Fuzzy Thinking in Non-fuzzy Situations: Understanding Students' Perspective

RINA ZAZKIS

> *If our reasoning has logic, it's fuzzy at best. We have only one decision rule: I'll do it if it feels right. The formal logic we first learn in tenth-grade geometry class has little to do with it. That's why we made it to tenth grade* [Bart Kisko, p. 17]

Consider the following statements:

(1) The product of any two negative numbers is negative.
(2) All even numbers are divisible by 4.
(3) All prime numbers are odd

Obviously, all three statements are false. However, for me it has always felt unjust to assign to all three statements the same truth value. The first statement is felt as "entirely false", the second one as "half false" and the third as "almost true". However, this is not the only piece of mathematical knowledge that disagreed with my intuition. The more I played with mathematical formalism the more it became my second nature that slowly but firmly pushed aside my "primitive" intuitions.

This note was inspired by students' responses to one of the exam questions in their mathematics course. The students were preservice elementary school teachers. The mathematics course was "Foundations of Mathematics for Teachers", a core course in the elementary teacher education program. The course included topics from introductory number theory, such as divisibility and prime decomposition. The exam question was:

"If a is divisible by 3,

(a) Is a^2 divisible by 6?
(b) Is a^2 divisible by 9?

If yes, provide a general argument for your conjecture (proof)
If no, provide a counterexample."

While students had no difficulty in claiming "yes" followed by an attempted proof in part (b), part (a) presented an unexpected challenge.

> "YES and NO" (wrote Stephanie) "If you take the number 9, its square 81 is not divisible by 6. But if you take for example 12, its square 144/6 = 24 is divisible by 6. This will work for every second number. So (a) is *half true* and *half false*"

Stephanie was not by any means the only one who provided both example and counterexample for part (a) of the question and had difficulty in choosing between YES and NO. However, it was her observation of "half true and half false" that rang a bell—Fuzzy Logic! In the following paragraphs I will briefly describe Fuzzy Logic and discuss its place in mathematics education.

Fuzzy logic

Fuzzy logic was introduced in the mid-sixties by Lofti Zadeh from the University of California at Berkeley as a study of fuzzy sets. According to Kosko [1993], Zadeh chose the adjective "fuzzy" over the traditional "vague" and the name has stuck.

A traditional "crisp" set S is a collection of objects. Any object (of the universal set) either belongs to set S or doesn't belong to set S. For example, the number 4 belongs to the set of even numbers between 2 and 20, and the number 24 doesn't. In other words, a statement of the form $x \in S$ is either true or false for any choice of x. In contrast, a fuzzy set is a set whose members belong to it *to some degree*. To specify such a degree of belongingness there is a need to mathematize this notion. Quantification is the chosen way and, as in the case of probability, [0,1] is the chosen interval. A membership designated by the measure 1 is an unqualified membership, while a membership of measure zero conveys that the element doesn't belong to the set; any number between 0 and 1 represents a partial membership. It is convenient to represent a fuzzy set as a set of ordered pairs, where the first element in each pair denotes the element itself, and the second denotes the degree of membership [Zimmerman, 1990]. The second element of the ordered pair must be a number in the closed interval [0,1]. However, elements having zero degree of membership are normally not listed. For example, if G is a (partial) list of grades indicating success in a course

$$G = \{(C, .65), (B, .75), (A, 1)\},$$

we interpret the above code by saying that the "degree of membership" in the fuzzy set G of the element A is 1, the degree of membership of the element B is .75, and of C is .65. In other words, the statement $A \in G$ is true, 100% true. The statement $B \in G$ is 75% true (and 25% false), and the statement $C \in G$ is 65% true (and 35% false). Fuzzy set theory further defines operations, functions, and relations on fuzzy sets. Fuzzy logic extends the idea that a statement can be true to a degree. Its advocates argue that the Aristotelian true/false dichotomy doesn't adequately represent many situations, since real situations are very often not crisp and deterministic. Even Russell recognized that symbolic bivalent logic is "not applicable to the terrestrial life but only to an imagined celestial existence" [Russell, 1923]. Real situations are often uncertain and vague. Claims of being tall, young, trustworthy, creditworthy, or being a good teacher, are true only to some degree, so they are also false to some degree (Modern society has created vague responses even to the traditional dichotomies of gender and marital status) Modern techniques of modeling with fuzzy logic create instruments that attempt to answer to what degree Mrs Z is a good teacher or a creditworthy customer. A simple model example of "creditworthiness" suggested by Zimmerman [pp 5, 366-367] shows a hierarchy that itemizes smaller subjective categories and assigns relative weights to each level of aggregation Modeling with fuzzy logic is frequent in areas where human judgment and evaluation are important. It appears more informative and is often preferred to traditional logic in meteorology, medicine, social sciences, and managerial decision making

However, while we may be happy to a degree, or young to a degree, or useful to a degree, the world of mathematics appears to many of us to be lacking in fuzziness A (real) number is either rational or irrational, a (whole) number is either odd or even, divisible by 17 or not, a perfect square or not There can be fuzzy numbers, that are taken as approximations, or to represent a spectrum of "non-fuzzy" numbers [Kosko, 1993, pp 123-125], or as an extension of the concept of an interval of confidence [Kaufmann, 1985] However, fuzzy numbers are beyond the scope of either the elementary or the undergraduate curriculum and I leave them out of this discussion Traditional mathematics is not fuzzy We learn that any mathematical statement is either true or false. Even if a statement has the status of a "conjecture", that is, we don't know whether the statement is true or false, we believe it is one or the other Not both

Fuzzy logic and mathematics education

Is there a place for fuzzy logic in mathematics education? I found it in two arenas First, there is a call for teaching fuzzy logic The paper by Alsina & Trillas [1991] introduces the idea of fuzzy sets and fuzzy logic and argues the value of teaching the subject based on the importance of its applications it also suggests a variety of activities and projects for students to get familiarized with the idea Second, mathematics education researchers have realized the advantages of applying fuzzy logic to describe and model subjects' behaviors [e.g. Lehrer & Franke, 1992]

I would like to add here a third perspective: understanding students' application of fuzzy logic to situations that are (traditionally) not fuzzy. Fuzzy logic has developed as an attempt to provide a mathematical model for the ambiguity of the "real world". In moving from bivalence to multivalence, simplicity has been sacrificed but further accuracy has been achieved. However, fuzzy thinking appears to be more intuitive, simple, and preferred by many students. Of course, by saying "fuzzy thinking" or "fuzzy logic" here I mean multivalent thinking/logic, not imprecise or vague reasoning Either lacking or ignoring the tools of bivalent logic, some students are applying fuzzy logic to situations that are not usually considered "fuzzy". These students are applying multivalence not to "real world" situations, but to mathematical, bivalent, situations.

All prime numbers are odd True or false? Following bivalent logic, obviously false, since 2 is a prime number However, following fuzzy logic, *almost true*. There is only one counterexample. *All even numbers are divisible by four True or false?* Following bivalent logic, obviously false, since 6 is an even number not divisible by 4 However, following fuzzy logic, 50% true and 50% false. The application of fuzzy logic to "pure mathematical" situations is consistent with students' empirical attitude and their experimental procedural approach to problem solving described in Zazkis & Campbell [in press]

In calculus we learn to distinguish between functions that are continuous at all points, functions that are continuous at all but a countable number of points. However, bivalent logic lacks a mechanism to distinguish between statements that are "true" and statements that are "true in all but a given number of cases" Fuzzy logic addresses this problem, both mathematically and psychologically. I have found that students have an emotional difficulty in saying that "all prime numbers are odd" is false, but more easily accept as true the statement "all prime numbers but 2 are odd" Similarly, I have found resistance to the claim that the set of rational numbers is not closed under the operation of division. The alternative claim: "the rational numbers excluding zero are closed under division" was easily accepted.

In true Piagetian fashion, excited by the resistance of my adult students to symbolic bivalent logic, I posed the following problem to my own children, 11 and 7 years old. I drew nine white circles and one black, and asked each child to determine the truth value of the statement: "All the circles are white". "Obviously, mom," claimed the 11 year old, "this statement is false since the last circle isn't white". "This is true but not quite true", claimed the seven year old, "without the last one it would be right". Further research is needed to determine whether these two reactions are typical for the age and development level of the interviewees. It is also an interesting question whether the ratio of black circles to white ones was critical to the response. However, while it may be a developmental component, or a step in mathematical maturity, to arrive at traditional bivalent logic, it may be as big a step to depart from it and alternate "crisp" logic and fuzzy logic, depending on the kind of situation being worked on

Kosko invites us to think of a segment [0,1] (See Figure 1) He claims that "the number line shows the math conflict between bivalence and fuzziness. Bivalence holds only at the "corners" or ends of the number line The bit values 0 and 1 stand as opposites Fuzziness or multivalence holds everywhere in between the corners" [Kosko, 1993, p 24] Everything between 0 and 1 is referred to as a "gray area"

Figure 1
Gray area in fuzzy logic

However, the perception that symbolic bivalent logic holds at the end points while fuzzy logic holds everywhere else is not quite exact. Bivalent logic also covers the full segment. However, "true" holds only at the right endpoint. "False" holds everywhere else. (See Figure 2.)

Figure 2
Asymmetry of *true* and *false* in traditional logic

This lack of symmetry between the antonyms true and false is one of the major pitfalls in students' understanding of formal bivalent logic. What is only 99% true is called false. This is hard to accept, especially when a student estimates the fuzzy truth value in the right half of the unit segment.

Conclusion

My call here is twofold: one, to recognize and value fuzzy (that is, multivalent) thinking in student's thinking, even when the mathematical situation does not invite fuzziness. Stephanie, in the example above, demonstrated a good understanding of divisibility and the distribution of numbers divisible by 3 whose squares are divisible by 6. What she didn't understand is that "half true" is considered to be "false".

Two, to build on fuzzy thinking instead of rejecting it. This could start with an attempt to emphasize the asymmetric roles of true and false in the game of symbolic logic: while true means "true for all", false doesn't means "false for all", it only means "true for not all".

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CORRECTION

The description of the formal basis for the "casting out nines" method in issue 15(2) became impossibly garbled somewhere along the path from the authors' manuscript to the final printed page, as many readers will have noticed. The passage immediately following the subhead in the first column of page 23 should read:

"For those who have never heard of casting out nines—and our experience suggests this may be a lot of people [2]—we begin with a brief description. It is only a special case of modular arithmetic applied to the detection of errors in arithmetic calculations. If $a = a'$ (mod 9) and $b = b'$ (mod 9), then

$$a + b = a' + b' \pmod 9 \quad \text{and}$$
$$a \times b = a' \times b' \pmod 9$$

Thus we can use the the possibly much simpler addition $a' + b'$ as a check on the correctness of the addition $a + b$."

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