Using History in Teaching Mathematics

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How many words can be formed from the letters of the Hebrew alphabet? When and where does the sun rise in Alexandria on August 16? What is the length of the circumference of a circle of radius 1? Why does $\sqrt[3]{(2+\sqrt{(-121)})} + \sqrt[3]{(2-\sqrt{(-121)})}$ equal 4? Where should a 150 lb. man sit on a seesaw to balance his 50 lb. son? How should the stakes in a game of chance be split if the game is interrupted? It is the consideration of questions like these which historically has provided the impetus for the development of mathematics. The consideration of such questions with our students is one way that we can motivate and excite them.

I have found this historical approach to the topics of the mathematics curriculum to be a profitable one. I will present here some concrete examples of the use of historical materials in developing certain topics from precalculus and calculus. Some of these are ideas which can be introduced easily in the course of a standard treatment of the material. Others would require some reformulation of the curriculum. In those cases, the curriculum would benefit from the reformulation. I will present these ideas in five general areas: algorithms, combinatorics, logarithms, trigonometry, and mathematical models. All are important areas in the curriculum and all have a relatively well-known history. It is the use of this history which I feel makes for a significant improvement in teaching.

We begin with algorithms. This is the new "buzzword" in mathematics education today, with conferences, institutes, and grants designed to see how algorithmic mathematics and the "algorithmic way of thinking" can be incorporated into the freshman college curriculum. But the idea of algorithms is hardly new. In fact, algorithmic mathematics is the oldest mathematics of which there are records. After all, the Babylonian texts are chiefly lists of problems and rules for their solutions, i.e. algorithms. I want to distinguish two types of algorithms here and discuss each in turn. The first type is the algorithm which produces a definite answer in a finite number of steps; often this is expressed as a formula, for example, the formula for solving quadratic equations. The other type of algorithm produces approximations to an answer; one usually stops the algorithm when one reaches a predetermined level of accuracy. As an example of this, we may cite the standard square root algorithm. Both of these types date from ancient times, but we want to take our examples here from a more modern period.

Our example of an algorithm of the first type is the cubic formula of Cardano (1545). The story of the Tartaglia-Cardano controversy is well known, and, of course, is a good story to tell a class. But we want to concentrate here on the mathematics itself. Recall that Cardano solved the equation $x^3 = px + q$ by setting $x = a + b$ where $ab = (1/3)p$ and $a^3 + b^3 = q$. When we try to solve these two latter equations for a and b, we get a quadratic equation in a^3 which leads to the solution

$$a = \sqrt[3]{(q/2 + \sqrt{[(q/2)^2 - (p/3)^3]})}$$

and

$$b = \sqrt[3]{(q/2 - \sqrt{[(q/2)^2 - (p/3)^3]})}$$

That is, we have a straightforward formula, or algorithm, giving us the solution x to the equation.

Once we have an algorithm that we have justified to some extent — and we can easily justify this one algebraically or geometrically — we need always to ask questions about it. Among the questions are first, does it always work, or are there restrictions on the input; second, does it give only one solution, or is there some ambiguity; third, can we generalize the algorithm to solve similar problems; fourth, is there a better algorithm which will solve the same problem. It is through the answers to questions such as these that mathematics progresses.

Let us consider the first question — and Cardano did exactly that. A quick glance at the formula shows that if $(q/2)^2 < (p/3)^3$, we get the square root of a negative number; and that, in 1545, did not make sense. On the other hand, if we take a concrete example where this problem occurs, such as $x^3 = 15x + 4$, the formula gives $x = \sqrt[3]{(2+\sqrt{(-121)})} + \sqrt[3]{(2-\sqrt{(-121)})}$, while it is obvious that a correct answer is 4. Is it somehow possible that 4 can be expressed as a sum of cube roots which include square roots of negatives? Cardano's answer to the dilemma was to find a different method to solve the equation. It was Raffaello Bombelli, however, who in 1572 gave a solution using Cardano's formula. His answer was to develop the formalism of complex numbers. That is, the search for solutions to the cubic becomes the motivation for the study of complex numbers. It is pedagogically better to follow this historical development and introduce complex numbers here rather than as solutions to certain quadratic equations, because here you know there must be a solution. In the quadratic case, it is simple enough to say that solutions don't exist. Bombelli, in fact, gave a rather straightforward discussion of the rules for operating with complex numbers, assuming that they were the same as those for real numbers with the added twist that $(\sqrt{-1})^2 = -1$. And it was only after he had justified the use of complex numbers to solve cubics that he also noted that one could use these same strange numbers to solve quadratic equations with negative discriminants

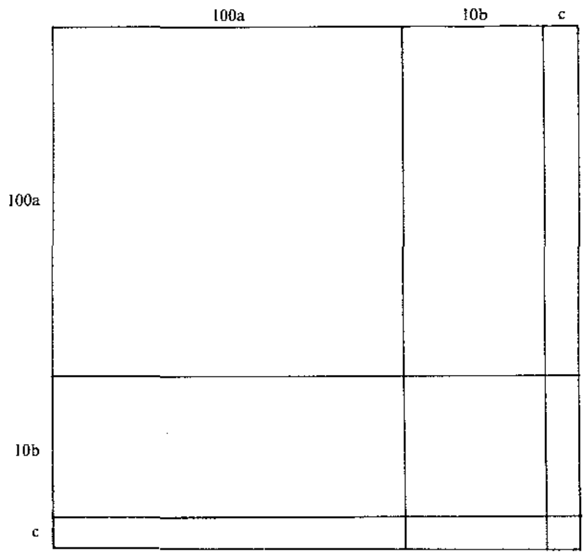
The second question on our algorithm, is there ambiguity in the solution, also leads to interesting mathematics. After all, the equation $x^3 = 21x + 20$ has three real solutions, $x = 5$, $x = -1$, and $x = 4$; but our formula looks like it only gives one. The answer to this dilemma lies in the meaning of the symbol $\sqrt[3]{\ }$. In fact, by analogy with the rule that positive real numbers have two square roots, one guesses that they also have three cube roots. One can now search for these.

The question on generalizing the results was answered in part by Cardano himself. Secondly, his student Ludovico Ferrari found the method of solving quartics. Though it would be difficult to develop the quartic formula in a freshman class, this discussion does provide an opportunity for mentioning the outcome of the search for formulas for polynomial equations of higher degree.

Our final question, is there a better method, again leads both historically and pedagogically to new mathematics. Descartes considered the solution to equations from a different point of view, building up new equations from factors of the form $x - a$. This leads to the remainder and factor theorems as well as the methods for finding integral and rational solutions to polynomial equations. Finally, since not all such equations have rational solutions, we are also led to numerical methods of approximating roots, including Newton's method It is not necessary to introduce calculus to discuss this; after all, Newton himself presented it in a purely algebraic fashion.

This method, of course, is a good example of the other type of algorithm, the one which produces approximations. Instead of discussing that example here, we want to look at some from earlier time periods The first such example, the square root algorithm, was covered in school courses years ago. Today it has apparently been replaced by the simpler rule: key in the number on your calculator and push the square root button This is unfortunate, because the algorithm provides a very simple example of a convergent sequence, and it is only by seeing such examples from an early time in one's education that the notion becomes familiar enough to use in the study of calculus. My suggestion then is to revive the teaching of the square root algorithm. We will not describe the method here. In a class, however, one needs to give a justification of it; the best way might be geometrical. Many scholars, in fact, believe that this was how the method was originally discovered [Mathews, 1985] For example, if we want the side n of a square with given area A, we seek digits a, b, c such that $(a\cdot 100 + b\cdot 10 + c)^2 = A$ (Figure 1) We first find a such that $(a\cdot 100)^2 < A$ Subtracting off that large square we now have the area of the remaining gnomon. We next find b so that $(10\cdot b)(100\cdot a)\cdot 2 < A - (a\cdot 100)^2$ (that is, first we ignore the double counting of the small square) and then adjust if necessary so that $(a\cdot 100 + b\cdot 10)^2 < A$ We continue similarly to find c. The pedagogical point to be made here is

Figure 1



that, assuming that there is no exact integral solution to this problem, we can continue this method to as many decimal places as we like. The answers we get at each stage form a sequence converging to $\sqrt{A}$. It is easy to convince a class geometrically after each step. It is a useful exercise next to consider our questions about algorithms in terms of this one In particular, one might look for generalizations.

Our next historical example of a convergent sequence occurs in the Archimedean work, *Measurement of the circle,* [Heath] where Archimedes calculates two approximations to π, one by using circumscribed polygons and one by using inscribed polygons. One must here analyze Archimedes' work and pull out the geometric lemmas he uses to construct his polygons. If we denote by t_n the length of one-half of one side of a regular polygon of $3\cdot 2^n$ sides circumscribed about a circle of radius 1 and by u_n the length of the line from the center of the circle to a vertex of that polygon, (Figure 2) Archimedes in effect shows that

$$t_{n+1} = t_n/(u_n + 1) \qquad u_{n+1} = \sqrt{(1 + t_{n+1}^2)}$$

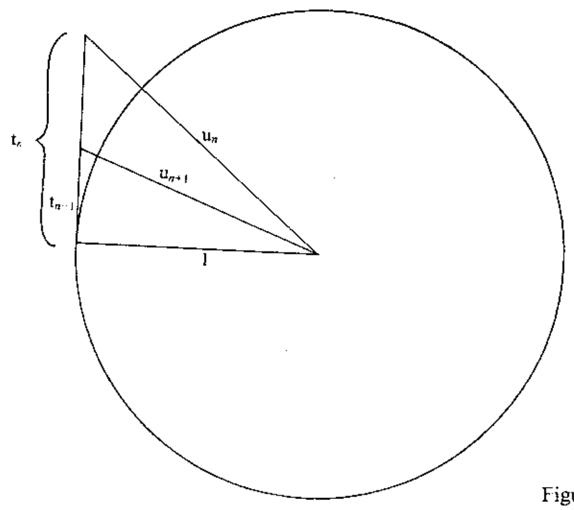


Figure 2

These formulas give us a good way of discussing a recursive algorithm. If we begin, as did Archimedes, with $n = 1$, then $t_1 = 1/\sqrt{3}$ and $u_1 = 2t_1$ or $t_1 = 0.5773502$ and $u_1 = 1.1547004$. Since at each stage the half perimeter P_n is equal to $3 \cdot 2^n \cdot t_n$, we calculate first that $P_1 = 6t_1 = 3.4641012$. The recursion formulas then allow us to calculate t_n, u_n, and P_n for as many successive positive integers n as we wish and hence to approximate π. We can use similar formulas to calculate the perimeter of the inscribed polygons. These give us another sequence approaching π. We can now discuss not only convergence, but also errors, for π is squeezed between two convergent sequences. At each step, therefore, we have precise bounds on the error. In fact, if we take as an even better sequence the mean of the two sequences, then the error at any stage is no more than half the difference between the values found from the two polygons.

A final example of an algorithm leading to a convergent sequence can be found in Archimedes' calculation of the area of a parabolic segment in *Quadrature of the parabola* [Heath]. Since Archimedes requires some detailed knowledge of the properties of a parabola, it may not be possible in the context of a precalculus course to justify the algorithm here completely. Nevertheless, the approximations in this case are the partial sums of a geometric series. Here one can follow Archimedes by both calculating explicitly the error term at any stage and by calculating the exact sum of the "infinite" geometric series, which provides the exact area of the parabolic segment. (Archimedes, of course, used a proof by exhaustion.) In each case, the students gain valuable insight into the meanings of approximation and convergence in the context of an interesting problem.

The next topic in which the historical roots lead to effective ways of developing the subject is that of elementary combinatorics. Though many of the basic counting formulas are found in essence in medieval Indian works, I prefer to use the treatment in European-Jewish sources, since there the justifications are given as well as the results. And, in fact, we find there the origins of mathematical induction.

The question of counting permutations and combinations does not seem to have interested the ancient Greeks. But it did interest the Jews; in fact, in a mystical Hebrew work, the *Sefer Yetsirah* or *Book of Creation*, written sometime during the first millenium of our era, the unknown author calculated the various ways in which the 22 letters of the Hebrew alphabet could be arranged. The reason for this calculation was that the Jewish mystics believed that these letters had magic powers; suitable combinations could therefore subjugate the forces of nature. In any case, the author understood that the number of possible arrangements of n letters was $n!$ In addition, he showed how many sets of two letters could be taken out of the 22. This is the number we today write as ${}_{22}C_2 = \binom{22}{2}$; it is equal to $(1/2) \cdot 22 \cdot 21$. In general, we easily see that ${}_nC_2 = n(n-1)/2$.

This idea of combinations of 2 elements out of n was generalized to other combinations by Rabbi Abraham ben Ezra in the early 12th century. [Ginsburg, 1922] He was interested in astrology; and an important occurrence for astrologers was the conjunction of 2 or more of the 7 "planets" (including the sun and the moon). Ben Ezra therefore showed how to calculate ${}_7C_k$ for each integer k from 2 to 7. In fact, he gave an argument which one can easily reproduce in class to show that

$${}_7C_3 = \sum_{i=2}^{6} {}_iC_2 \ , \ {}_7C_4 = \sum_{i=3}^{6} {}_iC_3 \ , \text{ etc.}$$

His argument easily extends to give the general rule

$${}_nC_k = \sum_{i=k-1}^{n-1} {}_iC_{k-1}$$

One needs to point out here that this formula is not very useful for calculating a given ${}_nC_k$, for one would need to know ${}_mC_j$ for every $m < n$ and every $j < k$. And of course Rabbi Ben Ezra did not prove this general result in any case. Better results, which allow not only the explicit calculation of ${}_nC_k$ but also of ${}_nP_k$ were stated and proved by Levi ben Gerson (Gersonides) in his 1321 text *Maasei Hoshev* or *The Work of the Calculator* [Lange, 1909]. It was in these proofs, in fact, that one finds one of the earliest explicit uses of the principle of mathematical induction. [Rabinovitch, 1970] One of the examples of such a proof is Levi's calculation of ${}_nP_k$. He states that this number is found by multiplying together the k factors beginning with n and decreasing in order to $n - k + 1$. His proof is in the two parts standard for an induction proof. Namely, he first shows that ${}_nP_2 = n(n-1)$ by a counting argument. Next he uses another counting argument to show that ${}_nP_{j+1} = (n-j){}_nP_j$. This latter is the inductive step. Gersonides concludes his proof by discussing why these two lemmas imply the conclusion. We should note here that Levi is concerned with proofs. In the introduction to his book he states that the number-theoretic books of Euclid are prerequisite to his own work and, in fact, the theorem-proof style of the book is based on the Euclidean model.

Levi ben Gershon goes on to give the calculation of ${}_nC_k$ in the form ${}_nC_k = {}_nP_k/k!$ and, of course, this needs to be presented to a class. But even more interesting is another theorem he proves by induction, namely the formula for the sum of the first n integral cubes:

$$\sum_{i=1}^{n} i^3 = \left[\sum_{i=1}^{n} i\right]^2 = [n(n+1)/2]^2$$

In a course, we might want to deal with other matters before getting to that proof, however, and again a historical approach is valuable. For instance, the results of Pascal concerning his arithmetical triangle help to reinforce some of the combinatorial calculations already made. [Pascal, 1654] And Pascal also proves many results by induction. In particular, it is useful to use Pascal's own motivation, the division of stakes in an interrupted game of chance, as a substantial example for both a beginning discussion of probability and a proof by induction.

General results on the sums of integral powers were necessary to establish the result that $\int x^n\,dx = x^{n+1}/(n+1)$. Therefore, if one wants to calculate that integral in an introduction to the integral calculus, one needs to deal with these sums of powers. The results

$$\sum_{i=1}^{n} i = n(n+1)/2$$

$$\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$$

$$\sum_{i=1}^{n} i^3 = [n(n+1)/2]^2$$

were all known by the middle ages. It is easy enough to give inductive proofs of these; as we mentioned, Levi ben Gershon gave such a proof for the third of these results. But for pedagogical purposes, one must first show the students how one could guess the results in the first place. It is important to point out that the inductive proofs, as any proofs, are only useful once there is something to prove. The next question is, can we generalize these results to give a formula for

$$\sum_{i=1}^{n} i^k$$

We have no records before the 17th century indicating an attempt to do this, perhaps because Greek tradition forbade dealing with fourth powers and because the algebraic notation was not sufficiently developed to give a good guess for such a generalization. But if we rewrite the formulas as

$$\sum_{i=1}^{n} i = n^2/2 + n/2$$

$$\sum_{i=1}^{n} i^2 = n^3/3 + n^2/2 + n/6$$

$$\sum_{i=i}^{n} i^3 = n^4/4 + n^3/3 + n^2/4$$

a pattern leaps to our eyes.

It was Jakob Bernoulli who first published the result giving this pattern in his *Ars Conjectandi*. [Bernoulli, 1715] Bernoulli found all the sums through the 10th powers. Since his derivation came from the use of the Pascal triangle, it can be presented. However, the derivation is a recursive one; that is, to calculate Σi^4 one needs first to know Σi^3. On the other hand, for use in calculus, one really only needs the simpler result

$$\sum_{i=1}^{n} i^k = n^{k+1}/(k+1) + n^k/2 + p_k^{(n)}$$

where $p_k(n)$ is a polynomial of degree less than k. One can easily adapt Bernoulli's derivation to prove this result by induction.

In these days of pocket calculators, the use of logarithms for calculation has become obsolete. Nevertheless, we must teach students something about logarithms. Logarithm functions, and particularly the natural logarithm function, are still important. Again the historical approach comes to our rescue. It is true that Napier, the inventor of logarithms, wanted to devise a method which simplified the complex calculations necessary to the astronomy of his day, but in fact his method, suitably modified, provides a possible introduction to the notion of natural logarithms and shows why they are "natural." This particular approach, however may be somewhat too difficult for all but the best students. On the other hand, it is certainly useful for the teacher to bear in mind. Napier began by considering the relationship of an arithmetic to a geometric sequence. In his case the arithmetic sequence increased while the geometric sequence decreased; in addition, he was dealing with sines, which in his day were simply lines in a circle of appropriate radius. As was common in his day, Napier used 10,000,000 as the radius; that value was the first term in his geometric sequence. Since we prefer to consider logarithms of arbitrary numbers, rather than of sines, and since it is useful for the logarithm of 1 to be 0 rather than that of 10,000,000, we modify Napier's program by comparing the arithmetic sequence 0, a, $2a$, $3a$, ... with the geometric sequence 1, r, r^2, r^3, ... where we can think of r as being a value close to 1. (Napier himself at the end of his life agreed that such modification was useful; it was, however, Napier's successor Henry Briggs who actually completed the program.)

Given the two sequences, we first note that addition in the arithmetic sequence corresponds to multiplication in the geometric one; that is, if we write $l(r^n) = na$ we have the result $l(x) + l(y) = l(xy)$ where x, y are any positive integral powers of r. Similarly, we get the other standard laws $l(x) + l(y) = l(x/y)$, $l(x^m) = ml(x)$ and $l(\sqrt[m]{x}) = l(x)/m$, where these laws only apply where they make sense. We can easily extend the correspondence, and the laws, to negative powers by extending the arithmetic sequence to the left with $-a$, $-2a$, $-3a$, ... and the geometric sequence with $1/r$, $1/r^2$, $1/r^3$, ... We then note that we can keep the correspondence $l(r^n) = na$ still valid if we put $1/r^k = r^{-k}$.

Next, we take Napier's momentous step of converting from arithmetic to geometry. [Napier, 1619] We conceive of two number lines on which these sequences are represented and consider points P and p moving on the two lines as follows. (Figure 3) P moves on the upper line with constant velocity v; hence P covers each marked inverval $[0, a]$, $[a, 2a]$, ... in equal time. The point p moves on the lower line so that it too covers each marked interval $[1, r]$, $[r, r^2]$, $[r^2, r^3]$, ... in the same time. We think of the velocity of p as constant in each marked interval. One then shows that this property of p implies that the velocity of p at any of the marked points is proportional to that point's distance from 0; i.e., its coordinate. It follows that for any two marked intervals $[\alpha, \beta]$, $[\gamma, \delta]$ such that $\beta/\alpha = \delta/\gamma$ the time for p to cover $[\alpha, \beta]$ is equal to the time to cover $[\gamma, \delta]$. This latter result was central to Napier's treatment as we shall see shortly. We further note that if P and p begin to move at the same time, P from 0 and p from 1, with the initial

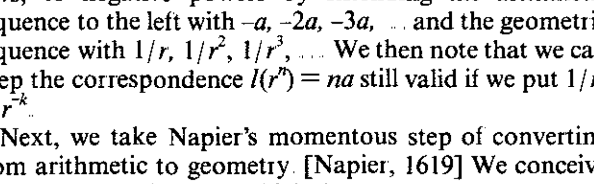


Figure 3

velocity of p equal to v, the correspondence $na \leftrightarrow r^n$ is exactly the correspondence between the distances of P and p from the respective 0 points at given times, assuming that at these times both are at marked points

With the above definitions, the velocity of p jumps at each marked point. To smooth out these jumps, we can put in new points on the lower line, such as $s = \sqrt{r}$ or $u = \sqrt[3]{r}$ as well as the integral powers of these, and also corresponding points on the upper line, such as $b = (1/2)a$ or $c = (1/3)a$ and multiples of these. We can then keep our velocity laws the same as well as our laws of correspondence In particular, we still have the property that for any two marked intervals $[\alpha, \beta]$, $[\gamma, \delta]$ such that $\beta/\alpha = \delta/\gamma$, the point p covers each in equal time

Napier himself moved quickly to the continuous case; that is, he made the assumption that the velocity of p increases smoothly so that at any point of the lower line that velocity is proportional to the point's coordinate. We continue to assume that P travels at constant velocity v while p starts with that velocity. If we suppose p is at x at the same time that P is at y, we can define with Napier the logarithm of x to be y. (Figure 4) From our property of p covering intervals $[\alpha, \beta]$, $[\gamma, \delta]$ in equal times when $\beta/\alpha = \delta/\gamma$, we conclude that $\log \beta - \log \alpha = \log \delta - \log \gamma$ Specialization of this result gives the basic logarithm laws.

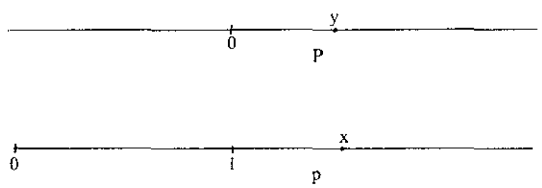


Figure 4

From this definition, Napier was able to calculate logarithms. I prefer to do that in a somewhat more modern way For our purposes here, however, we want to show why the logarithms defined this way are "natural" From the definition, we see that $\log 1 = 0$ and that as x increases so does $y = \log x$. It follows that for some number e, we get $\log e = 1$ The logarithm properties then show that we can rewrite $y = \log x$ as $x = e^y$; they also allow us to derive the exponential properties. The velocity relations can be translated as $\Delta y/\Delta t = v$ on the upper line, while $\Delta x/\Delta t = vx$ on the lower It follows that $\Delta x/\Delta y = x$, and an easy argument—one does not need calculus—shows that $e = (1 + \Delta y)^{1/\Delta y}$. Here we are thinking of Δy as a "small" number. We can then use calculators to approximate e. We now have our logarithms defined as exponents to base e, where $e \approx 2.718$; that is, they are natural logarithms.

We can now proceed to calculate these logarithms, in ways other than pushing the appropriate button on the calculator Napier's method is somewhat difficult. The method I use is to jump forward 50 years and show that the area under the curve $y = 1/(x + 1)$ has the properties of a logarithm. If we further do as Newton did and divide, we see that $y = 1 - x + x^2 - x^3 + \ldots$ Then, if no one knows the basic integral formulas, we conclude that $\log(x + 1) = x - x^2/2 + x^3/3 - x^4/4 + \ldots$ It is now easy to calculate values, at least for "small" values of x such as ± 0.1, ± 0.2 To calculate logarithms for larger values, we simply write these in terms of the known values and use the laws. For example, since $2 = (1.2 \times 1.2)/(0.8 \times 0.9)$, $\log 2 = 2 \log(1.2) - \log(0.8) - \log(0.9)$.

Since we are now well into calculus, we can discuss a useful historical approach there That is, I believe that we should take Newton's ideas seriously in his *De Analysi* and introduce power series into calculus at an early point [Newton, 1669] There are several advantages to this, particularly in the modern context of algorithmic thinking. Such series enable one to develop algorithms actually to calculate values of various interesting functions. These calculations in turn help to build the students' intuition about convergence. Further, the use of power series leads to a useful discussion of analogy—here between series and polynomials—as a method of mathematical discovery Finally, this approach provides a unifying concept to the development of the various transcendental functions.

It is possible to introduce power series immediately after showing that the derivative of x^n is nx^{n-1} and the integral of x^n is $x^{n+1}/(n + 1)$. As already mentioned, the series for the logarithm provides a good beginning to this study. From it we can also derive the series for the exponential function The next step is to develop the binomial theorem, but we will not discuss that here.

What about trigonometric functions? These are nicely handled the way Newton did them, by starting with the inverse functions. We begin with $x = \arcsin y$ as the arc whose sine is y From similar triangles in the diagram we see that $dx/dy = 1/\sqrt{(1-y^2)}$. (Figure 5) It follows that $x = \int 1/\sqrt{(1-y^2)}dy$. To calculate this, we expand $1/\sqrt{(1-y^2)} = (1-y^2)^{-1/2}$ in a power series by the binomial theorem to get

$$1/\sqrt{(1-y^2)} = 1 + y^2/2 + 3y^4/8 + 5y^6/16 + \ldots$$

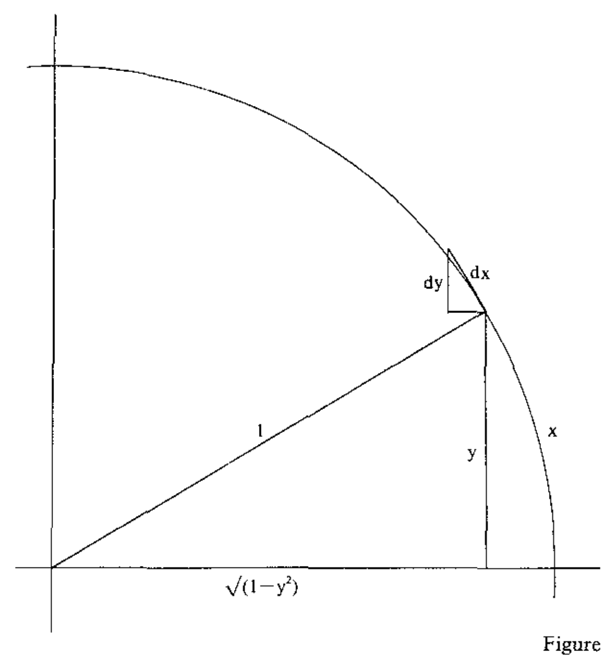


Figure 5

Integrating term by term we get

$$\arcsin y = y + y^3/6 + 3y^5/40 + 5y^7/112 + \ldots$$

Whenever dealing with power series, of course, we should discuss convergence. But in an introductory course, this can be handled intuitively. We might note here, though, that this series certainly converges for $y = 1/2$. Since arcsin $1/2 = \pi/6$, we can replace y by 1/2 and multiply the result by 6 to get an approximation to π. This procedure provides a new algorithm for π which could well be compared with the Archimedian algorithm discussed earlier.

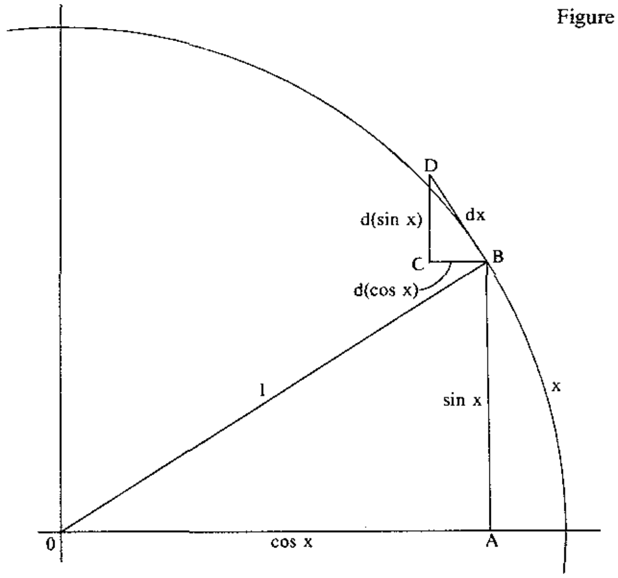
Once we have $x = \arcsin y$ we need to get a series for $y = \sin x$. This can be done by inversion, that is solving the first equation for y in terms of x. Though Newton did this often, it is a difficult procedure to teach in an introductory class. It is easier to use Leibniz' method of undetermined coefficients. This is, we assume that $y = \sin x$ is a power series $s(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$ and substitute the series for $x = \arcsin y$ into it using the property $\sin(\arcsin y) = y$. If we rewrite this combined series in powers of y, we get a power series in y which must be equal to y itself. Therefore, every coefficient must be 0 except the coefficient of y, which is 1. Solving the various equations for the a_i in turn, we get the well-known coefficients of the sine series. We notice, as did Newton, that these coefficients are reciprocals of the factorials of the corresponding powers, that is

$$\sin x = x - x^3/3! + x^5/5! - x^7/7! + \ldots$$

We can similarly calculate the power series for the arctangent, tangent and cosine functions, and in various ways. For example, we can derive the cosine series by using the binomial theorem and the sine series via the equation $\cos x = \sqrt{(1 - \sin^2 x)}$. We can then derive the tangent series by division of sine by cosine and the arctangent series by inversion. Alternatively, we can use a geometric argument similar to the one above to show that $\arctan y = \int 1/(1+y^2)dy$; then we calculate the series by division and term by term integration.

Once one has the sine and cosine series, it is obvious that the derivative of the sine is the cosine. It is curious that this analytic result was hardly mentioned before a 1739 paper of Euler's even though the series themselves were well-known. On the other hand, we can use the same diagram as before, relabeled, to show how this fact was first discussed geometrically in the works of Cotes. [Gowing, 1983] Namely, the two triangles BOA and CBD are similar. (Figure 6) It follows that $d(\sin x)/dx = \cos x$. Similarly, $d(\cos x)/dx = -\sin x$ (since $\cos x$ is decreasing). If we consider (Figure 7) and note that triangles EOF and HFG are similar, we get the analogous results that $d(\tan x)/\sec x dx = \sec x$ or $d(\tan x)/dx = \sec^2 x$ and $d(\sec x)/\sec x dx = \tan x$ or $d(\sec x)/dx = \sec x \tan x$.

Figure 6



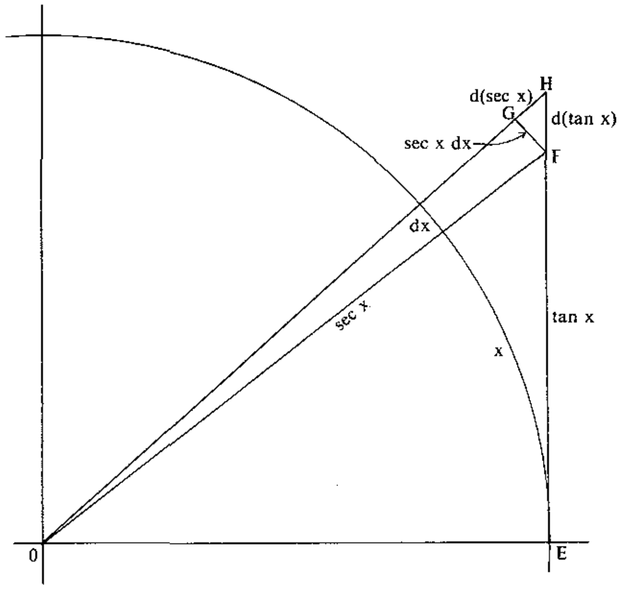


Figure 7

As a final example of the use of historical methods in the freshman curriculum, I want to consider another topic of current interest, namely mathematical models. The work of Archimedes on the law of the lever is very useful in this respect. Archimedes, being a good Greek geometer, set up an axiom system for his discussion of levers. This is similar to what we do today when we discard, in our first modeling attempt, complicating factors. That is, Archimedes assumed that the lever itself was rigid, but weightless, and that the fulcrum was merely a mathematical point. It is not difficult to present the essence of Archimedes' treatment to a class, since the axioms are quite self-evident and the theorems simple and easy to prove. One concludes with the result for weights A, B having a rational ratio r/s that they will balance at C when the ratio of the distance of A to C to that of B to C is s/r. Archimedes, in addition, proves this result for incommensurables as well, but that can be done in a freshman class by an appeal to continuity.

Once the law of the lever is proved, it should be used. One interesting result is the determination of the center of gravity of a triangle. Archimedes' argument here is somewhat tricky, but we can use our basic intuition on centers of gravity together with some elementary geometry to show that the center of gravity is at the intersection of the medians and specifically at a point two-thirds of the way from any vertex to the midpoint of the opposite side

A much more important mathematical model and one which is crucial to world intellectual history in many ways is the model of the universe. The early Greek model was that of two spheres, the sphere of the earth and the sphere of the heavens, with the latter revolving daily about the former. (It is interesting here to ask the class to give earthbound arguments as to why that is wrong; i.e., why in fact it is the earth that moves.) Since the development of astronomy was important in the development of mathematics, these ideas should be discussed in some detail. Some of the concepts to come out of this discussion include the idea of a coordinate system, on both the earth and the heavenly sphere, as well as methods for measurement of time, arcs, angles, distances, etc. Of course, trigonometry itself was developed for use in astronomy, and I believe that an introduction to trigonometry can well be taught along the lines of Ptolemy's own treatment. The Greeks themselves were more concerned with spherical trigonometry than plane trigonometry. For various reasons, that subject is no longer taught. On the other hand, the basic formulas for the solution of right spherical triangles are very simple to develop. These formulas include $\sin \alpha = \sin a/\sin c$, $\cos \alpha = \tan b/\tan c$, $\tan \alpha = \tan a/\sin b$, and $\cos c = \cos a \cdot \cos b$ (Figure 8). Once these formulas are known and the basic diagram from the model of the universe is understood,

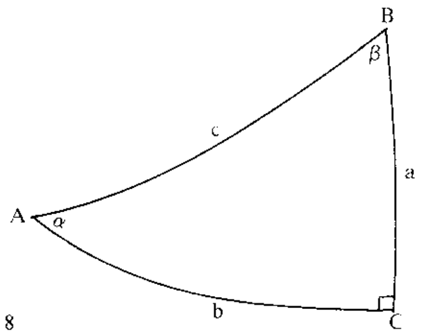


Figure 8

there are many very interesting problems which the students can solve. (In Figure 9, H is the point where the sun crosses the horizon on a given day, E is the east point on the horizon, and V is the vernal point. For our purposes, λ, the sun's longitude, is equal to the number of days after the vernal equinox. Also, ϕ is the latitude of our location; it follows that $\angle HEC = 90 - \phi$. Finally, δ is the sun's declination.) For example, they can calculate the time of sunrise and sunset on given dates at given locations on the earth, calculate the dates when the midnight sun is visible for places above latitude 66½° or when the sun is directly overhead at latitudes less than 23½° and calculate the location of sunrise or sunset at a given date and place. In fact, these calculations give the student good practice in dealing with the various trigonometric functions. Assuming they all have calculators, these provide excellent exercises in using the trigonometric buttons, exercises which are often more stimulating than solving plane triangles

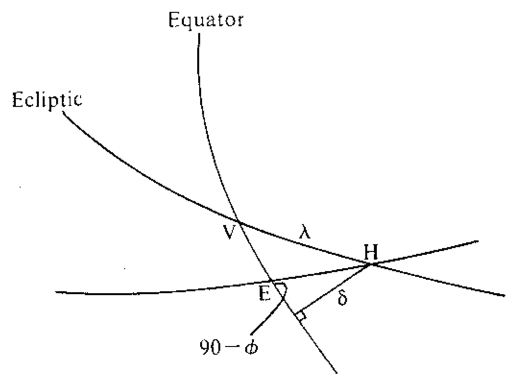


Figure 9

I could give many more examples of the value of a historical approach to mathematics. A study of the history of any particular topic in the curriculum often leads to many valuable pedagogic ideas. But besides the specific examples, I believe it is worthwhile to treat entire courses at this level from a historical point of view. Not only does this help the student understand the development of the subject, but it also provides him with ways to connect mathematics and other aspects of civilization as he masters the skills necessary to apply the techniques on his own

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