PERIMETER IN THE CURRICULUM

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Twenty years ago, Woodward and Byrd (1983) argued that, despite the presence of area and perimeter in the U.S. mathematics curriculum, something was quite wrong with students' understanding of the topics. They gave eighth-grade students (thirteen to fourteen years old) a task:

> Mr. Young had 60 feet of fencing available to enclose a garden. He wanted the garden to be rectangular in shape. Also, he wanted to have the largest possible garden area. He drew a picture of several possibilities for the garden, each with a perimeter of 60 feet (see Figure 1).

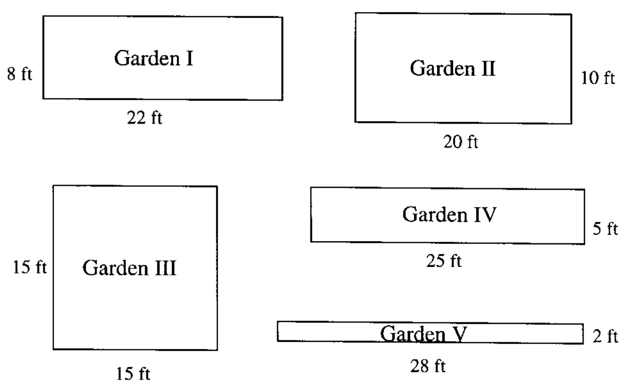

Figure 1: Several possibilities for the garden.

> Consider Mr. Young's drawings of the garden plots. Check the statement below that he found to be true.
>
> Garden I is the biggest garden
> Garden II is the biggest garden
> Garden III is the biggest garden
> Garden IV is the biggest garden
> Garden V is the biggest garden
> These gardens are all the same size. (p. 344)

Nearly 60% of these students chose, "These gardens are all the same size." The researchers concluded that the students had confused area with perimeter. More fundamentally, they argued that area and perimeter at that time were topics covered in classrooms (as they certainly are today!), but not concepts taught. That is, students were given definitions and formulas, but generally emerged with no understanding of what area and perimeter really meant, nor how they related to each other. They argued that the students in their study, in being unable to distinguish area from perimeter, were typical U.S. students and that this was likely to be the fault of the curriculum and its instruction.

The last fifty years have seen roughly three paradigms for school mathematics (K-12 - kindergarten to twelfth grade or students aged 5 to 18) in the U.S.:

- *the New Math* of the 1950s and 1960s, which emphasized the structures of the discipline of mathematics such as axioms, proofs and precise definitions
- the *traditional* curriculum, in which algorithms and skills took precedence
- the *reform* curriculum, where the stated goal is to increase the number of students who see the subject as relevant, useful and beautiful.

The topics in the curriculum vary some across the paradigms, but their treatments vary widely. This article analyzes three curricular approaches to one topic - perimeter - with an emphasis on the approach of one reform curriculum.

Test results seem to support the contention that conceptual learning of area and perimeter rarely happens in U.S. classrooms (*e.g.*, Carpenter *et al.*, 1981). Students often choose answers that correspond to superficial aspects of a task - adding all the numbers in a diagram, for instance, without regard to the prompt. Yet this performance is in no way attributable to an absence of area and perimeter from the middle school (ages12 to 14) curriculum. On the contrary, formulas for area and perimeter of rectangles are ubiquitous in textbooks. The current curricular reform seeks to address the lack of depth in student understanding of these topics by focusing on the concepts rather than on the formulas - teaching for understanding. This leaves open the question of what a deep understanding of perimeter might look like.

What is perimeter?

The standard conception of perimeter is, as the Greek root would suggest, *distance around*. But distance around may not tell the whole story of perimeter. Look at the *Punctured square* [1] (see Figure 2). What is its perimeter? There are arguably three reasonable answers [2] to the question of this figure's perimeter. I will present each as the logical conclusion based on a development of school curriculum.

Answer 1: This figure has no perimeter

According to the archetypal *New Math* curriculum, written by the School Mathematics Study Group (SMSG, 1965), perimeter is not defined for the *Punctured square*. SMSG develops perimeter in second grade (age 7) as part of the study of linear measure (SMSG, 1965). Students first measure the lengths of line segments, both directly and by measuring a string with the same length. They then move on to using a string to measure the length of a curve. Finally, the curve is closed and perimeter is defined as the "length of a polygon" (*ibid.* p. 526). The *Punctured square* is not a polygon. Therefore, perimeter is not defined for it.

Mathematically, the distinction here is fundamental. The two remaining curricular approaches treat this one figure as

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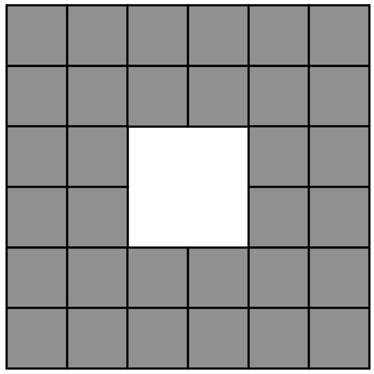

Figure 2: The 'Punctured square' – What is its perimeter?

having two simultaneous measures, area and perimeter. The SMSG approach makes a distinction between the 'object for which perimeter is a measure' and the 'object for which area is a measure'. The first object must by definition be a polygon. The second object can be any region [3].

A polygon has no area. A polygon is a simple closed curve composed entirely of straight edges. There are two polygons implied in the *Punctured square* – an outer square with perimeter 24 cm and an inner square with perimeter 8 cm (see Figure 3).

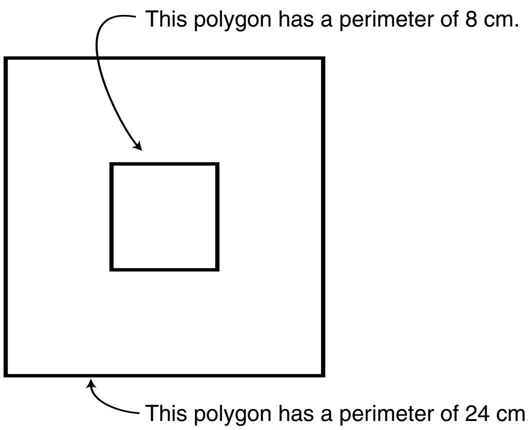

Figure 3: Two polygons – two perimeters.

Strictly speaking, neither of these polygons has an area [4]. We may consider the region that they bound, however. This is what is shaded in Figure 2. This region has an area of 32 square units.

Answer 2: 24 units

Nearly every middle school curriculum defines perimeter as the "distance around a figure" (*e.g.*, Charles, *et al.*, 2004, p. 441). "Figure" is not defined, but students are shown only polygonal and other simply connected regions. Students are often instructed, as in SMSG, to measure this by wrapping a string around the figure. The net impression ought to be that, whenever we want to find the perimeter of an object, we can wrap a string around it, measure the string and take that measure for the perimeter of the object.

Note, though, that "distance around a figure" does not explicitly exclude the *Punctured square*, given that "figure" remains undefined. Therefore, to find its perimeter, we should wrap a string around it and measure the string, which will be 24 cm long (Figure 4).

In this treatment, the *Punctured square* has both an area (32 square units) and a perimeter. Both measures are taken for the same object, in contrast with the SMSG approach.

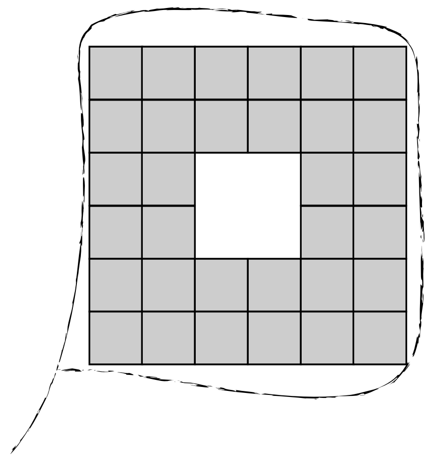

Figure 4: Measuring the distance around the figure by wrapping a string.

Answer 3: 32 units

Connected Mathematics (CMP; Lappan *et al.*, 2001), a *reform* curriculum, has a unit on area and perimeter in sixth grade (age 12). The topics are introduced in the context of bumper-car rides. Students begin by using square tiles to model the rides. Area is identified with the number of tiles covering the floor of the ride, whilst perimeter is identified with the number of unit rails needed to surround the ride to keep the cars from falling over the edge.

In my own classroom experience, students have quickly picked up on this idea and drawn two conclusions:

- wherever a tile's edge does not meet another tile's edge, a rail is needed
- wherever a rail is needed, we count one unit of perimeter.

For the usual geometric figures (*e.g.*, polygons), the number of rails keeping the cars on the track coincides with the distance around the track.

One question in the unit asks students to "design an interesting ride with lots of rails to bump against." It is quite common for students to remove tiles from the center of the ride, creating a hole – not unlike the *Punctured square* (see Figure 5).

We need to surround the hole with rails to keep the cars on the ride. The *Punctured square* ride requires eight units of rail around the hole, which are added to the 24 around the outside for a total perimeter of 32 units.

Discussion

Of the three definitions, SMSG's is the most mathematically precise. The development towards the definition is consistent

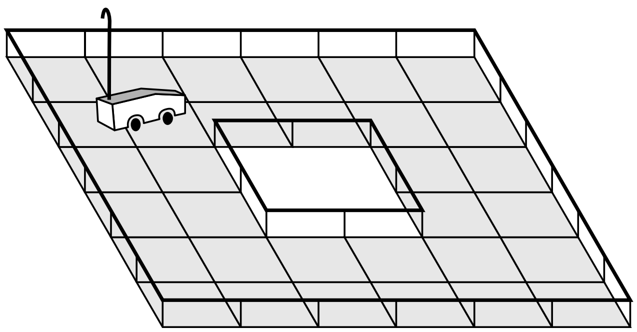

Figure 5: The 'Punctured square' as a bumper-car ride.

and logical. Yet it is limited. A student might wonder whether a circle, which is not a polygon, has a perimeter. This can be covered with an extension of meaning; any figure, like a circle, that can be closely approximated by a many-sided polygon can be said to have a perimeter. But many (*e.g.*, Kline, 1973) have criticized this sort of mathematical formalism in the elementary school (ages 5 to 12) curriculum. The traditional definition is at most a pseudo-definition, relying on ambiguous undefined terms (what is a "figure", anyway?).

This sort of imprecise definition pervades the traditional curriculum, but is rarely critiqued. CMP's definition is not really a definition – being conceptual, not formal – but it also does not masquerade as a mathematical definition. By encouraging students to imagine a plane figure as a bumper-car ride, it intends to draw students' attention to an important feature of 2-dimensional figures and to distinguish that feature (perimeter) from others (*e.g.*, area).

Once students have imagined figures as bumper-car rides, the extensions are not problematic: a hole in the track needs to be surrounded by rails, and, for instance, the rails could be bent to allow non-polygonal edges. CMP's reliance on context, to help students identify and make sense of mathematical ideas such as perimeter, is a hallmark of the current U.S. mathematics curriculum reform. Whereas the students interviewed by Woodward and Byrd were likely to have learned formulas for area and perimeter of a rectangle and were then asked to solve problems such as finding the amount of fence needed for a garden with particular dimensions, students in today's reform classrooms are more likely to learn about area and perimeter by considering questions about gardens and fences before deriving a formula.

Contexts introduce ambiguities. In the bumper-car example, we may ask whether the perimeter is really the total number of rails or whether it is the number of rails around certain special kinds of tracks.

To illustrate these ambiguities further, imagine the *Punctured square* as a lake. In the middle of the lake is an island, unconnected to the surrounding land. On a nice summer evening, one might stroll around the lake. This walk would be of length 24 units (the distance around). One can claim to have walked around the lake without setting foot on the island. On that same evening, however, a park manager is trying to quantify the sand on the lake's beaches [5]. She will need to know the distance around the island as well as the distance around the lake. Which of these measures is truly the perimeter? Those who object to finding the perimeter of the *Punctured square* would need to account for why lakes without islands have perimeters, but lakes with islands do not.

The mathematics (but not necessarily the contextual ambiguity) is resolved by the concept of *boundary*. In the bumper car and beach examples, we are interested in measuring what lies between the figure (bumper-car ride or lake) and everything else. This between place is the boundary of the figure. For young children, this is not a problematic concept. Children know about fences, rails, curbs and edges. All these are instantiations of boundary. Yet rails do not lead directly to a mathematical definition. For this, we require an understanding that plane figures are made up of points and consider the points individually – each either is or is not a boundary point and so the boundary of a figure is a set of points with particular properties. The argument here is not for more mathematical precision, it is for giving students intuitive notions about important mathematical ideas. These notions can be formalized later.

On a bumper-car ride, the rails make up the boundary. For a lake, the boundary is called the shoreline. In its standard curricular incarnation, a garden has no holes [6]. There is no need for internal fencing. The length of the garden's boundary is the same as the garden's perimeter. The same holds for lakes without islands and bumper-car rides without holes in the middle. For *simply connected* regions such as these, the length of the boundary coincides with the distance around. Perimeter is a special case of boundary. In the precise mathematical spirit of SMSG, perimeter may be defined as the length of the boundary of a simply connected region. Boundary, however, is the more general mathematical concept.

Boundary applies in all dimensions. The boundary of a line segment consists of its two endpoints. The boundary of a cube consists of its faces. Boundary is also a big idea mathematically. In complex analysis, the behavior of an analytic function on the boundary of a region puts limits on its behavior for the interior of the region. Many important sets consist entirely of boundary points. The Mandelbrot set, among other fractals, consists of the boundary points of a set in the complex plane. The rational numbers consist entirely of boundary points [7].

In a research interview, a twelve-year old student offered "the peel of an orange" as an example of perimeter. Strictly speaking, this is incorrect. If we think of perimeter as a boundary, however, the student is on the right mathematical track. The peel certainly is the boundary of the orange. We would use surface area to measure the peel of the orange in precisely the same way we use perimeter to measure the boundary of a garden. The concept is the same. The difference is in the dimensions. Surface area is the measure of the boundary of a (simply connected) 3-dimensional object. Perimeter is the measure of the boundary of a (simply connected) 2-dimensional object.

Conclusion

It is an obvious truism that curriculum is one of the fundamental ways we communicate mathematical ideas to students. In the New Math era, mathematicians considered quite carefully the message about perimeter they wished to communicate to students – perimeter is a linear measure of a simple closed (polygonal) curve.

The treatment in current standard U.S. mathematics textbooks (those that are often called *traditional* to distinguish them from the *reform* curricula) is problematic. "Distance around a figure" has the flavor of a mathematical definition. Yet, beneath this surface of definitional certainty is ambiguity. What, in fact, should count as a figure? There is no definition of "figure". Yet surely some readers will object to the idea of discussing the perimeter of the *Punctured square*.

The CMP treatment is also ambiguous. Students and teachers frequently deal with shapes like the *Punctured square* because the context allows for the possibility that students will create such a figure. In so doing, they bring the ambiguity to the forefront. From this ambiguity can come definition. Thinkers such as Lakatos (1981) have argued that this is the nature of doing mathematics.

In Lakatos's language, the *Punctured square* is a *monster*. We always have two choices when monsters arise – we may hide them in the closet (the *traditional* curriculum) or we may deal with them directly. SMSG dealt directly with the monsters in advance by defining them out of existence. In CMP classrooms, the *Punctured square* monster is dealt with directly when it is born. Some teachers define the monster out of existence ("That is a lovely bumper-car ride. When we talk about perimeter, though, we will only deal with figures that do not have holes.") Some consider the larger construct of boundary ("That is a lovely bumper-car ride. Sure, you can count the rails around the hole in the perimeter. But most of the figures we will study will not have holes.")

In the current reform effort, contexts have been carefully chosen to promote student thinking about the same mathematical ideas that SMSG promoted. Authors have chosen contexts because they are interesting to students and draw attention to these important mathematical ideas. As a result, students are sometimes pointed in the direction of more general, more mathematical concepts than appear at first glance. It seems a reasonable conjecture that many CMP students are developing a boundary conception of perimeter. The small study cited above supports this. Students in this classroom were evenly split on whether the *Punctured square*'s perimeter was 24 units or 32 units, but none expressed reservations about measuring the perimeter of such a figure. This ought not to be troubling. With relatively little effort it seems possible to teach an important mathematical idea (boundary) at the same time that we teach a trivial special case (perimeter).

Moreover, a boundary conception of perimeter might explain some of the obstacles that occur in classrooms.

Consider two examples.

Firstly, in my own classroom, students often observed at the beginning of our study of surface area that surface area was "like perimeter". Given that perimeter is a one-dimensional measure and surface area is two-dimensional, this seemed wrong and I discouraged the idea. I now understand that each is an instance of boundary and, as a result, I would encourage discussion of the similarities and differences. The boundary conception might lead to a better understanding of both surface area and perimeter.

Secondly, my informal observation that students had trouble shifting from finding the perimeters of their tiled shapes by counting edges to finding perimeters of irregular shapes using a string. If a perimeter is a distance, then the string preserves the perimeter once we have straightened it to compare to a ruler. However, if perimeter is a boundary, then the string does not preserve perimeter. A straightened string bounds no region. To a student with a boundary conception of perimeter, the straightened string might bear no relation whatsoever to the original figure. Measuring that straightened string may seem arbitrary. Neither of these hypotheses has yet been investigated.

Woodward and Byrd (1983) argued that we ought to teach the concept of perimeter, not just to cover it as a topic. The bumper-car context is one curriculum's attempt to do this, but perimeter is not conceptually rich. Instead, students seem to be working on the much richer concept of boundary. As long as teachers are prepared to help students to understand the relationship between perimeter and boundary, this is a step in the direction of conceptual understanding of important mathematics.

Notes

[1] Named this with the acknowledgment that mathematicians generally mean something else by *punctured* – that only one point is removed from the interior of the region.

[2] Assuming the conventional agreements about textbook diagrams: that the small squares are unit squares and that we are considering the shaded region to be the figure.

[3] This is not quite correct. A *region* is an open, connected set in the plane. In the current context it is not necessary that the set be open. *Connected set* feels too technical for an analysis of curricula for 5 to 14 year olds, but the reader is welcome to make the substitution wherever region appears.

[4] Alternatively, each polygon has an area equal to zero. The point is that a polygon is a one-dimensional figure whilst area is a two-dimensional measure. Note that this discussion is based on the definition of polygon given above, and so it is quite formal mathematically.

[5] Indeed, the ambiguities multiply. Are there beaches on the island? Can people get to the island to swim on those beaches? Is it a beach if no one swims there?

[6] You might at this point ask, "Where then is one to plant one's tomatoes?"

[7] To see this, we need the formal definition of a boundary point. A point, a, is a boundary point of a set, S, if every small neighborhood of a contains both points in S and points outside of S. Any neighborhood of any rational number contains both rationals and irrationals, thus every rational number is a boundary point. This should make clear that the formal definition of boundary is not a reasonable goal for elementary school mathematics.

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