Problems with the Language of Limits

JOHN MONAGHAN

The phenomena reported on here were part of a wider study [Monaghan, 1986] which investigated mathematically able adolescents' conceptions of the basic notions behind the calculus. This paper reports only on those aspects of the study that examined students' understanding of the language used by teachers to communicate calculus concepts. Mathematicians use many everyday words and phrases with specialist meanings. This can confuse students. This article deals solely with ambiguities inherent in the four phrases *tends to, approaches, converges* and *limit*. These phrases are commonly used in calculus courses where their mathematical meanings are equivalent. Students can, and do, moreover, construct images of limiting processes that are at variance with mathematicians' concepts. The interference of everyday meanings in mathematical situations occurs in both arithmetic and geometric contexts.

Related issues are addressed in the papers of Cornu [1980], Davis and Vinner [1986] and Tall and Vinner [1981].

Methodology

The study was conducted in England between 1982 and 1984 with students from state and independent schools. They were mathematically competent in that they had passed the English Ordinary-level examinations in mathematics at 16 years of age and were in their first year of Advanced-level (A-level) studies. Three subjects are usually studied for two years at A-level courses. The sample included both those studying mathematics and those who had dropped mathematics.

The first formal questionnaire was administered to 27 A-level mathematics students and 27 A-level non-mathematics students all from the same school. This questionnaire had 37 questions (many with several parts) and was administered in September and the following May. The A-level mathematicians had received instruction in most of the techniques of the calculus by May. Interviews were conducted in the month following each administration. A second questionnaire, an amended version of the first, was administered to 190 students (114 studying A-level mathematics) from six schools in the following May/June period. These schools followed the same O-level and A-level course as the school in the first sample.

The following questions common to both questionnaires are used for reporting on language problems in this paper.

1) Consider the sequence 0.9, 0.99, 0.999, 0.9999, ...
Which of the following sentences is true of this sequence?

i) It tends to $0.\dot{9}$	ii) It tends to 1
iii) It approaches $0.\dot{9}$	iv) It approaches 1
v) It converges to $0.\dot{9}$	vi) It converges to 1
vii) Its limit is $0.\dot{9}$	viii) Its limit is 1

The next four questions refer to the six curves in Figure 1. Answer each question six times, once for each curve.

2) Can we say "the curve TENDS TO 0" as x gets larger and larger?
3) Can we say "the curve has 0 as a LIMIT" as x gets larger and larger?
4) Can we say "the curve CONVERGES to 0" as x gets larger and larger?
5) Can we say "the curve APPROACHES 0" as x gets larger and larger?

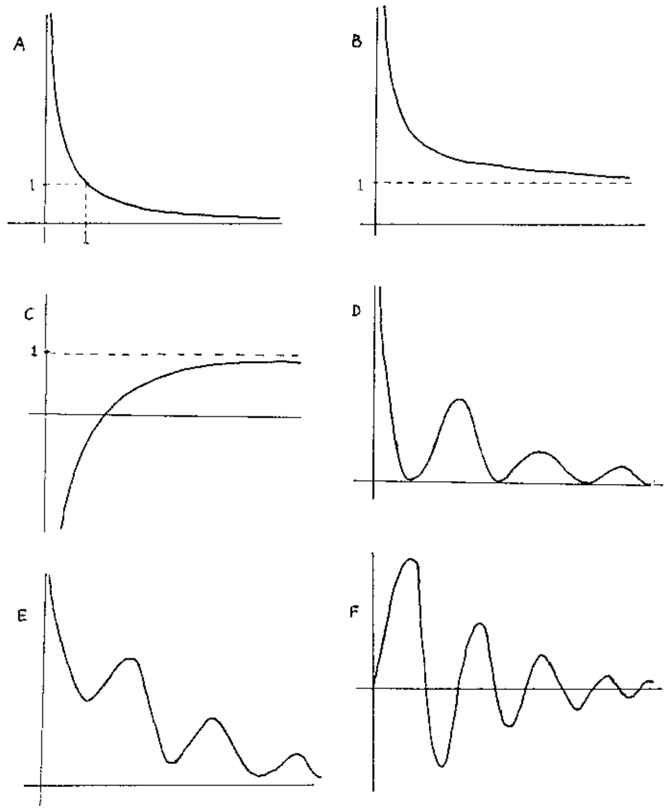

Figure 1

For the Learning of Mathematics 11, 3 (November 1991)
FLM Publishing Association, White Rock, British Columbia, Canada

Students were presented with a five point scale— yes / think so / unsure / think not / no— for each question and were encouraged to use *unsure* only in cases of extreme doubt. In addition to these questions several open questions were presented.

In the first questionnaire the students were asked to write four sentences, each using one of the four phrases *tends to, approaches, converges* and *limit*. The context could be mathematical or non-mathematical. (The idea belongs to Cornu [1980].) In the May administration students were asked to write only one sentence using the word *limit* but not using *speed limit*.

At the end of the second questionnaire students were asked if they found any of the four phrases confusing and if so to say which ones and why.

Results

Due to the descriptive nature of this article and to avoid pages of tables, a resumé is given of responses to questions 1 to 5. Percentages are given for some responses but then only for the larger sample from questionnaire 2. The results for the smaller sample were very similar. A more detailed picture is available in [Monaghan, 1986]. The percentages represent collapsed *yes* and *think so* responses. No response or *unsure* accounted for about 10%, on average, of the group doing A-level mathematics and 15% of those not studying mathematics. When two figures are given, the first represents those doing A-level mathematics.

Question 1 The sequence 0.9, 0.99, 0.999,... did (74%, 66%) tend to $0.\dot{9}$ and tend to 1. It approached $0.\dot{9}$ and 1. Both groups were equally divided on whether the sequence converged to $0.\dot{9}$ but did not think (22%) it converged to 1. The limit was seen (60%) as $0.\dot{9}$ but was not seen (36%, 24%) as 1.

Questions 2 to 5 Curve A tended to (92%, 70%) and approached 0, but a roughly equal division resulted with the phrases *limit* and *converges*. None of the phrases applied to the curve B—*approaches* was (32%, 47%) and *tends to* was (6%, 37%) but the other phrases received less than 18%. Responses for curve C were very similar to responses to curve B. Curve D had 0 as a limit (75%, 63%), approached 0 but the students were equally divided in their responses with the phrase *tend to* or *converges.* Curve E approached 0 (73%, 68%) and tended to 0 but did not have 0 as a limit (39%, 25%) nor did it converge to 0. Curve F tended to 0 (68%, 63%), converged to 0 and was generally thought to approach 0 but the students were not sure that 0 was the limit.

The open questions: The first questionnaire

Each phrase is examined in turn and the interpretations described starting with the most common. Frequency counts seem unnecessary for such a descriptive task and relative weightings are merely stated. The examples come from the students.

LIMIT Every student used *speed limit* in the first administration so they were asked to repeat the limit sentence with another context in the second administration. In most cases a speed limit is a conventional law—the legal limit it is forbidden to exceed. Most people, however, do exceed it sometimes. In a graph drawing question (on drawing $y = 1/x$ with positive x increasing, say) this image of a limit would suggest that one can get to 0, if this is desired, but the rule is DON'T. In contrast the mathematician regards the rule as a necessary feature of the curve. $1/x$ is $1/x$ — you can't just suddenly jump up or down!

After speed limit came physical limits and mental limits. Physical limits are boundaries that are technically highly unlikely to be passed, such as the limit of the amount of alcohol one can consume or the height one can jump. They need not be concerned with humans. Planes have a limit (ceiling), radar has its limit of detection and there are physical limits to cars' speeds. These limits imply a boundary. They may represent the most extreme realization or the point just past the possible. In mathematical situations this can be thought of in two ways, with, say, the sequence 0.9, 0.99,... The limit may be the boundary, $0.\dot{9}$, or just past the boundary, 1 (from the students' view, of course).

Mental limits have no mathematical analogue. They are the limits of people's patience, nerves or intellectual abilities. They can also be what people drive themselves to, their breaking points. Fewer instances occurred of: conventional limits (social customs); restrictions (*you must limit your salt intake*); and expressions (*you are the limit*).

APPROACHES 8/9 of the group doing A-level mathematics and 2/3 of the other group used *approaches* in the sense of "drawing nearer": *the train approaches the station; the car approaches the traffic lights; winter approaches; the dog approaches the cat.* In the first three examples the object being approached will, eventually, be reached though it has not been at the time the sentence is uttered. Mathematicians do not view a convergent series in a temporal light but students may: 0.9 + 0.09 +... approaches 1, but it will never get there. The "dog approaches the cat" example has a connotation implying that it may not reach it. If a vicious dog is to be moved away from children then one will, presumably, approach it, but one would have to be desperate to touch it. A safe distance would be a limit. In this sense $y = 1 + 1/x$ approaches 0 as positive x increases.

The remainder used three other meanings of approaches. A method of doing or of thinking of something: *different approaches to mathematics; several approaches to the question of abortion.* A route or way into something (note the indefinite article): *there are many approaches to London; several approaches to my house.* Resembling: *Racism approaches Fascism; his behaviour approaches the ridiculous.*

CONVERGES *Converges* has fewer everyday meanings and was mainly used in three common examples: *the light rays converge; the roads converge; the lines converge.* In each instance two continuous objects come nearer and in most cases, touch. If these are students' dominant or only concept image of *converges,* then it is difficult to see how they will make sense of a sequence converging to a number. Graphs will make more sense but if $y = 1/x$ converges to 0, then we must think of 0 as the line $y = 0$ and not a number.

The remainder used examples where individual (discrete) objects come into contact or close proximity: *the cars converged; the footballers converged on the ball; the crowd converged on the politician*. Interesting isolated examples were: *my thoughts converge to Christian thought; a straight line converges the farther away you look; two lines converge to a point; two objects which converge eventually meet*.

TENDS TO With eight exceptions all examples were of personal inclination (*she tends to drink a lot; he tends to wear jeans*) or of general trends (*holiday weather tends to be bad; eggs tend to break when dropped*). These two senses have considerable overlap (*chemistry tends to be hard; I tend to eat breakfast at 8:00*). As a general trend *tends to* may be used in a mathematics class but would be more suited to comparing bar charts (*the frequencies tend to be low in the early graphs*) than discussing the behavior of algebraic sequences or curves.

Apart from caring (*the nurse tends to the patient*) the remainder used mathematical examples: *1/9 tends to 0.1; 1, repeatedly divided by 10, tends to 0; a sequence may, eventually, tend to a limit.*

At this point it is useful to compare these results with those of Cornu [1980]. He examined the hypothesis that the initial stages of acquiring the notion of a limit is "contaminated" by earlier concepts. He shows that these ideas remain for some time. He examined student sentences using *tend vers* and *limite*. With *tend vers* he obtained, amongst others:

> Le présent tend vers le passé. Ses rêves tendent vers la mort. Le temps tend vers l'orage. La mère tend vers l'enfant. Mon opinion politique tend vers celle de mon père. Le cours de Maths tend vers l'indifférence totale.

Cornu grouped these phases into three categories (apart from mathematical examples): i) Those indicating a tendency to grow closer. ii) Those approaching a goal or end. iii) Those indicating a 'tension'. He noted, moreover, that the expression *tend vers* is not part of the usual vocabulary of students and is somewhat forced.

In English *tends to* is not unusual when it is used as a trend or inclination but does seem somewhat forced in a mathematical context. In fact, for *tends to,* asking students to use the phrase independent of context may not be very useful. In the interviews (which had a mathematical context) a meaning similar to Cornu's "growing closer" or "approaching a goal" were common. Typical responses to "What does *tends to* mean to you?" were:

- Approaches is similar to tends to but unlike a limit it just has to go nearer and nearer to it.
- It tends to 0. It's getting nearer all the time but it's never actually going to get there.
- I think approaches and tends to mean the same thing.
- Tends to, to me, means it doesn't actually reach it but it gets very close.

Limite, in French, is used more naturally by students:

> La limite est un point à ne pas dépasser.
> La limite est un point très dur à atteindre.
> La limite est l'état le plus proche de l'impossible.

It is, to French students, something static or fixed or something that it is forbidden or impossible to cross.

This it appears that the connotations of *limite* in French are similar to those of *limit* in English, whereas *tend vers* and *tends to* have some differing connotations. This can partially be explained by the difference between the noun and the verb. Moreover, to my knowledge, there are more variations with the verb, e.g. "tend" can mean "take care of" in English, whereas this is "soigner" in French. Both languages, however, generate ideas that are at odds with mathematical ideas.

The second questionnaire

The responses were classified by first reading them all and then putting them into piles according to the dominant response. This is a rather rough and ready response analysis technique but it was useful in that definite types of responses were easily isolated. There was much overlap between the types of response identified, but little difficulty in sorting questionnaire papers into appropriate piles was experienced.

The percentages are rough guides (rounded to the nearest 5%). The types of response that emerged are:

No response, 20%—blank spaces or just the word *No* in the space for comments on the phrases.

All the same, 20%—some put *all are confusing,* others put *I can't see the difference* (some qualifying this with *but I suppose there must be*). A typical comment was:

> All of them. Approaches—does it actually reach 0? Has a limit? What sort of limit?
> Tends to? Absolutely no idea what the difference is between tends to, approaches and converges.

Converges and *approaches* seen as the same or equally confusing, 15%.

Converges, approaches, and *tends to* are all the same (or, in fewer cases, are all confusing), 10%—this means *limit* is seen as somehow different.

Converges seen as confusing, 10%.

Tends to and *approaches* seen as the same (sometimes qualified with *both are vague*), 10%.

Others, 15%—e.g. isolating *converges* and *tends to* as phrases they did not understand. Most combinations not mentioned above were included here.

Several subjects gave extended responses. We quote one in full not for its typicality but for the range of impressions. The student was doing A-level mathematics.

> Yes. The similarity in the meanings of the phrases is itself confusing. Further, the term converges to can mean many different things and depending upon which definition or meaning is put into practice, the answer to any question can differ. The actual definition of to converge was clear enough but in here, it is more difficult to decide what the answer should be. The term tends to is slightly confusing and apparently exactly similar to has as a limit. The phrase approaches is the source of confusion, as to whether a number

which the function approaches more closely as x increases but which it can never reach are in a supposed infinite limit can be supposed to be approached by the function. It seems that these terms in normal everyday mathematics have little notion of their significance or meaning in fact.

Discussion

From the response to the open questions it is clear that the four phrases generate everyday connotations that are at odds with the mathematical meanings. As in Cornu's study it was observed that these everyday meanings persisted well into a first course in elementary calculus. This phenomenon was also observed by Davis and Vinner [1986] who found that pre-university students who, after two years of study, performed test questions correctly retained "naive misconceptions".

To emphasise this point the protocols used to support observations below all come from students studying A-level mathematics and were made at the end of the first year (there was, in fact, very little difference, with respect to language, between either the mathematicians and non-mathematicians, or between the beginning and the end of the year).

The first thing to notice is that the three verbs *tends to, approaches* and *converges* have a different sense to the noun *limit*. This is not surprising in that verbs are, simplistically, "action words". The action in this mathematical setting is "getting to a limit"—a dynamic interpretation is set up. This dynamic interpretation exists with sequences and functions. Of the three verbs, however, *converges* stands out as different because the images it evokes are not always applicable to the mathematical settings. These claims will not be examined in more depth.

Tends to and *approaches* were very often seen as the same. They represent a movement towards a terminus without ever getting there. Despite the different uses of the phrases in the open questions, the questionnaire results show only minor differences in the responses with these two phrases. This was supported by protocols:

> (In the sequences questions)
> It never actually gets there, which is what *tends to* means to me. It means it approaches it or comes close to it but it won't actually finally get there.

> (In the functions questions (curve A))
> Interviewer Why does the curve tend to 0 but not have a limit 0?
> Student Because it doesn't actually reach 0.... (it approaches 0) because it gets closer as it goes along.

Protocols revealed that *converges* was sometimes grouped with the other verbs as a word indicating a "growing closer". Students very often, however, stated that they were unsure what *converges* meant in mathematical situations. The questionnaire results support the view that *converges* is generally seen as different to the other two verbs in the context of sequences and functions. The responses to the open questions, moreover, show that the dominant everyday meaning is of two continuous objects coming together and touching. Protocols reveal some of the confusions:

> I don't really see how numbers can converge. Converge means light from a thing coming in It's two separate parts. You'd have to have two sequences coming in on each other. I don't think you can have one sequence converging.

Although *converges* may appear more suitable for functions, there are still problems arising from everyday meanings:

> When I think of converge it seems to me that it (curve A) is going to sort of touch 0. Two lines are going to touch each other.

> (Curve D doesn't converge to 0 because) I was thinking of the word converges as coming from two sides, whereas that's only coming from the top.

The term *limit* was generally seen as more specific than the other phrases. This accords with Cornu's conclusion that students view *limite* as more precise than *tends ver* which is vague. I believe the vagueness of the verbs (especially *tends to* and *approaches*) results from the verbs representing the movement towards the terminus whereas the limit is the terminus. Despite being more specific, however, the limit was dually seen as the final point ($0.\dot{9}$) and as an unreachable boundary point [1].

> Its limit is its final point that it will get to. I think the limit is $0.\dot{9}$ and there again there the limit is 1 but it won't actually get to one, so you can't have 1 as its limit.

This dualism was noted above in the discussion in the first set of open questions. The results of question 1 imply that the realized boundary ($0.\dot{9}$) was more often seen as the limit, but by no means always:

> I didn't really see the limit as what it is. I saw the limit as what it's very close to but it isn't actually 1. So you have got $0.\dot{9}$ eventually, but you haven't got 1. 1 is its limit it can't reach.

Whichever view students take, the everyday meaning of a limit as a boundary is clearly present. The confusion of a limit with a boundary is discussed by Davis and Vinner [1986] at some length.

Conclusion

To the mathematician the phrases *tends to, approaches, converges* and *limit* are interchangeable. To a large extent this is seen by students but there are many disturbances in the pattern.

Approaches appears to present the least difficulty to students because it is a vague term. *Tends to* is often seen as similar in meaning to *approaches* in mathematical contexts although its everyday use does not suggest limit situations. Both phrases are given a dynamic interpretation. *Converges* is confusing in that its everyday meaning is strongly associated with lines converging. Many students cannot see how a sequence of numbers can converge. *Limit* is often viewed as a boundary point. It is either like the terms of the sequence, such as $0.\dot{9}$, or the closest term beyond the boundary, such as 1. It must be stressed that students experience very real difficulties in the mystery of this jump to the infinite.

All of these statements are difficult to verify in a strong sense because the interpretations vary so much. These are, I hold, general trends in an area rich in multiple interpretations due to context and the mood of the student. Tall and Vinner [1981] state:

> We shall call the portion of the concept image which is activated at a particular time the evoked concept image. At differing times, seemingly different conflicting images may be evoked.

Thus it seems here that what is evoked in one context may not be evoked in another.

Language is not the only source of student difficulties in this area.

Other important factors are: the location of our reasoning schemes (used to interpret these infinite processes) in finite experience; and the fact that instructional paradigms intended to assist students' early understanding can create conceptual obstacles later, e.g. monotone sequences. Language is, however, a crucial tool everyone uses to build their mathematical frameworks and the false cues it can generate must be analysed.

What are the instructional implications of all of this? I have not investigated how these difficulties can be overcome, but one avenue is obvious. Students should be led to explore and discuss their own conceptions and to realise how everyday meanings of mathematical phrases can direct them into fallacious interpretation.

References

[1] Cornu, B. [1980] *Interférence des modèles spontanés dans l'apprentissage de la notion de limite*. Seminaire de Recherche Pédagogique (no. 8), Institut National Polytechnique de Grenoble

[2] Davis, R. and Vinner, S. [1986] The Notion of Limit: Some Seemingly Unavoidable Misconception Stages, *Journal of Mathematical Behavior*, 5(1)

[3] Monaghan, J. [1986] Adolescents' Understanding of Limits and Infinity. Unpublished Ph D thesis, University of Warwick

[4] Tall, D., and Vinner, S. [1981] Concept Image and Concept Definition in Mathematics with particular Reference to Limits and Continuity, *Educational Studies in Mathematics*, 12

A woman teacher one of my students receives a well-deserved Distinction for her Master's degree She received more or less straight As for all her work but she still cannot believe that the Distinction belongs to her: it is as though the person with her name exists somewhere else outside her body; this powerful person whom she cannot recognize as herself Instead she feels that she is hopeless consistently panics about her performance and appears to have little confidence in herself She can however express her views clearly and forcefully and the external examiners in her *viva* thanked her for the tutorial! [...] How is it that for many women the powerful part of themselves has been so split off as to feel that it belongs to someone else? Here is no simple passive wimp femininity but a power which is desired, striven after yet almost too dangerous to be acknowledged as belonging to the woman herself

Valerie Walkerdine
