Communications

But are students communicating mathematically?

JENNI INGRAM, ANNE WATSON

We aim to open up discussion about the intertwined roles of teachers and tasks that involve students communicating about mathematics when working in groups. Over many years we have observed, researched and ourselves have taught students working on mathematics in groups and find that it is often easier to pay attention to the forms of communication rather than its mathematical content. The notion of 'groupworthy' tasks has become a popular focus and communication an explicit component of curricula in many countries; it is timely to consider further the mathematical content of communication while working in groups on such tasks.

Context

There has been considerable interest both in research and classroom practice in tasks that encourage and support students' communication within mathematics lessons. Some published tasks emphasise communication, team working skills, and the contributions individuals can make to the task, but the mathematics that can be learnt by participating is less obvious. For other tasks the focus is clearly on the mathematics and the communication arises from working with others on the task. However, for many group tasks it is teacher intervention that enables students to develop mathematical understanding through communicating mathematical ideas (Moschkovich, 1999; Ryve, Nilsson & Pettersson, 2013). The theoretical background of such considerations is that mathematics is socially constructed, and we are not contradicting this, but we have observed, in classrooms where group work is becoming the norm, acceptance of this stance can be superficial and uncritical if it is assumed that any and all communication around tasks necessarily benefits learning. As Pimm (1987) asked:

> Is more pupil talk, purely measured quantitatively, to be seen as a desirable end in its own right? Is pupil talk better seen as a vehicle to certain ends? And, if so, is every type of pupil talk equally important, or should certain styles of talking be encouraged and others discouraged (p. 38).

There is considerable interest at the moment around the design and use of tasks that support students to communicate mathematical ideas (*e.g.* Johnson, Coles & Clarke, 2017). One aspect of this is the popularity of 'groupworthy' tasks. Most definitions of this term derive originally from Lotan (2003), that is, tasks are:

> open-ended and require complex problem solving; they provide students with multiple entry points to the task and multiple opportunities to show intellectual competence; they deal with discipline-based, intellectually important content; they require positive interdependence as well as individual accountability; and they include clear criteria for the valuation of the group's product. (p. 72)

This definition avoids the question of whether and how anyone learns any new-for-them mathematics, the focus being on 'showing intellectual competence' which can often be interpreted as the completion of the task or a group product. In our research, and when working with teachers, we observe, in the design of tasks, differences in the purposes for communication that afford differences in the mathematical content of the ensuing communication.

Good task design is driven by, among other things, underlying principles about how students learn (Watson & Ohtani, 2015). These principles often include participation in generic mathematical practices such as conjecturing, explaining, justifying or evaluating solutions. Malcolm Swan (2006) lists 11 design components of which two specifically address the mathematical content of discussion:

> The teacher's role is to encourage articulation of intuitive viewpoints, challenge with alternative viewpoints when these do not arise spontaneously, and facilitate reformulation of ideas by mediating learning through language which enables the student to construct his or her new concepts.
>
> The teacher may provide 'scaffolding'—conceptual resources necessary for a *higher level of cognitive functioning* (p. 79, our italics)

In his tasks, working and discussing in groups is assumed, tasks shape learners' experience and pedagogy focuses on cognition such as by introducing dissonance or technical terms to be used and particular representations to be compared (see also Ryve *et al.,* 2013). Crucially, the term 'scaffolding' here does not mean breaking a task into steps but structuring the reformulation of old ideas, forming new ideas, and using new conceptual resources, *i.e.,* individual learning. Swan emphasises the mathematics in the discussion; the structure and social features of discussion are the pedagogic context for the cognitive aim, not the aim itself.

Our observation is that teachers' practice, when adopting the norms of group work, often focuses on the mechanics of discussion, rather than its mathematical content. Professional development materials often focus on rules and roles for discussion rather than tools for mathematical communication such as giving examples, drawing diagrams, constructing equivalent expressions, comparing symbolic formulations, suggesting extensions of domains of application, offering counterexamples and co-constructing chains of reasoning as analysed by Sierpinska (2004). For example, in Mercer's widely promulgated work about talk, students are asked to reflect on their turn-taking and participation and are not asked to (also) reflect on the mathematics they have learnt (*e.g.,* Mercer & Sams, 2006). Chan and Clarke (2017), like Moschkovich (1999), emphasise the importance of considering the mathematical aspects alongside the social aspects of interaction.

In this communication, we explore these issues by referring to three tasks that have been considered groupworthy for at least 30 years, having been used widely in the UK and elsewhere.

1. Getting together with squares

The focus of this task is cooperation between group members to complete five squares when 5 participants are given a jumbled partial set of pieces (as shown in Figure 1). Each person has to have a complete square in front of them at the end. People can give a piece to another person but cannot take one or communicate that they need a particular piece. The task has the qualities listed by Lotan.

Students learn how to recognise and respond to the needs of others; social inclusion can also be an aim. Indeed, we originally came across this task in a personal development context (Kingston Friends, 1989) having little to do with mathematics. But what mathematics is likely to be learnt? As the task is completed in silence there is no requirement to verbalise fractions of the squares, lengths of sides or symmetries within the jigsaws. Whilst some students may recognise and enact them (such as silently ‘measuring’ a diagonal against a potential side length) the squares can also be completed, even by one person alone, as a trial-and-error jigsaw. Of course, any individual might learn something new from individually considering others’ actions and the outcomes, but there is nothing within the task as designed that makes this necessary, and no overt communication. These features have to be orchestrated by the teacher.

2. Great dodecahedron

The second task appears in a range of publications (*e.g.,* the cover and text of Pimm, 1987). A 2D image of the great dodecahedron coloured in a specific way is displayed and students are invited to describe anything that they can see, without pointing or gesturing. Individuals must communicate what they are seeing precisely enough for the other listening students to identify what is being described. The extent that students use mathematical language to describe what they can see depends on how the teacher structures the task. Some students may describe cubes, pyramids or triangles drawing on their mathematical properties, whilst others describe seeing ‘a duck’s beak’. The mathematical language and learning must be managed by the teacher. Without the group there is no task; it is the refinement and precision necessary to communicate and appreciate alternative mathematical views that provides the mathematical content.

3. Card matching tasks

Carefully designed card-matching tasks can focus students’ attention on mathematical relationships between their contents. In one particular task students need to match the different representations of a function: a graph; a table of values; an equation; a covariation description (Swan, 2008). The design of Swan’s tasks, underpinned by his design principles mentioned above, means that students must discuss mathematical features in order to connect the representations; the task cannot be completed using one-to-one matching and groups may have to construct extra cards to complete the task. The matching could be undertaken by individuals, but the presence of alternative viewpoints and the need to appreciate and agree these to complete the task requires explanation and justification based on mathematical properties and relationships. A well designed set can draw participants’ attention to common misconceptions and notational ambiguities and requires these to be resolved. Communication has to be about mathematical ideas, whether in formal or informal language, to complete the task.

Discussion

With all three tasks it is possible for a teacher to orchestrate post-task or within-task communication of mathematical processes and ideas and of the argumentation that took place or could have taken place.

In Case 1 it is possible to complete the group task by using only what is already available round the table, plus social group work and generic problem-solving strategies. In Case 2 there is no group production to complete; the aim is to extend the understanding of individuals. In these cases individuals might learn something new, either from each other or from subsequent review. Case 3 is, in our experience, different in that completion of tasks like this depends not only on existing knowledge, but on transforming knowledge to fit the scenario of making connections. The content on the cards requires resolution of conceptual distinctions and extension beyond what is familiar. It is possible to design card-matchings that are soluble using non-mathematical reasoning. Our interest is not, however, primarily about what new learning the tasks might afford but in whether and how mathematically focused communication is necessarily afforded by the task, rather than relying on teaching skills.

All the tasks afford the opportunities to use ‘intellectually important content’ and to generate ‘positive interdependence’. All three tasks are frequently deemed groupworthy by teachers and used for group work, but only Case 3

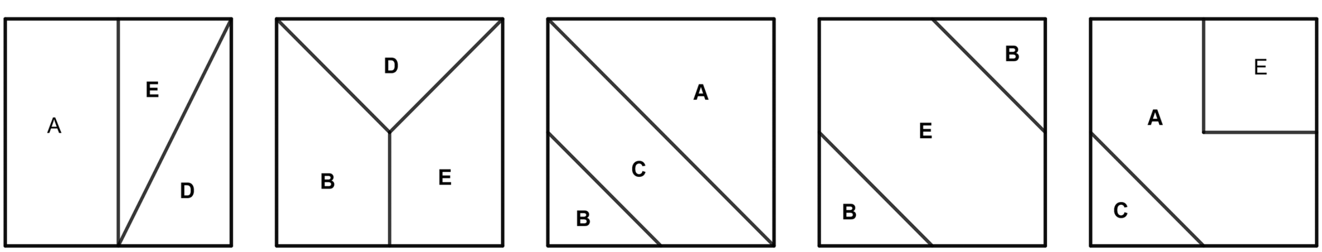

Figure 1. Getting together with squares.(SMP, 1992, p. 14; similar to nrich.maths.org/6936)

requires a variety of mathematical communication. It is the range of mathematical communication and interactions within this range that is of interest to us as researchers, rather than the more social aspects or the impact on specific individuals.

Concluding remarks

In these three tasks, only the card matching tasks with their mathematically specific design features appear to embed a need for mathematical communication rather than this being dependent on a teacher's pedagogical insight and strategy. The cognitive transformations that might be actualised for individuals through the use of groupworthy tasks are often not essential components in their design. In explorations of websites we have found 'groupworthy' tasks for which the nature and roles of inclusive participation seem to be the priority, but we have also found examples of tasks where the mathematical content is likely to lead to specific discussion of mathematical ideas, as in Swan's tasks [1]. A key ingredient is the need to discuss which of several possible mathematical properties are necessary. Research about the range and nature of mathematical communication needs to include attention to the how and what of mathematics in talk, not only the who and why.

Note

[1] See, *e.g.*, http://francesharper.com/complex-instruction/groupworthy-mathematics-tasks/

References

Chan, M.C.E. & Clarke, D. (2017) Structured affordances in the use of open-ended tasks to facilitate collaborative problem solving. *ZDM* **49**, 951-963.

Johnson, H.L., Coles, A. & Clarke, D. (2017) Mathematical tasks and the student: navigating "tensions of intentions" between designers, teachers, and students. *ZDM* **49**, 813-822.

Kingston Friends (1989) *Ways and Means.* Kingston, UK: Kingston Friends Workshop Group.

Lotan, R.A. (2003) Group-worthy tasks. *Educational Leadership* **60**(6), 72-75.

Mercer, N. & Sams, C. (2006) Teaching children how to use language to solve maths problems. *Language and Education* **20**(6), 507-528.

Moschkovich, J. (1999) Supporting the participation of English language learners in mathematical discussions. *For the Learning of Mathematics* **19**(1), 11-19.

Pimm, D. (1987) *Speaking Mathematically: Communication in Mathematics Classrooms.* London: Routledge and Kegan Paul.

Ryve, A., Nilsson, P. & Pettersson, K. (2013) Analyzing effective communication in mathematics group work: the role of visual mediators and technical terms. *Educational Studies in Mathematics* **82**(3), 497-514.

Sierpinska, A. (2004) Discoursing mathematics away. In Kilpatrick, J., Hoyles, C. & Skovsmore, O. (Eds.) *Meaning in Mathematics Education,* pp. 1-26. Dordrecht: Kluwer Academic Publishers.

SMP (1992) *SMP 11-16 Using Groupwork Sample Pack: Activities for the Mathematics Classroom.* Cambridge: Cambridge University Press.

Swan, M. (2006) *Collaborative Learning in Mathematics: A Challenge to our Beliefs and Practices.* London: National Institute for Advanced and Continuing Education (NIACE).

Swan, M. (2008) Designing a multiple representation learning experience in secondary algebra. *Educational Designer* **1**(1), unpaginated.

Watson, A. & Ohtani, M. (2015) Themes and issues in mathematics education concerning task design: editorial introduction. In Watson, A. & Ohtani, M. (Eds.) *Task Design in Mathematics Education. New ICMI Study Series,* pp. 3-15. New York: Springer.

Aesthetics as philosophy for mathematics education

NATHALIE SINCLAIR

As Leonard Koren (2010) makes clear in his book *Which 'aesthetics' do you mean: Ten definitions,* the word 'aesthetic' can mean very different things, including both the quality of experience one might have solving a particular problem and the particular style one might ascribe to a work of art. In my recent work, I have been interested in Jacques Rancière's conceptualisation of aesthetics in relation to politics. This connection arises when considering the relation between what can be sensed—what can be perceived, seen, felt—and what is taken as 'common sense', that is, as a social shared understanding of what makes sense. Art can function politically when it disrupts what is taken to make sense by changing the distribution of the senses. According to Rancière (2004), the tension of aesthetic sense—an ongoing tension in how the autonomy of the aesthetic sense is set against the aspiration to *live* it as a sensibility—is the source of its political power.

Rancière argues that aesthetic practices operate through an uneasy mix of *autonomy* and *dependence*. Such practices are, on the one hand, free from the demands of functionality and explanation (not being about utility) and, on the other, completely reliant on sensory perceptions (seen, heard or touched). As de Freitas and Sinclair argue, "The aesthetic operates through the dream of an unavailable ideal form that must be made flesh and possessed as reality" (2014, p. 179). We show how this same conjoining of autonomy and dependence is at work in the mathematics aesthetic. For example, the visual or numerical pattern that the mathematician perceives through sensory modalities is granted a certain autonomy as it comes to operate according to an intrinsic logic, independent of and impervious to human intervention. Of course, a mathematician might 'explain' a pattern by referring to operations she used to produce it (tripling a number or adding another diagonal to the polygon), but this activity does not engender the pattern.

My growing sensibility to autonomy in mathematics was at play as I read Stott's (2018) description in the previous issue of FLM of the aesthetic experience of a young girl named Anathi solving a mathematical problem. Stott describes the girl's emotional response, both verbal exclamations and bodily movements, as she figures out the value of the club symbol in her mathematical puzzle. I was especially interested in the first panel of Stott's comic strip (the first scene of the aesthetic experience), where Anathi can be seen counting with her fingers, from 13 all the way to 25 [1], an activity that would have stretched over a period of time, which the snapshot of the panel does not capture well: thirteen, fourteen, fifteen, sixteen, seventeen, eighteen, nineteen, twenty, twenty-one, twenty-two, twenty-three, twenty-four, twenty-five (32 syllables).

In the practice of Anathi, both autonomy and dependence are at work. There is a ritual enactment of the autonomy of counting and, at the same time, the body is complicit in performing this autonomy through number naming (and hence, through the movement of the mouth, throat) and finger pointing and touching. In the autonomy lies a mechanical process—without thinking, she can just point to and touch with her fingers, saying number names aloud. Moreover, this process will work *no matter which numbers* she starts and ends with: she just needs to read the answer off her hands at the end.

The hands, in fact, epitomise the delicate interplay of autonomy and dependence, as Anathi counts on her fingers while also counting with them (one pointing, one being pointed at). As Sinclair and Pimm (2015) write, this dual role is well captured by the expressions 'using fingers to count with' and 'using fingers to count on'. Fingers are both what Rotman (1987, p. 27) calls the 'one-who-counts' (counting with my fingers) as well as the 'things-to-be-counted' (counting on my fingers). Fingers are thus:

> simultaneously subject and object, both of the person and of the world. In inhabiting this dual status (being both me and not-me), fingers provide echoes of the analyst Donald Winnicott's notion of 'transitional object': "an intermediate area of experiencing to which inner reality and external life both contribute" (1971, p. 2). (Sinclair & Pimm, p. 100)

Interestingly, the many mathematics educators who admonish the use of fingers in children's counting may also inadvertently be thwarting the kind of aesthetic experience reported in Stott's article, by preventing the conjoining of autonomy and dependence.

By recruiting Radford's (2010) version of poēsis, Stott emphasises the creative nature of Anathi's aesthetic experience and its emergence from a "complex arrangement of gestures/exclamations, language, gaze and body movements/postures" (p. 8). These material embodiments seem to be necessary for the establishment of dependence, for the living of mathematics through the senses. But if we follow Rancière, it is through examining the instances of or impulses towards separation that we can understand how the aesthetic might function in school mathematics. How might the dream of the ideal which instigates autonomy be triggered within or through sensory-rich mathematical activity which dwells in dependence?

This question, and its relation to the experience of Anathi, seems relevant to the findings of research on children's number fluency. For example, in discussing strategies that learners can use to add and subtract numbers, Geary (2011) asserts that decomposing numbers (solving 8+5 by first doing 8+2 and then adding 3) is more sophisticated than concrete counting with fingers and manipulatives because it depends on mental processes and memory-based mental representations. Indeed, children in the fourth grade had higher standardised test scores if they showed *a preference for* mental strategies such as decomposition when they were tested, two years earlier, in the second grade (Carr & Alexeev, 2011). Might these learners have succeeded in pursuing autonomy at the price of giving up dependence? Might there be a different kind of autonomy that arises from the new distribution of the senses in which fingers and manipulatives are abandoned?

This line of questioning echoes some of the issues that have come up in the long-standing discussion around the place of the concrete and the abstract in mathematics education. The language of the researchers mentioned above is clearly one that privileges the abstract, while the conversation around the use of fingers and manipulatives in early mathematics education tends more towards the concrete. If one goal of school mathematics is to occasion aesthetic experiences or to expose learners to the mathematical aesthetic (as a significant aspect of the discipline), then perhaps our aim is to figure out how to place the concrete and abstract side-by-side to make them live together. In Stott's example, asking Anathi to use decomposition to accomplish her subtraction (25–13) may well have been more 'sophisticated' and perhaps even more efficient (quicker then uttering 32 syllables) [2], but would that delicate tension between autonomy and dependence have survived? I think probably not.

Returning to the beginning of this commentary, a major contribution of Leonard Koren to the question of aesthetics has been in exploring the *wabi-sabi* aesthetic that is characteristic of Japanese pottery and that values imperfections, temporality, incompleteness, earthly crudeness and contradiction. Maheux (2016) describes how this kind of aesthetic might be at play in teachers' appreciation of the tentative, vague and perhaps cumbersome practices of students which contrast with the more precise, clear and symmetric style that might be valued in the dominant mathematical aesthetic. In a *wabi-sabi* aesthetic that values process (over speed or product), for example, counting on your fingers could be seen as the 'slow math' counterpart to fast food which may be efficient and sophisticated, but have shed an appreciation for process.

On a final note, it is perhaps not surprising that the first panel of the comic strip was the one that caught my attention because, along with Rancière, I have been trying to grapple with Harman's (2007) thinking about aesthetics which directs attention to that which *precedes* or even causes perception, apprehension, comprehension. Harman argues for "aesthetics as first philosophy" (not ethics, nor metaphysics) inasmuch as aesthetics is about how "individual substances interact in their proximity to one another"—where those individuals are not only humans, and not only animals (whose aesthetic appeal we already recognise), but also non-human beings such as rocks and milkweed and "snowflakes [that] rustle the needles of the quivering pine". For Harman, this interaction occurs through a proximal relation in which one object affects another, where one object "*allude[s]* to the reality of the other even while brushing against its surface". Allure thus becomes a phenomenon of causal relations, the underlying principle of change.

But allure, in belonging to the world, also occupies its multiple scales, including the individual scale of human movement (Anathi's fingers) and the micro-phenomenological scale of what Whitehead (1967) might term pre-individual 'vibration'—a concept he used to describe the agency of the environment itself. In that environment, Anathi's fingers are entangled in an atmospheric complex of heat, light and pressure jumbled together with pencil, paper, desk and also her classmate's and the researcher's bodies. These vibrations can be thought of as pre-affective potentialities of the universe that we are now starting to see/capture/measure with the help of new technologies (eye-tracking, mass spectrometry tracing organic compounds in exhaled breath, fMRI, *etc.*). We might thus ask how does Anathi's hand allude to the reality of the mathematical task, what causal relations are at play before she consciously decides to count, laugh, throw up her arms?

Harman and Rancière may seem worlds apart, but their work can both be seen as exploring and exploding the conjunction *and*. For Rancière, the aesthetic is about autonomy *and* dependence; for Harman, it is about touching *and* not touching, communicating through proximity *and* emerging as an independent entity. As is evident in the history of western civilisation, we have a tendency to turn *ands* into *ors*, that is, to resolve into binaries and exclusiveness. An aesthetic philosophy for mathematics education could provide an alternative: a way to sustain more inclusive coordinating conjunctions.

Notes

[1] It is not clear in Stott's article how Anathi gets to twelve, even though she says all the numbers between 13 and 25, inclusive (which would be 13 numbers in all). Perhaps she says 'thirteen' but does not count it on her fingers.

[2] In the same way that direct algebraic expressions might be more efficient than recursive ones—and thus often preferred historically by mathematicians—technology always matters. With a spreadsheet at hand, recursion is just as quick.

Acknowledgements

Thanks to David Pimm and Ofer Marmur for reading and commenting on earlier drafts of this communication.

References

Carr, M. & Alexeev, N. (2011) Fluency, accuracy, and gender predict developmental trajectories of arithmetic strategies. *Journal of Educational Psychology* **103**(3), 617-631.

de Freitas, E. & Sinclair, N. (2014) *Mathematics and the Body: Material Entanglements in the Classroom.* New York: Cambridge University Press.

Geary, D.C. (2011) Cognitive predictors of achievement growth in mathematics: a 5-year longitudinal study. *Developmental Psychology* **47**(6), 1539-1552.

Harman, G. (2007) Aesthetics as first philosophy: Levinas and the non-human. *Naked Punch* **9**. Retrieved from http://www.nakedpunch.com/articles/147

Koren, L. (2010) *Which 'Aesthetics' Do You Mean: Ten Definitions.* Point Reyes, CA: Imperfect Publishing.

Maheux, J.-F. (2016) Wabi-Sabi mathematics. *The Journal of Humanistic Mathematics* **6**(1), 174-195.

Radford, L. (2010) The eye as a theoretician: Seeing structures in generalizing activities. *For the Learning of Mathematics* **30**(2), 2-7.

Rancière, J. (2004) (G. Rockhill, Trans.). *The Politics of Aesthetics: The Distribution of the Sensible.* New York: Continuum.

Rotman, B. (1987) *Signifying Nothing: The Semiotics of Zero.* London: MacMillan Press.

Sinclair, N. & Pimm, D. (2015) Mathematics using multiple senses: Developing finger gnosis with three- and four-year-olds in an era of multi-touch technologies. *Asia-Pacific Journal of Research in Early Childhood Education* **9**(3), 99-110.

Stott, D. (2018) Observing aesthetic experiences and poēsis in young students. *For the Learning of Mathematics* **38**(1), 7-11.

Whitehead, A. (1967) *Adventures of Ideas.* New York: The Free Press.

Alternative perspectives on cultural dimensions of proof in the mathematical curriculum: a reply to Shinno *et al.*

JENNIFER A. CZOCHER, PAUL CHRISTIAN DAWKINS, KEITH WEBER

In **38**(1), Yasuke Shinno and his colleagues explored a way to account for cultural dimensions when analyzing how proof is taught in different countries. The authors proposed a framework, Bosch and Gascon's (2014) 'reference epistemological model' (REM), to serve as a basic theoretical lens for "clarify[ing] what constitutes proof in the curriculum of a given country" (Shinno *et al.,* 2018, p. 29). The authors framed representation of proof in the Japanese curriculum in terms of the cultural and linguistic factors that may have influenced its development.

We believe that Shinno *et al.* have made an important contribution by drawing attention to this pivotal issue. In particular, we were persuaded by the authors' argument that one must attempt to detach oneself from one's own cultural institutions when analyzing the treatment of proof in a curriculum. Characteristics accepted as normative or necessary in one educational institution may not be shared or even acknowledged by scholars from institutions in other nations. Recognizing a need to articulate epistemological assumptions underlying a curriculum can expand our understandings of the conditions that may impact teaching and learning of proof. Further, as we will argue in this commentary, cross-cultural explorations of how proof is taught and learned can generate meaningful hypotheses pertaining to teaching of proof.

From our positions as scholars in the United States, we found that several of the suggestions made by Shinno *et al.* (2018) would not be relevant to the way that proof is taught in the United States or would not align with the research programs of American mathematics educators. We found these points of divergence interesting, and we use these differences to reveal implicit assumptions, both in the authors' arguments and in how proof is taught and investigated in the United States. We then use the assumptions that we highlighted to propose specific areas of future research.

Is there a single frame to investigate the teaching of proof?

A key argument in the authors' article is that researchers should identify and use REMs to investigate the treatment and development of proof in national curricula, something that the authors undertook with the Japanese curriculum. We observed three assumptions underlying this recommendation:

> (1) Proof *is* an object that is represented in the curriculum.
>
> (2) There *is* a single national curriculum that can be analyzed—or at least there is enough consistency *among* the curricula used in a nation that shared characteristics can be highlighted.

(3) For a given curriculum, there is an appreciable degree of consistency in and a progression for the treatment of proof across grade levels.

These assumptions appear to be satisfied in Japan, perhaps because educational researchers there play a central role in developing the mathematics curriculum (as Shinno *et al.*, 2018 discussed on p. 29). However, in the United States these assumptions do not hold. By and large, proving opportunities are limited or non-existent in the curriculum as enacted in many American classrooms. When proof is present in the intended curriculum, its role and expression can vary greatly. Because proof is viewed as secondary to other learning goals, it is often omitted or deemed appropriate only for the advanced students (*e.g.,* Bieda, 2010). Further, the program of study in American secondary schools tends to be segmented into content areas such that instances of proving usually do not build on the justifications students produce in prior courses or build toward what students encounter should they enroll in collegiate precalculus or calculus. This is in contrast to the Japanese curriculum's evolution from local to quasi-axiomatic. Similarly, whereas Japanese educational researchers contribute to the creation of a coherent national curriculum, in the United States, both research-based curriculum materials and commercial curricula generated by large publishing firms compete for adoption at the state and local levels. We do not mean to say that REMs cannot be a useful tool for exploring how proof is developed in United States classrooms, but given that less attention is paid to proof in general and proof's coherence in particular and given the variety of ways that proof is taught, a focus on REMs would shed less light on American teaching practice than Japanese teaching practices.

We raise these issues to suggest a promising avenue for research. It could be useful to compare the role of empirical research in shaping curricula in the United States and Japan and other nations (with regard to proof and in general). What are the differences in the fidelity of research recommendations and representation in the curricula? In the United States, innovative curricula often do not lead to learning gains if they are not accompanied by teacher development, amendments to high-stakes assessment and other factors. To what extent is this true in Japan and elsewhere? What nations encourage an integrated effort to incorporate research findings into curriculum? How have they done so? Answers to these questions would provide researchers with a better sense of the limitations and the consequences of the lack of coherence we have among theory, policy, and practice while also raising awareness about alternative arrangements.

Alternative proof progressions

The authors propose a specific REM to highlight what proof means in Japanese classroom and how this meaning evolves through the curriculum. Proof is initially based on 'the logic of the real world' where proofs employ pragmatic argumentation. Proof then shifts to 'local theory' where arguments are locally organized within the mathematical and classroom contexts in which they are discussed. Finally, proof is situated within a 'quasi-axiomatic theory' in which proofs are deduced from explicit axioms and definition. (The authors use the qualifier 'quasi-' to denote that words like 'axiom' and 'postulate' are not explicitly used in the Japanese curriculum). The authors claimed that these elements "essentially determine the teaching and learning of mathematical proof" (p. 29), at least in a Japanese context. We see the author's epistemological analysis as naturally situated within Chevellard's (1991) broader research program describing the transformation of mathematical objects and practices as they are recast as pedagogical objectives within educational institutions. Some scholars in this research program agree with Shinno *et al.* that how mathematical objects are transformed into pedagogical objects largely dictates how they will be taught, learned, and understood. In contrast, many American researchers adopt sociocultural approaches that foreground the emergence of mathematical activity from classroom interactions. These different research epistemologies suggest different accompanying progressions and REMs with respect to proof. For instance, one possible progression based on a sociocultural analysis would be: acceptable justifications in a classroom proof may initially have a pragmatic or axiomatic basis (depending on the classroom context), but over time, the backing for permissible inferences is neither based on a local theory or a global axiomatic theory. Instead, the backing derives from classroom mathematical practices. Through the students' collective work, certain inferential schemes become taken-as-shared and require no further justification (*e.g.*, Stephan & Rasmussen, 2002). Another progression, rooted in a constructivist approach, might be from preliminary theorems to more sophisticated theorems. In this case, later proofs rely on previously proven results so there would be decreasing reference to axiomatic structure. As Reid (2011) found, proofs become less explicit as more justifications and proof methods become part of the classroom toolbox. In this case, the progressively increasing explicitness illustrated by the authors' selections from the Japanese curriculum may not be visible. Indeed, the opposite trend may be more salient as explicit appeals to axioms and definitions may decrease over time. We offer these alternative progressions to illustrate why the progression of REMs Shinno *et al.* (2018) posited in the Japanese curriculum conflicts with some of our experiences in the American context. This further demonstrates the value of international comparisons, as the different theoretical dispositions of researchers from different cultures lead them to foreground different phenomena.

The relationship between language and REMs

The authors suggested that linguistic features of the Japanese language may have influenced the way that proof was developed in the Japanese curriculum. For instance, the Japanese words *shomei* which (roughly) means a proof that adds understanding and *ronsho* which (again roughly) means a proof based on logic. *Shomei* seems to characterize the proofs within local mathematical theories and is useful for deciding the truth of a proposition, while *ronsho* seems to characterize proofs within an axiomatic theory and is useful for evaluating the truth or falsity of a set of propositions. The authors suggested that perhaps the prevalence of the words *shomei* and *ronsho* in Japanese mathematical parlance contributed to local theories and axiomatized theories playing such a large role in framing proof in the Japanese curricula.

We note here that English-speaking authors have proposed similar learning trajectories. For instance, the British mathematician and mathematics educator, David Tall, proposed that students might first justify mathematical assertions in the embodied world (through perceptions and actions), then in the symbolic world (using arithmetic, algebra, and the 'rules' of arithmetic and algebra), and finally in the axiomatic formal world (through deduction from set theoretic definitions) (Tall, 2013, p. 14). From our perspective, Tall's progression parallels Shinno *et al.*'s progression from real-world logic to local mathematical theories to axiomatic theories. Similarly, Harel and Sowder (1998), mathematics educators who investigated proof in American classrooms, proposed a trajectory where students begin with empirical proof schemes, then progress to transformational proof schemes, and finally to axiomatic proof schemes. Again, we see strong parallels between the proof schemes trajectory and Shinno *et al.*'s progression. To us, it was notable that Tall, Harel, and Sowder's investigations of English speaking students led to progressions that aligned well with how proof is treated in the Japanese curricula, suggesting that the learning progression identified by Shinno *et al.* may have been due to the invariant aspects of mathematical reasoning rather than to peculiarities of the Japanese language.

Similarly, Shinno *et al.* observed that universal statements in Japanese textbooks were not explicitly quantified. The authors asserted that "this is probably due to the fact that the Japanese language does not use articles" (Shinno *et al.*, 2018, p. 27). However, universal statements in American classrooms are often not explicitly quantified either. In a popular undergraduate textbook, Hammack (2013) directly addressed this point:

> Now we come to the very important point. In mathematics, whenever $P(x)$ and $Q(x)$ are open sentences concerning elements x in some set S (depending on context), an expression of form $P(x) \Rightarrow Q(x)$ is understood to be the statement $\forall x \in S (P(x) \Rightarrow Q(x))$. In other words, if a conditional statement is not explicitly quantified then there is an implied universal quantifier in front of it. This is done because statements of the form $\forall x \in S,\ P(x) \Rightarrow Q(x)$ are so common in mathematics that we would get tired of putting the $\forall x \in S$ in front of them. (Hammack, 2013, p. 46).

We interpret this excerpt to mean that focus on universal statements is an invariant part of mathematical practice. As a result, mathematicians frequently do not explicitly quantify statements because it is taken-as-given that we are less concerned with material implications about specific objects. Again, this suggests that the issues that Shinno *et al.* highlighted were due to an invariant aspect of mathematical reasoning rather than the peculiarity that the Japanese language does not use articles. We fully support the authors' aim of trying to understand aspects of a nation's curriculum in terms of interesting features of that nation's language. What we suggest is that by looking across different nations' curricula, we can better understand what aspects are particular to that nation or are shared across geographies and cultures. This can provide insight into whether a feature of a nation's curricula is based on cultural or linguistic peculiarities or invariant aspects of mathematical practice.

References

Bieda, K.N. (2010) Enacting proof-related tasks in middle school mathematics: challenges and opportunities. *Journal for Research in Mathematics Education* **41**(4), 351-382.

Bosch, M. & Gascón, J. (2014) Introduction to the anthropological theory of the didactic (ATD). In Bikner-Ahsbahs A. & Prediger S. (Eds.) *Networking of Theories as a Research Practice in Mathematics Education,* pp. 67-83. Dordrecht: Springer.

Chevallard, Y. (1991) *La Transposition Didactique. Du Savoir Savant au Savoir Enseigné.* Grenoble: La Pensée Sauvage.

Hammack, R.H. (2013) *Book of Proof.* Richmond, VA: Richard Hammack.

Harel, G. & Sowder, L. (1998) Students' proof and schemes: Results from exploratory studies. In Schoenfeld, H., Kaput, J. & Dubinsky, E. (Eds.) *Research in Collegiate Mathematics Education III,* pp. 234-283. Providence, RI: American Mathematical Society.

Reid, D.A. (2011) Understanding proof and transforming teaching. In Weist, L. & Lamberg, T. (Eds.) *Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education,* pp. 15-30. Reno, NV: University of Nevada.

Shinno, Y., Miyakawa, T., Iwasaki, H., Kunimune, S., Mizoguchi, T, Ishii, T. & Abe, Y. (2018) Challenges in curriculum development for mathematical proof in secondary school: cultural dimensions to be considered. *For the Learning of Mathematics* **38**(1), 26-30.

Stephan, M. & Rasmussen, C. (2002) Classroom mathematical practices in differential equations. *The Journal of Mathematical Behavior* **21**(4), 459-490.

Tall, D. (2013) *How Humans Learn to Think Mathematically: Exploring the Three Worlds of Mathematics.* Cambridge, UK: Cambridge University Press.

On making epistemological inferences based on linguistic observations: a commentary on Shinno *et al.*

BORIS KOICHU

> Japanese language has two words for 'proof' and does not have articles, traces of these linguistic facts can be found in Japanese mathematics textbooks and may have epistemological nature? How interesting! And what about Israel?

This was my first reaction to the Shinno *et al.* (2018) article in **38**(1). Indeed, as Mason (2010) has argued, brief-and-vivid recollections of observations that would resonate or trigger the others' own recollections has special value in mathematics education research. For me, the article was definitely resonating and triggering. My second thought was about Shinno and colleagues' approach to situating their observations in international context, by means of a proposal for a particular *reference epistemological model* (REM). I asked myself: how can the proposed three-layer REM account for the complexity of teaching and learning proof in school mathematics instructions in different countries? I briefly discuss these two questions in this commentary.

How interesting! And what about Israel?

For me, the masterful discussion of *shomei* and *ronsho* in the Shinno *et al.* article suggests that the Japanese language is sensible to expressing differences between some of the functions of proof. Based on the contextual translations presented in the article, I have an impression that *shomei* alludes, in de Villiers' (1990) terms, to validation and explanation, and *ronsho* to validation and systematization. I also learned that many Japanese mathematics educators treat *shomei* as a category containing *ronsho*.

Two Hebrew words are frequently used in the official Israeli mathematics curricula: *hohaha* (translates as 'proof') and *hanmaka* (translates as both 'argumentation' and 'reasoning'). Similarly to Japan, Israeli curricula treat *hohaha* as a particular case of *hanmaka* where the former notion presumes mathematical rigor and the latter one is somewhat vague and alludes to 'providing an argument' without strictly prescribing what type of argument may be used. At this point, it is important to note that Israel has three levels of high-school mathematics curricula attuned to diverse aspirations of Israeli students. The advanced-level curriculum emphasizes proof as one of its central objectives and explicitly states that students must attain understanding of differences between proof, explanation and justification based on consideration of examples. The intermediate-level mathematics curriculum also indicates the importance of proof, but then uses the combined term *hohaha/hanmaka* for formulating the following objective: students must understand that mathematical statements are justified by inferences from what is given as well as from the previous knowledge, including definitions, theorems and 'previously made conclusions'. The introduction to the low-level curriculum does not contain the *hohaha* and *hanmaka* notions, but uses instead 'skills of logical thinking' that students must develop in school in order to properly function in the Israeli society as adults.

It would not be accurate, however, to equate *hohaha* with *ronsho* and *hanmaka* with *shomei* based on the above examples. In particular, the distinction between *hohaha* and *hanmaka* seems to relate to types rather than functions of argument in mathematics. To this end, I would like to mention that the Israeli advanced level curriculum adopts the word *hohaha* to make the distinction between "proofs that prove and proofs that explain" (Hanna,1990, p. 9).

Linguistic distinctions, though different, are explicit in the official curricula and research publications in Japan and Israel, but seem to be less visible in the mathematics textbooks of both countries. Indeed, as mentioned by Shinno *et al.* with reference to Miyakawa (2017), the *ronsho* notion does not appear in the Japanese textbooks. In Israel, Silverman (2017) analyzed justification-related aspects of eight middle-school mathematics textbooks. He found that a sizable portion of justifications in the textbooks consisted of one of two combinations of arguments: experimental demonstrations combined with deductions using a general case, and deductions using a specific case combined with deductions using a general case. For me, this finding sounds as if the idea of *shomei* was put forward in Israeli textbooks, but *ronsho* was in the minds of the textbook writers as part of their horizon knowledge. It also sounds as if the *hohaha* and *hanmaka* notions are complementary in Israeli middle-school textbooks, as well as in the intermediate-level high-school curriculum, as mentioned before.

My next comment is about the role of linguistic issues when considered in the transition from mathematics textbooks to classroom discourse and practices. In brief, I am going to argue that this role is probably even more subtle than is argued (though in a very careful manner) by Shinno and colleagues. To begin with, Israeli textbooks contain tasks requiring either *lehohiah* (to prove) or *lenamek* (to provide argument), but these two requests do not necessarily solicit essentially different classroom practices. The Israeli practice seems to conform to Stylianides' (2007) conceptualization of *proof in school.* To recall, this conceptualization takes into account modes of argumentation and modes of argument representation as developed in a particular classroom community. That is, it can involve more or less mathematical rigor as well as more or less demonstration, depending on the stage and level of mathematics study, on characteristics of the students and on the teacher's knowledge and preferences.

Further, Shinno *et al.* connect linguistic characteristics of the mathematics formulations in the textbooks with ways by which students are engaged with proof in a classroom. Specifically, they observe that universal quantifications rarely appear in Japanese textbooks and suggest that this fact may provide an explanation of the difficulties that prospective elementary school teachers have with universal aspects of proof. I find this connection intriguing. After all, Shinno *et al.* acknowledge that words such as 'any', 'arbitrary' and 'all' might be used orally by teachers when interpreting the textbook's mathematical statements and that it is difficult to account for the influence of such discursive moves on students' perception of proof. The authors use the Mejía-Ramos and Inglis' (2011) study on semantic contamination in support of this suggested connection, but it should not be overlooked that in that study the focus was on the use of noun and verb referents of 'proof' in natural language and not in mathematics textbooks.

My point is that cultural influences are important, but the role of the polished language in mathematics textbooks on students' perceptions of proof should not be overestimated. I further illustrate this point by the following anecdote. Modern Hebrew language includes instances of inventing new verbs that signify some widespread practices. For example, walking along Dizingof Street in Tel Aviv, popular for dining and shopping, is informally known as '*lehizdangef*' (the prefixes *le-* and *la-* appear in the infinitive form of verbs in Hebrew). This fact reminded me that a data set of one of my past studies contained instances of turning, by Israeli middle-school students, some frequently used proving moves into verbs. For example, the idea of proving that two triangles are congruent by applying the standard triangle congruence theorems is signified by a verb '*lahfof*', literally, 'to make triangles congruent'. The (invented) verb '*lahfof*' is a derivative from the noun '*hafifa*' and/or the adjective '*hofefim*', which in the context of geometry mean 'congruence' and 'congruent', respectively. The verb '*lahfof*' is broadly understood and taken-as-shared in Israeli classrooms, and one can attribute to this fact something important about the epistemology of proof in an Israeli context. Needless to say, no Israeli curriculum document or textbook endorses any such 'verbification' (Lunney Borden, 2011).

How can the suggested REM account for complexity of teaching and learning proof in international context?

The REM developed by Shinno *et al.* creates a clear image of gradual progression, apparently across the grades, from the logic of the real world to the local theories and towards axiomatic theory, in relation to students' engagement with proof. The scholars indicate that the three-layer REM "is based on some prior research results in the international context" (p. 29). Therefore, it is just reasonable that the REM looks aligned with some other three-layer frameworks for understanding and teaching mathematical proof offered by representatives of the Western tradition (*cf.* Tall, 2008, for a paradigmatic example). After all, the Western and Eastern traditions of doing mathematics, as different as they are, have also a lot in common. As Man Keung Siu noted in his panel presentation at ICMI-19, "there is something about mathematics that is universal, irrespective of race, culture or social context" (quoted in Hanna & de Villiers, 2014, p. 10). In addition, Shinno *et al.* discuss how the REM is informed by the *shomei-ronsho* distinction, and thus they make a case for the suggested REM as a curriculum development tool that can either be used nationally (in Japan) or internationally.

In my view, an extent to which the suggested REM is sensitive to linguistic and cultural peculiarities of teaching proof in different countries requires further elaboration. As I was trying to illustrate in the previous section by examples based on an Israeli context, some linguistic issues can be visible in the policy-making documents, but less visible in the mathematics textbooks and even less visible in classroom discourse and practice. I have also presented arguments in support of the opposite direction, namely, culturally loaded classroom discourse, though less rigorous and coherent than textbook discourse, may convey important messages about epistemological aspects of mathematical proof.

Shinno and colleagues conclude their article by the call for future international studies aimed at investigation of linguistic influences on curriculum development. In conclusion of this commentary, I would like to reiterate this call and mention two candidate ideas for further development of nationally-internationally suitable REMs in the realm of mathematical proof. The first one is stimulated by the Wittgenstein (1953) idea of meaning as use: it might be important to ground the refined REMs on thorough investigations of uses of proof-related vocabularies in different countries, including the use of proof-related words in natural language and in language that belongs to mathematics registers. The aforementioned study by Mejía-Ramos and Inglis (2011) seems to me an important step in this direction. The second (related) idea is stimulated by Stein, Remillard and Smith (2007) who argued that written and experienced mathematics curricula are mediated by intended and enacted curricula. To this end, systematic exploration of linguistic differences between the national policy-making documents and textbooks and between the textbooks and spoken mathematical language can bring us to interesting insights on cultural influences on mathematics instruction in different countries.

References

De Villiers, M. (1990) The role and function of proof in mathematics. *Pythagoras* **24**(1), 17-24.

Hanna, G. (1990) Some pedagogical aspects of proof. *Interchange* **21**(1), 6-13.

Hanna, G. & de Villiers, M. (2014) Aspects of proof in mathematics education. In De Villiers, M. & Hanna, G. (Eds.) *Proof and Proving in Mathematics Education,* pp. 1-10. New York: Springer.

Lunney Borden, L. (2011) The 'verbification' of mathematics: using the grammatical structures of Mi'kmaq to support student learning. *For the Learning of Mathematics* **31**(3), 8-13.

Mason, J. (2010) Mathematics education: theory, practice and memories over 50 years. *For the Learning of Mathematics* **30**(3), 3-9.

Mejía-Ramos, J.P. & Inglis, M. (2011) Semantic contamination and mathematical proof: can a non-proof prove? *The Journal of Mathematical Behavior* **30**(1), 19-29.

Miyakawa, T. (2017) Comparative analysis on the nature of proof to be taught in geometry: the cases of French and Japanese lower secondary schools. *Educational Studies in Mathematics* **94**(1), 37-54.

Silverman, B. (2017) *Explanations and Justifications in Israeli Mathematics Textbooks and the Textbook's Contribution to Shaping Classroom Learning.* Unpublished Ph.D. dissertation. Rehovot, Israel: Weizmann Institute of Science.

Shinno, Y., Miyakawa, T., Iwasaki, H., Kunimune, S., Mizoguchi, T, Ishii, T. & Abe, Y. (2018) Challenges in curriculum development for mathematical proof in secondary school: cultural dimensions to be considered. *For the Learning of Mathematics* **38**(1), 26-30.

Stein M., Remillard J. & Smith, M. (2007) How curriculum influences student learning. In Lester, F. (Ed.) *Second Handbook of Research on Mathematics Teaching and Learning,* pp. 319-368. Reston, VA: NCTM.

Stylianides, A. (2007) Proof and proving in school mathematics. *Journal for Research in Mathematics Education* **38**(3), 289-321.

Tall, D. (2008) The transition to formal thinking in mathematics. *Mathematics Education Research Journal* **20**(2), 5-24.

Wittgenstein, L. (1953) *Philosophical Investigations.* Oxford: Blackwell.

If Trump were an applicant to your mathematics education program, would you accept him? A response to Rodriguez, Kitchen and Harding

EGAN J CHERNOFF

No.

Words can be offered seriously and can also be taken seriously. They can be worked at by a reader, re-read and thought about, until they yield up meanings that may have escaped a first scrutiny.

— David Wheeler, **1**(1) p.2
