (1781/1996) *Critique of Pure Reason* (trans. Pluhar, W. from the 1781 and 1787 editions), Indianapolis, IN, Hackett Publishing Company

Klein, W. (1983) 'Deixis and spatial orientation in route directions', in Pick, H. and Acredolo, L. (eds), *Spatial Orientation: Theory, Research, and Applications*, New York, NY, Plenum Press, pp. 283-311.

Laborde, C., Puig, L., and Nunes, T. (1996) 'Language in mathematics education, in *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education*. Valencia, Spain, University of Valencia, **1**, pp. 53-84.

Lévi-Strauss, C. (1966) *The Savage Mind*. London, Weidenfeld and Nicholson

MacGregor, M. and Stacey, K. (1995) 'The effect of different approaches to algebra on students' perceptions of functional relationships', *Mathematics Education Research Journal* **7**(1), 69-85

Mason, J. (1996) 'Expressing generality and roots of algebra', in Bednarz, N., Kieran, C. and Lee, L. (eds), *Approaches to Algebra*, Dordrecht, Kluwer, pp. 65-86.

Núñez, R. (2000) 'Mathematical idea analysis: what embodied cognitive science can say about the human nature of mathematics', in *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education*, Hiroshima, Japan, Hiroshima University, **1**, pp. 3-22

Nyckees, V. (1998) *La Sémantique*, Paris, Beilin.

Otte, M. (in press) 'Does mathematics have objects? In what sense?', *Synthèse*

Presmeg, N. (1998) 'Metaphoric and metonymic signification in mathematics', *Journal of Mathematical Behaviour* **17**(1), 25-32.

Quine, W. (1960) *Word and Object,* Cambridge, MA, M.I.T. Press. (10th printing, 1976)

Radford, L. (1998a) 'On culture and mind, a post-Vygotskian semiotic perspective, with an example from Greek mathematical thought', paper presented at the 23rd Annual Meeting of the Semiotic Society of America, Victoria College, University of Toronto, October 15-18

Radford, L. (1998b) 'On signs and representations: a cultural account', *Scientia Paedagogica Experimentalis* **35**(1), 277-302.

Radford, L. (1999) 'The rhetoric of generalization', *in Proceedings of the 23rd International Group of the Psychology of Mathematics Education Conference,* Haifa, Israel, **4**, pp. 89-96

Radford, L. (2000a) 'Signs and meanings in students' emergent algebraic thinking: a semiotic analysis', *Educational Studies in Mathematics* **42**(3), 237-268.

Radford, L. (2000b) 'Students' processes of symbolizing in algebra', in *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education,* Hiroshima, Japan, Hiroshima University, **4**, pp. 81-88

Radford, L. (2002) 'On heroes and the collapse of narratives: a contribution to the study of symbolic thinking', *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education,* Cockburn, A. and Nardi, E. (eds), Norwich, Norfolk, University of East Anglia, **4**, pp. 81-88.

Radford, L. (2003) 'Gestures, speech and the sprouting of signs', *Mathematical Thinking and Learning* **5**(1), 37-70

Rowland, T. (2000) *The Pragmatics of Mathematics Education: Vagueness in Mathematics Discourse,* London, Falmer Press.

Russell, B. (1976) *An Inquiry into Meaning and Truth,* London, George Allen and Unwin Ltd.

Sfard, A. (1994) 'Reification as the birth of metaphor', *For the Learning of Mathematics* **14**(1), 44-55

Steinbring, H. (2001) 'Mathematical symbols: means of describing or of constructing reality?', Paper presented at the Third Nordic Conference on Mathematics Education, Norma 01, "Conceptions of Mathematics", Kristianstad, Sweden, June 8-12

Talmy, L. (1983) 'How language structures space', in Pick, H. and Acredolo, L. (eds), *Spatial Orientation: Theory, Research and Applications,* New York, NY, Plenum Press, pp. 225-282

Van Gogh, V. (1996) *The Letters of Vincent van Gogh* (sel. and ed. de Leeuw, R.; trans. Pomerans, A.), New York, NY, Penguin Classics.

Weiner, J. (1999) *Frege*, Oxford, Oxford University Press