Transference of Objects

JAN VAN DEN BRINK

It is striking how children, when doing arithmetic, manage to achieve the transference from numbers and operations in contextual frameworks to "bare" numbers and operations and then back again. This transference is also found between one contextual situation and another. Transference is the major activity in learning to do arithmetic It can also be called *carry-over*. (We do not mean here, by the way, the transference of knowledge from the teacher to the student) The significance of *transference* for arithmetic and mathematics education can be seen from the fact that it forms the core of *mathematisation*: through application of already acquired knowledge and skills, hitherto unknown regularities, connections and structures can be located [Treffers, 1986, p. 55]. The "acquired knowledge and skills," present in the characterization of mathematisation, are transferred from the contexts in which they are learned to the new context.

The transference is not limited to numbers and operations. Symbols, words, meanings, activities and structures, too, in use within a familiar context, are transferred to another, hitherto unknown, context Sometimes they will fit into the new context; at other times it turns out that the student has erroneously seen a similarity between the two contexts.

Moments of transference are therefore accompanied by conflicts. At times, a student will consciously try out a transference, at other times it will appear through a sudden insight In any case, it nearly always constitutes an insecure situation.

We consider the role of the particular arithmetic language used (arrow-language, for instance) to be essential The form of the language, both written and oral, must be able to be adapted to another context by the addition or removal of certain characteristics.

Both arithmetic language and conflicts are instrumental in learning arithmetic via transference

Two types of transference

Objects are transferred from one "context" to another.

a The other context may be part of the original one By posing restrictions on the original, a new — usually more symbolic — context is created. Take, for example, the bus-model within the bus-context.

b. The other context may also be broader, enclosing the original context Take, for example, bus-problems within the context of the student-made arithmetic textbook [Van den Brink, 1987]

For the learning process it is essential that the two contexts (bus-context and arithmetic-context, for example) together form a cohesive entity and mutually restrict or expand one another [see Freudenthal, 1987].

Transference can be characterized by terms such as *application, adaptation, embedding* and is supported by a language which, *qua* form, can be adapted to various contexts Moreover, it should be noted that "acquired knowledge and skills," which are part of the definition of mathematisation, are not the same for each child. A considerable variety of ideas are present in early education which produce a great diversity of transferences The way in which one regards this diversity characterizes the type of education Mechanistic arithmetic education experiences this diversity as a hindrance; realistic education, on the contrary, makes use of the differences in constructive play-acting and confrontational conflicts. The examples of children's "incorrect" transference can, in fact, be used functionally in conflict situations.

Interviews and observations

First of all, we present a number of examples of transference by children of assorted ages and from various school-types. Each example will be accompanied by commentary

ERIC

Eric tried to find 7×4 on the number-line. At 5×4, however, he got stuck and wasn't confident enough to find 7×4 by leaping in this way. After a short while Eric suddenly said:

"Wait a minute, 7×4, that's 7 *groups* of 4 So then you have to . ." He then counted leaps of 4 on the number-line up to 28 " . 7×4 is 28."

Eric emphasized the word "groups," with which he was familiar. Apparently he connected repeated addition exclusively with "making groups."

The transference from the group-structure to the leap-structure on the number-line convinced him that he would find the solution. The transference is often carried by a spoken word (in this case *groups* instead of *leaps*).

DEBBIE

While using a computer program illustrating all sorts of multiplication models, Debbie let the sum 8×8 pass through the various models: the group-model, the dash-model, the grid-model and the number-line. She referred to all the models, however, as *groups of 8*.

Debbie simple used the term "groups" in all situations and was supported in this by the computer program in which, after all, only those models are included that are mutually exchangeable. It was not necessary for her, as was the case with Eric, to independently see the correspondence between *groups* and *leaps*.

GERLINDA and KATTY

The girls were playing the game "closer and closer" on the calculator Katty was adding, Gerlinda subtracting. They

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had arrived at the positions of, respectively, 12 and 13 and had to approach each other even closer without passing Katty suddenly had a bright idea:

"You can also do 12 and a half. But that's not on the calculator."

Gerlinda said: "But you can do one point two," and pressed 1.2 as the notation for ½. But then she saw the notation 1.2 and, remembering decimal numbers from common addition, said: "Oh, no, not one point two but one point five. That's one and a half, isn't it?" she asked me.

A little later, having reached the positions of 12.312 and 12.311, Katty performed 12.311 + 0.001 = 12.312. Gerlinda exclaimed: "Now you're on mine, you can't do that. You have to work with *thousands*, with four numbers behind the decimal point."

Katty corrected herself: "12.311 + 0.0001 = 12.3111." She read the figure aloud: "Twelve point three thousand a hundred and eleven."

So she worked with thousands and Gerlinda was right: 4 figures behind the decimal point

Gerlinda had reached 12.312, Katty 12.3111. It was Gerlinda's turn.

"You have to work with quarters," she said. She again meant "with 4 figures behind the decimal point," only now she said it in fraction-language.

From this example it is clear that words from all sorts of familiar contexts (natural numbers, fractions), based on one situation or another, are transferred in order to describe the decimal numbers.

PAUL AND JEROEN

Paul and Jeroen were solving 365 × 24 on the calculator through repeated addition: 24 + 24 = + 24 = etc. (The problem concerned how many hours there are in a year.) They wrote down the number of times that they had pressed 24. They had reached 120 (5x24) and continued unperturbed. This would take a while.

I intervened: "You have 365 days; and each day has 24 hours."

(Just a comment in passing to keep the objective in sight.)

"Hey, wait a minute," Jeroen exclaimed. "24 times that ..." He pointed to the ×-key and to 365.

Jeroen had discovered the ×-key on the calculator for solving this problem and for the repeated addition of 24

The discovery that a *multiplication* on the calculator need not necessarily be performed as repeated addition, but that the machine has a times-key, was introduced by the *oral* expression "times" induced by the × sign on the calculator.

Alongside words such as "groups," "quarters" and "times," written symbols, too (×, +), can cause transference of operations from familiar contexts to new ones, such as the calculator

CARO

First I gave Caro the following inkblot sum:

■ – 2 = 11

"That's hard," Caro sighed. "... nine!"

She had done 11 – 2 = instead of ■ – 2 = 11

"You have to 'take away,' " I said.

"14, no 13."

"How do you know that?"

"You've got 11. First add two: 11, 12, 13. And then take it away again."

Next I gave Caro the sum:

3 + ■ + 3 = 8

"That's 11," said Caro. "3 + 3 is 8 and then you add 3 and take it away again. Then it'll be 11."

Caro was copying the procedure from the previous sum.

"No," I said, "Try it again."

"1"

"3 + 1 + 3 is 7," I told her.

"Then it's 2."

In order to make a transference possible, the children often assume at first that a previous procedure is universal. The idea of neighbouring sums rests on this concept. But sometimes it goes wrong.

MARTIJN

* – 2 = 1
4 + 1 = *
3 + * = 5

In answer to the normal sum 4 + 1 = which stands in the middle of a number of fill-in-the-blank sums, Martijn said:

"3, because you have 1 and you have to get 4." (1 + * = 4)

It is startling how strongly children tend toward universal procedures: an already familiar way of calculation is considered applicable to *all* kinds of problems. This sometimes holds true, too, for model words, in the manner in which Debbie, for instance used the word "groups," and for models in themselves, such as, for example, the clock-model.

HEDWIG

Hedwig was given some apple juice

"It was only half a glass," she said, having finished it.

She was given a little more and drank that up, too.

"How much apple juice did you drink now?," asked the interviewer

Hedwig measured with finger and thumb on the glass and answered, "A quarter of a glass."

"How much juice did you drink altogether?," asked the interviewer

"Three-quarters of a glass."

"How do you know that?"

"It's just like the clock," Hedwig replied. "You have a quarter of an hour ... um .. there's already a quarter of an hour and then another half hour, that's two more quarters. So together it's three quarters of an hour." [Streefland, 1978]

Familiar models (clock, bus, etc.) are applied in context problems. It can also take place the other way around.

DAVE

Dave gave meaning to the reverse equation 3 = 4 – * by reading it as part of a bus ride

"5," said Dave. "1 gets off, then there are 4." He pointed to the 4 in the sum and raised four fingers. "And then 1 more gets off ... 3," he said, showing 3 fingers. "So it's 1."

Dave was working at school during this period with bus-sums. He used the term "get off" for the minus-sign. But this context can also be seen in the fact that Dave embedded the sum in a bus-ride. Later in the school-year, too, this transference could still be observed with many children.

It is fascinating to look at the examples in which children discover transference procedures in order to apply them themselves.

KIKKIE
The computer asked Kikkie to think up a sum. She typed in 16 + 16 =

The computer gave the answer and then in turn asked her the answer to 16 + 18 =

Kikkie must have discovered the procedure the computer was following because when it was her turn she typed in 13 + 12 = , commenting, "I've moved it up just one." She evidently meant that, instead of 12 + 12 = she was asking 13 + 12 = And when, later on, the computer asked her the answer to 16 + 9 = she said, conscious of the relation between neighbour sums, "also 25," meaning, just like the previous sum 13 + 12 = .

Suddenly it dawned on her: "I get it now; you have 13 and then you add 3 to it from the 12, and so you're left with 9." That's how the computer does it.

The transference procedures for making one sum out of another are discovered here in the context of question and answer between child and computer.

Objects of transference

If we look at the examples given in section 2, the objects transferred are the following:

— spoken words (Eric's "groups")
— written symbols (Jeroen's times-key)
— visual models and diagrams (Hedwig's clock and Dave's bus-model)
— arithmetic procedures (Caro and Martijn)
— structures (Kikkie's neighbour-sums)

The objects are evidently not limited to signs, words and meanings. Procedures and structures, too, are transferred. Moreover, there is no sharp dividing-line between objects of different natures. Carrying a word from one context to another may also involve change or expansion of the meaning of that word. (See examples: Jeroen's times-key, Eric's "groups")

This is what makes transference so interesting for mathematics and mathematics education.

Expanding insight
In mathematics, in particular, the possibility is kept open that what is now known about a concept may still be incomplete. A concept can always be expanded by all sorts of related meanings. This standpoint gives significance to further research into how concepts occur, both for students and for educational researchers. Transference from one context to another is one way to achieve greater insight. Addition, for instance, comes in all kinds of shapes and forms that have ostensibly nothing to do with one another. For example: *dividing* (5 passengers in a double-decker bus), *increasing* (the number of passengers in a double-decker bus), *increasing* (the number of passengers in a bus at a bus stop), *combining* (two disjoint groups: girls and boys). There are countless other contextually embedded versions of addition and subtraction: *marbles, number-line,* etc.

The same holds for the is-sign: '=' has a variety of meanings:

"... is equal to ..." (5 + 2 =)
"... is equivalent to ..." (5 + 2 = 3 + 4)
"... becomes ..." (5 +2 = 7)
"... can be written as ..." (5 = 2 + 3)

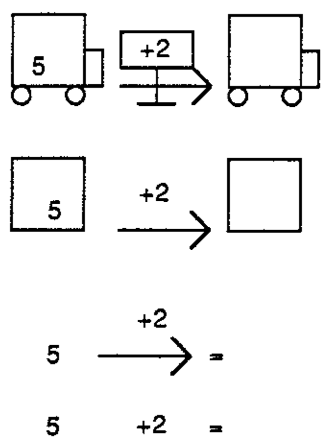
An important question is: in which ways does the transference take place. Linguistic imagery of all kinds can be very valuable: metaphor, metonymy, substitution, *pars pro toto*.

Pimm [1987] has the following to say on transference and metaphor: "The carry-over of terms to the context is particularly interesting as it is precisely how extension of the meaning of terminology by means of a metaphor (which is so widespread in mathematics) actually happens."

Transference gives a wider meaning to words and symbols, especially in mathematics.

In mathematics education, however, metaphors may in fact prove to be obstacles which hamper insight. The unconscious use of metaphors causes misunderstandings between students and teachers.

Treffers calls attention to the notes designed for the Wiskobas project and remarks on the importance of *metonymy* in the languages used in realistic arithmetic education. Metonymy is a form of daily language use in which calling something by another name is not based on similarity, but rather on some kind of mutual relation. Metonymy is found, for example, when only the arrows, a small aspect of the whole (bus) context, remain in use (*pars pro toto*). Or, when an arrow is added to the sums written in the is-language.



Adapting the notation by erasing or adding contextual frills can be of great help in stimulating the transference to other contexts. The transition, too, from the real world to the symbolic world within one sole context can be brought about through arrow-language.

Substitution
Young children also use substitution in transference. They not only transport words, symbols and their meanings

from the old contexts to the new ones, but also the procedures, structures and ways of calculation [Pimm, 1987; Lehrer, 1974]

These phenomena correspond to our own research experiences, which we have described as context substitution: the entire context is replaced Calculators offer a striking example of how children use their own machine as a familiar contextual framework when investigating another type of calculator

Substitution is not, however, abstraction. In context substitution the contexts themselves are still present, as opposed to abstraction which becomes more and more a bare framework This has to do, for instance, with the transference from the bus-context to the bus-model in order to draw the children's attention to that model as being one among many

Incorrect transference

The essence of transference is that children see connections They come to understand the unknown by means of its similarities with what is already familiar to them Which is why it is a regular part of children's learning arithmetic But the shift does not, unfortunately, always take place flawlessly An apparent similarity may prove deceiving and the transferred objects will not then fit the new context The objects are, after all, connected in the first place to the original context and not to another, arbitrary one

There are a variety of causes for incorrect transference

a *literal and figurative usage*
Figurative and literal usage for instance may create differences in the significances ascribed by students and by teachers

b *homonyms*
Language usage may cause difficulties, such as where the symbols or words found in two contexts are identical but not their meanings

Examples

- The word *more*. *More* may signify *more/less*, or it may mean *in addition* as in *5 more*.
- The is-sign and its numerous significations.
- The "fill-in-the-blank disaster," where children fail due to a misuse of key-words such as *plus* and *minus*: for instance, in * − 2 = 5 it turns out that you have to add instead of subtract.
- The expression *a third,* which may mean either *1/3* or *3rd*
- The different functions of natural numbers: amount, ordinal, measure, etc

SYLVIA
Sylvia was trying to figure out the sum * + 1 = 4

"Zero," said Sylvia, "Cause then comes 1" She had the number-line in mind: 0, 1, 2, 3, 4, . . .

"But that doesn't make 4," I said (0 + 1 = 4 ?)

"No," answered Sylvia, "it's 3." (3 comes before 4 on the number-line)

The sum was solved by accident The problem is that one can't tell from the (cardinal) number in which context or in which model it should be regarded

c. *Isolation*
No transference takes place because one of the contexts is too unfamiliar or may even cause anxiety

DAVE
I presented Dave with the inkblot sum

■ + ■ = 5

and he answered with 6 + 8

"How can you tell?" I asked

"I just think so," said Dave

I reminded him of the double-decker bus game that five of them had played in class.

"Yeah, I know," said Dave "But *that's* arithmetic, isn't it?" he continued, pointing to the inkblot sum

"Yes," I said

"I can't do that, 'cause I can't do arithmetic."

"But remember about the double-decker? With that diagram?"

"5 downstairs, none on top," said Dave "2 with 3 on top; 1 downstairs and 4 on top."

"You can do arithmetic just fine!" I exclaimed

"Yeah, but I thought that it was: 1 + 5 is more than 4 And I can't do that"

The double-decker game was perfectly clear to Dave Nonetheless, he didn't use this context as a model for solving the inkblot sum He had decided in advance that he wasn't able to solve these sums with the equal sign and he lacked the necessary self-confidence for applying the double-decker model [see also Cobb, 1985]

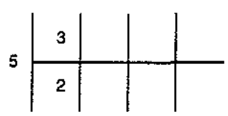
ANNET
To Annet I also gave the inkblot sum

■ + ■ = 5

and reminded her of the double-decker game

"But that was upstairs and downstairs," she replied, "And this isn't"

She refused to accept the similarity The notation ■ + ■ = 5 did not indicate the double-decker, as far as she was concerned Apparently, a certain graphic similarity is required between the notation and the context such as is the case, for instance, with the double-decker notation:

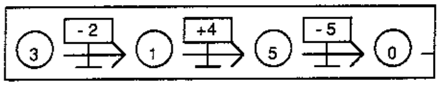


This notation is referential

So one does not always succeed in using a familiar context as a model in another context A notation that does not refer by its form to the situation may be the cause of transference failure Notational forms that emphasize a gradual transition from one context to another are didactically important (arrow-language, for example)

d *Universal model*
On the other hand, children may treat a familiar context as a universal model, as a remedy for all ailments Debbie, for instance, simplified each model to the familiar groups-model with no trouble at all But it didn't always work

DAVE
I gave Dave the following chain-sum



"What's happening here?" I asked.

Dave described the sum in terms of a bus-ride: "Three in the bus, two get out . ."

"And if this was a game of skittles," I asked, "what then?"

"Three skittles, two get out . ."

"Two *get out?*" I interrupted. "How's that possible?"

"Yeah," Dave replied, "at the bus stop two get out."

"Two *skittles* get out," I protested, "can't you see it as a game of skittles?"

"No," said Dave, "I can only do it with a bus stop "

Dave simply couldn't manage to interpret the chain-sum either as a game of skittles or (later) as marbles. He quite obligingly illustrated the chain-sum with a skittles-context, but continued to let the skittles or marbles behave as passengers The bus-context served as a universal model, the skittles functioning merely as illustration

FREDERIK
Frederik was atempting to solve the following equation:

$$5 + 3 = 10 - *$$

He chose 10 – * as "substitute sum" and solved it by referring to the arithmetic kite:

"If two blocks of 10 lie one on top of another, then the sum is zero," said Frederik, writing zero in the blank.

The universal model of sticks in the arithmetic kite hampered Frederik's comprehension of what the sum actually meant

In summary, we can state that transference does not take place, or incorrectly takes place:

— when using *homonyms* without emphasizing the different meanings of these words, examples of procedures; (*contextual* or, better yet, *referential* arrow-language can be of assistance here.)
— when the arithmetic context remains too *isolated* from an illustrative or referential notation
— when a *universal model* dominates all contextual situations.

Transferences in realistic arithmetic education

Children are quite enterprising and skillful in transferring objects, sometimes correctly, sometimes incorrectly In realistic arithmetic education, incorrect transferences are not avoided. The connection between *transference* and *realizing (becoming aware)* comes about through similarities being seen between contexts and by calling attention to the illusory resemblances between them. This corresponds to the components of *realizing*: construction and confrontation.

Educational situations in which transference takes place show the following characteristics:

1 Children are given the liberty to find similarities between that which is new and that which is already familiar to them
2 Children are given the opportunity to actually carry out a transference or a reduction of objects.
3. Time is given in order to determine whether the transference undertaken does, in fact, fit the new situation.

These characteristics of transference situations can, indeed, be found in realistic education.

Constructing the context
During the introduction of addition and subtraction, using play-acting, children are given unlimited freedom to broach the subject of all sorts of contextual frills in order to construct the context in class. This corresponds to the first characteristic of transference

Various other subjects, such as reading, playing, writing, drawing, are used in order to realize the context in class (characteristic 2)

Confrontation with other ideas within one context
Somewhat later, it is made clear to the children that not all facets of the bus-context are significant. The children become aware through arrow-language of the transference from bus-context to bus-model, consisting in getting on and off (characteristic 2)

Confrontation with other contexts
Use of the bus-arrows in skittles causes problems: do the numbers indicate a score or the number of fallen skittles? (characteristic 3)

Ambiguity (adding scores or subtracting numbers of skittles) can be avoided through agreement on the meaning of the arrows used

Towards a bare arithmetic context
The language itself (arrows for buses and skittles) can prove a useful means for notating similar contexts Here lies a point of departure for vertical mathematisation: the original (bus) meaning blurs [Von Glaserfeld and Steffe, 1986], but the arrow-language remains transparent [Polyani, 1962] so that the bus-context is still recognizable and the arrows can be alternately used as "bare" arithmetic language and as language in a contextual framework [Freudenthal, 1987].

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