Communications

Schoenfeld's problem-solving models viewed through the lens of exemplification

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Schoenfeld's work on mathematical thinking and problem-solving in mathematics education is well-known and has been influential for decades. For example, in *Confessions of an accidental theorist* (1987) he discusses problem-solving strategies that involve a combination of making the heuristics explicit and providing a managerial strategy for students. Schoenfeld draws on Pólya's (2014) macro level approaches adding more layers of detailed instruction, leading to the position "Knowing the strategies isn't enough. You've got to know when to use which strategies." (Schoenfeld, 1987, p. 32).

In *How We Think* (2011), he refines his ideas regarding the complexity and layers of strategies. Strategies break into sub-strategies, which are either successful, replaced or broken down further into sub-sub-strategies. These are all based on what the individual learner knows. Schoenfeld stresses that complex "in the moment" decision-making (which subsumes problem solving) is iterative, with iterations at finer and finer levels of granularity, "down to individual utterances or actions" (p. 18). In Schoenfeld's world, the 'problemness' of a task doesn't reside in the task. Rather, it resides in the relationship between the individual and the task—so individual knowledge makes a huge difference.

The examples discussed in detail by Schoenfeld (2011) were mostly of teaching and at a level of generality that remains sheltered from more mathematically direct pedagogical perspectives. Motivated by this gap, the purpose of the present work is to provide readers with a useful, concrete and tangible perspective by connecting Schoenfeld's models with some particular examples that can be abstracted at the level of strategy. Let us discuss some examples.

Example 1.1.

Consider the following problem from a course on integration:

$$\text{Evaluate:} \quad \int \frac{2}{x^2-1}\,dx.$$

One well-known problem solving strategy for this example is for students to factor and recast the integrand such that:

$$\frac{2}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$$

for some constants A and B. On a macro level, this strategy might be derived as a special case from Pólya's (2014) decomposing and recombining strategy, *e.g., Can you decompose the problem and recombine its elements in some new manner?* The particular algebraic method is known as 'partial fractions'. Each term in the right-hand side can then be integrated to form logarithms.

The algebraic line of attack represented above may be identified as forming a sub-strategy of the macro-level strategy and embodies the next iterative layer of complexity in the process at a finer level of grain size. This forms but one of the couple of hundred problem solving techniques acknowledged by Schoenfeld (1987, p. 32).

There is yet another layer of complexity that a closer inspection and discussion of the above example will unmask. Although a line of attack has been identified for Example 1, it is important to recognize that *there are several ways a student may determine A and B*. Reframed from Schoenfeld's perspective regarding the importance of the connection between individual knowledge and problem solving, *different students may determine A and B using different sub-sub-strategies.* What might these look like in practice?

For some students, the most direct way to calculate A and B will be to draw on and implement a sub-sub-strategy termed as the Heaviside cover-up method. That is, their individual knowledge equips them to implement the follow sub-sub-strategy. To compute A a student covers $(x + 1)$ in the left-hand side and substitutes $x = -1$ which reveals $A = -1$. Similarly, to compute B, a student covers $(x - 1)$ in the left-hand side and substitutes $x = 1$, which reveals $B = 1$.

Other students may choose to play with the numerator in the left-hand side so that terms in the numerator align with factored terms in the denominator, forming:

$$\frac{2}{(x+1)(x-1)} = \frac{x+1-(x-1)}{(x+1)(x-1)} = \frac{1}{x-1} - \frac{1}{x+1}.$$

So their individual knowledge and relationship with the problem equips them to implement a different sub-sub-strategy.

When applied appropriately, the above discussion illustrates that either of the above two techniques can be used by students to calculate A and B in mere seconds.

Yet still other students may apply what is known as the method of undetermined coefficients to determine A and B. The idea is for a student to multiply both sides by $(x + 1)(x - 1)$ and then rewrite the equation by grouping like terms together to form:

$$2 = (A+B)x + (B-A).$$

Equating the coefficients then reveals two equations for A and B. For students whose individual knowledge and relationship with the problem equips them to implement this alternative sub-sub-strategy it may involve more calculations and represent a longer path to take in tackling this

For the Learning of Mathematics **39**, 1 (March, 2019)
FLM Publishing Association, New Westminster, BC, Canada

problem.

There are still yet more ways that students (and teachers) can solve this problem, forming personal, mathematical *heterotopias*: “places in the pedagogical landscape where different or alternate mathematical and pedagogical ordering is performed” (Tisdell, 2018). Indeed, it is virtually impossible to capture them all and we clearly see that each strategy may be as individual as the student who solves the problem, furnishing “individual utterances or actions” (Schoenfeld, 2011) at the micro sub-sub-strategy level for this particular example.

In addition, due to the algebraic nature of the solution methods, these ideas could also be seen as recursively applying Pólya’s (2014) decomposing and recombining strategy where a student is repeatedly performing algebraic operations to decompose and recombine the expression in slightly different forms.

The previous problem exemplifies and connects Schoenfeld’s models with a mathematically direct perspective that can be abstracted at the level of strategy.

Two themes run through Schoenfeld’s early work (1987): the amount of time involved for students to solve the problem; and how ‘far’ a strategy or method takes a student away from the original problem. Shorter methods are favoured over more time-consuming ones; and Schoenfeld also recommends that students stay close to the original problem as a first priority. When we consider the importance of a learner’s individual knowledge and relationship with the problem under consideration, this implies that different students will have individual perspectives regarding the concepts of time and distance when adopting strategies. For one student, adopting a certain strategy might form a quicker method of solution than pursuing a different strategy. Yet, for another student, this very situation may be reversed. What might these look like in practice? To explore this idea, let us revisit two examples from Schoenfeld’s classical work.

Example 1.2.

Determine a formula in closed form for the (finite) series:

$$s_n := \sum_{k=1}^{n} \frac{k}{(k+1)!}.$$

In Schoenfeld’s brief discussion he identifies the following observations as being important.

> The special cases that help are examining what happens when there the integer parameter n takes on the values 1, 2, 3, . . . in sequence; this suggests a general pattern that can be confirmed by induction. (1987, p. 31)

A particular student’s individual knowledge and personal relationship with the problem may indeed enable him/her to pursue this suggested pathway. For example, if n takes on integer values 1, 2, 3, . . ., then one sub-strategy employed by a student might involve the following considerations:

$$\begin{aligned}
s_1 &= \frac{1}{2!};\\
s_2 &= \frac{1}{2!}+\frac{2}{3!}=\frac{1}{2!}+\frac{2+1-1}{3!}=\frac{1}{2!}+\frac{3}{3!}-\frac{1}{3!}=1-\frac{1}{3!};\\
s_3 &= \frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}=1-\frac{1}{3!}+\frac{4-1}{4!}=1-\frac{1}{4!}.
\end{aligned}$$

The emerging pattern is now becoming clearer and the student could pursue a proof of the form:

$$s_n = 1 - \frac{1}{(n+1)!}$$

via induction.

Alternatively, a different student’s individual knowledge and personal relationship with the problem may mean somewhat distinct strategies are employed. Perhaps the following direction would be seen as more favourable to some students due to personal choices about the length of time associated with the strategy. For example, a learner may recognize that the summand can be recast in a suitable form that produces a telescoping sum. This can involve a simple re-writing of the numerator in the summand so that (part of) it bears a stronger resemblance to the denominator:

$$\begin{aligned}
\sum_{k=1}^{n}\frac{k}{(k+1)!} &= \sum_{k=1}^{n}\frac{(k+1)-1}{(k+1)!}=\sum_{k=1}^{n}\left(\frac{1}{k!}-\frac{1}{(k+1)!}\right)\\
&= \left(\frac{1}{1!}-\frac{1}{2!}\right)+\left(\frac{1}{2!}-\frac{1}{3!}\right)+\cdots+\left(\frac{1}{n!}-\frac{1}{(n+1)!}\right)\\
&= 1-\frac{1}{(n+1)!}
\end{aligned}$$

These learners may solve this quickly (in a matter of seconds) by applying a method that keeps the learner very close to the original problem, forming a telescoping sum.

Thus, we suggest the following strategy and acknowledge that all of the above sub-(sub)-strategies can be linked back to Pólya’s (2014) decomposing and recombining strategy.

> *Strategy 1.* Some students may find it useful to mathematically recast the summand of a series.

Example 1.3.

Let $P(x)$ and $Q(x)$ be polynomials whose coefficients are the same, but in ‘backwards order’:

$$\begin{aligned}
P(x) &= a_0 + a_1x + a_2x^2 + \cdots + a_nx^n;\\
Q(x) &= a_n + a_{n-1}x + a_{n-2}x^2 + \cdots + a_0x^n.
\end{aligned}$$

In Schoenfeld’s brief discussion (1987, pp. 31–32) he identifies the following observations as being important:

> If you try to use special cases in the same way on the second problem, you may get into trouble: Looking at values n = 1, 2, 3, . . . can lead to a wild goose chase. It turns out that the right special cases of $P(x)$ and $Q(x)$ that you need to look at for Problem 2 are easily factorable polynomials.

A particular student’s individual knowledge and personal relationship with the problem may indeed enable him/her to pursue this suggested pathway and Schoenfeld furnishes the following example in this regard:

$$P(x) = (2x+1)(x+4)(3x-2).$$

Alternatively, a different student’s individual knowledge and personal relationship with the problem may mean somewhat distinct strategies are employed. Perhaps the following direction would be seen as more favourable to some students due to personal choices about the length of time associated with the strategy.

If n takes on integer values, 1, 2, then for some students these cases may seem to suggest a general pattern very easily.

Consider $n = 1$:

$$
\begin{aligned}
P(x) &= a_0 + a_1 x = a_1\left(x + \frac{a_0}{a_1}\right); \\
Q(x) &= a_0 x + a_1 = a_0\left(x + \frac{a_1}{a_0}\right).
\end{aligned}
$$

Thus, this seems to suggest that there is an inverse relationship between the roots. The case $n = 2$ may take a learner a little longer, but the quadratic formula can be used for general coefficients to obtain the same relationship between the roots.

Thus, in contrast, Problem 2 can involve different students examining the special cases $n = 1$ and $n = 2$ to map out a general case. Such learners can solve this quickly (in a matter of seconds in the case $n = 1$) by applying a method that keeps the learner very close to the original problem. For some individuals, their personal perspective may mean that the special cases $n = 1, 2$ can be particularly helpful.

> *Strategy 2.* When dealing with problems that concern the roots of polynomials, some students may find it useful to look at easily factorable cases $n = 1$ and $n = 2$.

Our discussion of the previous three problems has aimed to connect Schoenfeld's models with some particular mathematical examples that can be abstracted at the level of strategy. In particular, we have seen the importance of the individuality of the learner throughout, their unique perspectives and their personal knowledge through the lens of exemplification. We have seen that a key consideration is a learner's individual relationship with respect to distance and time in problem-solving. We hope readers find the discussion useful and tangible.

Acknowledgements

The Author is primarily affiliated with the School of Mathematics & Statistics, UNSW, Sydney, Australia. This work was partially supported by funding from an ITaLI Visiting Fellowship at the University of Queensland during 2018. The feedback from Alan H. Schoenfeld regarding how this article could be improved is warmly acknowledged.

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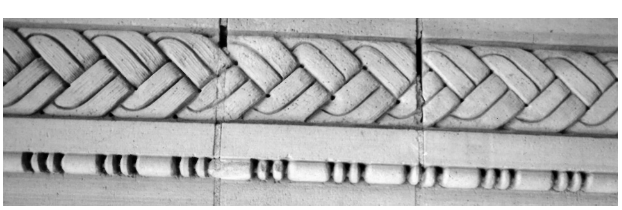

What makes a mathematician?

OSNAT FELLUS,
FLORENCE GLANFIELD

One of the plenary speakers in the 2018 meeting of the Canadian Mathematics Education Study Group was Professor Donald Violette, a mathematician from the Université de Moncton, whose topic "What if we teach passion?" [1] took us on a journey of becoming a mathematician and of teaching others to fall in love—passionately—with mathematics.

Professor Violette's talk generated some thoughts, for us, about the process of becoming a mathematician through multiple, distinct yet inseparable, identity-related dimensions. We connected this process with four networked dimensions formulated as an identity model by Roz Ivanič (1998):

> *Autobiographical identity:* the experiences and the interpretations of these experiences that one has in learning, using, and doing mathematics;
>
> *Discoursal identity:* the way one is spoken *to* or *about* as a learner or user of mathematics;
>
> *Authorial identity:* the appropriation of mathematics-related ideas and knowledge; and
>
> *Socioculturally available identities:* socioculturally available possibilities for self-hood.

For easy reference, we use the acronym ADAS to frame our comments about identity making in the context of mathematics.

Violette's account suggests an illustrative example of ADAS. Through his description of the passion he developed toward mathematics, Violette repeatedly demonstrated the positive experiences he had in learning and doing mathematics. These experiences, we suggest, are framed within the autobiographical dimension (first A in ADAS) of identity work where one harnesses personal experiences and the interpretations ascribed to them to construct a mathematical identity.

Discoursal identity (D in ADAS) is constructed by how one is spoken *to* and *about* as a learner of mathematics. Violette recounted how from early on he was spoken to by address and attribution, within his school and outside of his school, as a person who knew mathematics and could do mathematics. This discoursal identity reified his identity as knowledgeable in mathematics so much so that during high school when his mathematics teacher fell ill, the school principal asked Violette to teach mathematics in his teacher's stead. The Ministry of Education then granted him a teaching certificate so that he could teach his classmates for the months the mathematics teacher was absent.

Constructing a mathematical identity in and through the autobiographical (first A in ADAS) and the discoursal (D in ADAS) dimensions does not happen in a vacuum. Rather, one needs to have opportunities to appropriate mathematical knowledge thus engaging in identity work in the authorial dimension (second A in ADAS) where the learner develops his authorial voice in the context of mathematics. This notion of authorial identity builds on Bakhtin's (1981) concept of appropriation:

> The word in language is half someone else's. It becomes 'one's own' only when the speaker populates it with his own intention, his own accent when he appropriates the word, adapting it to his own semantic and expressive intention. Prior to this moment of appropriation, the word does not exist in a neutral and impersonal language (it is not, after all, out of a dictionary that the speaker gets his words!), but rather it exists in other people's mouths, in other people's contexts, serving other people's intentions: it is from there that one must take the word, and make it one's own (pp. 293–294).

In a similar vein, Wenger (1998) affiliates authorship-related activity to the "ability, facility, and legitimacy to contribute to, take responsibility for, and shape meanings that matter in a social configuration" (Wenger, 1998, p. 197).

Going back to Violette's account, we hear how, in the act of asking him to teach mathematics, he was recognised as someone who has developed an authorial voice. This occurred through the process of adapting mathematical concepts "to his own semantic and expressive intention" in Bakhtin's words, thus gaining "facility, and legitimacy to contribute to, take responsibility for, and shape meanings that matter in a social configuration" in Wenger's words. Violette's autobiographical, discoursal, and authorial identities are unified to represent a network of tightly related dimensions in which he developed as a mathematics learner, mathematician, and mathematics educator.

In regard to the fourth dimension in ADAS, *i.e.,* the socioculturally available identities, we heard Violette discuss how socioculturally shaped beliefs about who can and who cannot learn mathematics either lead students to or divert students from paths of learning mathematics. Specifically, Violette offered illustrative examples of sociocultural beliefs within which it was not expected that people of Acadian descent would engage in mathematics. These examples caused us to think about the sociocultural beliefs that continue to inform conversations about who can and who cannot do mathematics, and the ways in which those conversations shape the identities of individual mathematics learners.

Violette's talk suggests to us as mathematics educators and mathematics teacher educators, that using ADAS as a model to understanding mathematics identity may have the potential to shape the identities of others as learners and doers of mathematics. Violette's work with students, for example, was a strong piece of his talk. His commitment to his experiences as an Acadian and his sharing of his fascination and love of the Möbius strip as an illustration of the beauty of mathematics are instructive. Violette's ADAS identity as an Acadian emerges in the mathematics summer camps that he now hosts for young Acadians. He engages in public talks with school children of all ages to illustrate the fascination of the Möbius strip. We believe that it is Violette's *story to live by* (Connelly & Clandinin, 1999) as it is manifested within the framework of ADAS that poses a vicarious identity for Acadian children. This would be in contrast to the number of people that do not see mathematics as part of their story to live by but who tell stories of the ways in which mathematics is a barrier to living the story they wanted to live.

In our work as mathematics teacher educators, we also suggest that the notion of constructing an identity as a mathematics educator or mathematics teacher is equally complex. Murphy and Glanfield (2010) describe the complex notion of specialist and generalist identities as *plotlines given to us* as mathematics teacher educators—plotlines that position all of us within prescribed ways of being that is, in fact, in constant flux. We invite readers to think of their own examples or counterexamples of the ADAS identity-related dimensions, and the plotlines given to you, so that we, as a community, can potentially use a unified framework to better understand the making of a mathematician.

Violette's talk further provided support to the understanding that developing an identity as a learner of mathematics is a process, not a product. That is, Violette demonstrated that one is not born a mathematician. Rather, it is through a continuous development through autobiographical identity, discoursal identity, authorial identity, and socioculturally available identities (read ADAS) that one becomes a mathematician. The talk turned our attention to the need to intentionally—and carefully—construct a constellation of ADAS that lays down each student's road toward developing one's identity as a user and doer of mathematics. This careful identity-related work, we argue, may be made possible by dispelling myths about mathematics such as: that mathematics has no human subject; that mathematics is culture-free and values-free; that mathematics is hard to do; that only a few can do mathematics; and that mathematics is powerful (Wagner, 2011). Within such a framework to teaching and learning mathematics, students' main role would be to develop their identities as mathematicians and to begin to see that mathematics plays a role in their story to live by.

Note

[1] The text of this plenary will appear in the conference proceedings, available at http://www.cmesg.org in mid 2019.

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Points of contact, points of intersection: recalling David Henderson

DAVID PIMM

I do mathematics to find out who I am.
(David Henderson)

I seem to be someone who needs to write about the deaths of significant teachers, colleagues, friends (although, in this instance, perhaps 'friendly' rather than 'friend'). [1] This is not just to acknowledge their lives and influence on me (though that is certainly part of it), but also to try to provide some reasons why writing about them could possibly engage you, the reader, as well. To offer potentially vicarious memories, perhaps, for you to engage with. These pieces are not autobiographies, though there are certainly autobiographic elements in each of them. Neither are any of them biographic obituaries as such: those are for other people to undertake, in other places, at other times. Instead, they are here, in FLM, both to mark and to acknowledge a certain end and a non-end at the same time. For these people's writings and their presence in the (conscious and subconscious) memories of living others continue to exist, alongside the still-emerging (ever-emerging?) discipline, field, profession (whichever you wish to call it) of mathematics education and its intricate, complex and far-from-unproblematic cognate relationship with mathematics itself.

In the early autumn of 1975, aged twenty-two-and-a-half, I arrived at Cornell University (in Ithaca, New York, USA), entering the Ph.D. programme in pure mathematics. By Christmas of that year, I had moved over into Education, at least in terms of my programme, and started a masters' degree in mathematics education, though I remained a TA in the mathematics department for the two years that I was at Cornell. Early in that first semester, I met David Henderson, a geometer and geometric topologist in the mathematics department, who would subsequently supervise my masters' thesis (on language, symbols and meaning in mathematics). [2] But it was another twenty years before I found out about certain challenging aspects of his life that had taken place just a couple of years before I met him, through a powerful plenary lecture he gave at the Canadian Mathematics Education Study Group (CMESG) annual conference in 1996 (see below). That conference was the last time I recall meeting him in person (though we still had occasional textual contact). It was more than twenty years after that when I heard about his death, on Dec. 20th, 2018, resulting from a collision with a car on a pedestrian crossing in a town in Delaware.

In 1974, David had started teaching an upper-level undergraduate geometry class that he continued to teach for forty years. [3] And, in 1976 and 1977, I was in that geometry class (the first year as a student, the second as his TA). I recall an astonishing sequence of results that comprised a non-numerical theory of polygonal area in the plane, based on finite dissection. The first result (which I'll return to shortly) involved the triangulation (and subsequent separation) of any polygon, thus turning it into a finite set of triangles. The second required placing any triangle horizontally onto its longest side (or one of them, if isosceles or equilateral), bisecting the height and rotating each of the two top vertex triangles down until the top vertex met each of the bottom two vertices, and then proving it to be a rectangle. Then, thirdly, we were to show every rectangle can be finite-dissected into an equivalent square (decidedly challenging). So, at this point, the original polygon has been dissected into a finite number of squares. Then, finally, we derived a theorem that provides a means of turning two squares into an equivalent square (a form of geometric addition, of geometric arithmetic if you like) to be repeated a finite number of times until there is a single square equivalent by dissection to the original polygon. And this final square *is* the initial polygon's area.

This last part, of course, is Pythagoras' theorem, and there are proofs of it by dissection. (For much more on this, see Chapters 12 and 13 in Henderson, 1996a.) And it certainly gave me a new sense of why Pythagoras' theorem is worth knowing. Yet one related thing I remember most was my offering a proof by induction of the triangulation of any polygon and then David asking me to think about how I could be sure that there would *always* be at least one pair of vertices (to be the pivotal segment of the separation) that could be joined directly without crossing another existing side of the original polygon. That provoked quite a discussion between the two of us, in front of the rest of the class.

For this was how David taught. Being the academic grandson of the topologist R. L. Moore (he of the Moore Method of mathematics teaching) had led him to work directly, collectively with students, no textbook, simply providing a sequence of tasks for students to explore *ab initio.* (Even his later geometry textbook, first published in 1996, primarily offered sequences of tasks and discussion.) And this, unquestionably, fed into his profound belief about the development of mathematical meaning as well as a teacher's openness to learning *about mathematics* from his students. During that same time, through David, I met and became friends with Larry Copes (who was then teaching mathematics at Ithaca College, unsurprisingly also in Ithaca). Once or twice a semester, Larry and I (and sometimes David) would drive up to SUNY Buffalo to spend a couple of days visiting Stephen Brown and his doctoral students (Larry was a former doctoral student of Brown's—see Copes, 1983).

These were some of my very first encounters with mathematician-mathematics-educators [4] and, as far as I know, Larry coined the phrase 'educational mathematics' (see Copes, 1982), which Henderson adopted as the first category of his three prime areas of work on his vita (and so much more). In passing, putting the label 'educational mathematics' alongside Caleb Gattegno's precept that 'only awareness is educable' suggests that only certain sorts of mathematics can educate human awareness. And, for Henderson, that prime source was unquestionably (albeit not exclusively) geometry.

A second point of contact with David came indirectly, and over time, through the four articles he published in FLM (three singly, one with others): one in 1981, in **1**(3) [5]; two back-to-back in **16**(1) and **16**(2); one in **28**(3). Right from the start, he was focusing intensely on mathematical meaning and understanding, about how proofs should explain 'why', about who is a mathematician, and on listening acutely to students as they worked on mathematics, learning himself about mathematics from what they said, wrote, drew and did. One small instance of this that he wrote about concerned the vertical angle theorem and a slim and elegant student proof arising from a 180˚ rotation around the point of intersection of the two lines forming the angle, which placed one angle on top of the vertical opposite one. (I also recall him mentioning another such proof, though I have not seen it written up, where a point projection through the point of intersection transforms—in a very literal sense—one angle into the other.) But, in that same paper, he also wrote:

> But: *What is the consistency of mathematics? Is it all arbitrary?* 'Arbitrariness' is a dangerous notion. There may be more than one possible starting point, but that does not mean that the starting points are arbitrary. Differing contexts, differing points of view, and differing why-questions bring with them a demand for differing starting points. If the choice were arbitrary, then that does away with the need for any discussion. It is easy when discussing mathematics to slide from 'if it is not absolute' to 'then everything is completely relative/arbitrary'. In most of these discussions there does not appear to be any middle ground. (1996b, p. 51)

The third point (another one of intersection) came when he and I together led a (nine-hour) Working Group at the 1995 CMESG [6] conference, held at the University of Western Ontario. We had pre-arranged (and pre-posted) a dozen questions, including 'How can we be rigorous without axioms?', 'Are precise definitions always desirable in geometry?' and 'How do we geometrically understand the real numbers?' The three days' encounters were grouped under *circles, straightness* and *the great stellated dodecahedron* respectively, but the discussion (also detailed in participants' written responses) was richer (as ever) than the prompts, that had nonetheless served their purpose (see Henderson & Pimm, 1995).

My fourth point of contact came at the very end of May, 1996, less than two months after Ted Kaczynski (known as the Unabomber) was finally arrested in the US, when David delivered a plenary lecture (Henderson, 1996c) at CMESG in Halifax, Nova Scotia. His opening couple of minutes concerned his sense of his significant similarity to Kaczynski, being significantly direct about his experiences in the early 1970s.

> I want to mention specifically one mathematician in my generation and that is [...] the suspected Unabomber. He had very much the same kind of university mathematics education that I had. [...] Both [Kaczynski] and I accepted tenure-track professorial positions at major research mathematics departments (Berkeley and Cornell). And we were initially both successful with professional mathematics. Then, in the early 1970's, both [Kaczynski] and I quit mathematics. I got angry with mathematics—I got very furious about what mathematics had done to me. It is too complicated to go into all my feelings then (even if I could retrace them accurately)—but if someone came up to me at that time and called me a mathematician I felt strongly like punching them in the face! The evidence indicates that [Kaczynski] had a different, more violent reaction but his writings express feelings very similar to the ones I had at that time. He and I both went into the forest and built a cabin and lived there alone and we isolated ourselves. But, there was a huge difference—I made a constructive positive breakthrough and [Kaczynski] didn't. It was geometry, the many friends I made, and my family that brought about this breakthrough and in many ways saved my life. (p. 28)

This was not the primary focus of his lecture (that was 'proof'). But it nonetheless reflects significant aspects of his encounters with and reactions to formal mathematics (despite having an early, renowned reputation as a professional mathematician). And unlike some mathematicians who moved their careers into mathematics education, David stayed resolutely rooted in both.

My final point concerns our reciprocal interactions through writing: first, me working on the chapter he wrote with his wife, Daina Taimina, that appeared in the 2006 book *Mathematics and the Aesthetic* and then his extensive corrections, commentary and questions (over 200 items in all) on two NCTM books on geometry I had worked on (Sinclair et al., 2012a, b), not least encouraging an emphasis on finite dissection as supporting a direct and non-numerical notion of area. I could hear his always-thoughtful, confidently hesitant, voice through his writing. A few examples: "This is only true if an angle is the union of two segments. This is not my view nor, I think, the view of most people who see the angle as something at the point", "Sometimes the diagram is only possible in the mind not on paper", "This only works (for me) if I am focusing on a point on the circumference as the circle grows" and "I think not: Thm. Any figure with half-turn symmetry and a line of reflection symmetry has at least one more line of reflection symmetry".

And I still remember his astonishing eyes: eyes the colour of glacial meltwater; eyes both intensely direct and yet not penetrating or intimidating; eyes that always seemed two-way mirrors, him looking both out and in at every single moment. (And, curiously enough, now that I think about it, in the final chapter in *Mathematics and the Aesthetic,* Nathalie Sinclair and I wrote about eyes in relation to mathematics and mathematicians.)

On the very first page (before even the title page) of his book *Experiencing Geometry on Plane and Sphere* there is a 1978 poem written by David entitled 'Geometry' (though it had disappeared by the third edition). Here are a couple of elements from it:

> If after studying I am not changed—
> if after studying I still see the same—
> then all has gone for naught.
>
> Space isn't made of point and line
> the points and lines are in the mind.

I studied with David Henderson, and I studied for him (as well as for myself) and, as with many students of their significant teachers, I also studied him. And I was changed. Nothing went for nought with me in our encounters. The lines in this short remembrance piece are about certain points of contact, points of intersection, that I had with him over the forty-plus years I knew him, even though, after the first two years when I saw, talked and met with him regularly, we very seldom met in person. He was central, a core pivot for my entire life (a centre of rotation, for sure), starting with a prolonged interaction that has unquestionably led me to where I am today. I was fortunate, perhaps, in coming to knowing him before I knew of him.

In a short epitaph to David Wheeler (before Dick Tahta and I engaged in a conversation about him, and about a book, published in FLM **21**(2), a special issue marking Wheeler's demise), Dick wrote, "The bereaved are often angry with the dead for having left them" (p. 20). Despite a sense of becoming bereft of yet another intellectual parent, I find I have the luxury, if that is the word, of not being angry, simply sad. David Henderson's death has, as often happens, triggered memories, things I find I still know but did not (in the present at least) know I knew ('unknown knowns', perhaps).

The novelist William Gibson wrote, "Time moves in one direction, memory in another". Memories are never simple, never transparent, never actual of the past, and only somewhat present in the present. I end this piece with the opening few lines of a memoir poem I have recently written, entitled *Recollection.*

> Memories. Carrion. Shrapnel.
> Seeds embedded in the present flesh. Fossilised
> mites the past once fed. Not reflections in a mirror,
> nor tree limbs leached beneath the surface of the lake.
> More like votive candles blown: all that remains—
> ghost smoke.

Notes

[1] In FLM, there have been pieces I have written, directly or indirectly, about Rita Nolder, in **13**(1); about David Wheeler, in **21**(2); about David Fowler, in **24**(2); about Dick Tahta, in **27**(3); and here, now, about David Henderson.

[2] In his on-line CV, under 'Scholarly interests', David declares: "Starting in 1970, I became a member of the Graduate Field of Education at Cornell and starting supervising Master's and Doctoral Thesis in Mathematics Education in 1974. Since then, I have had a significant supervisory role in 40 graduate theses in the Field of Education and the chair of 4 Ph.D. theses and 14 M.S. theses in mathematics education, almost all of these theses (as a condition for my supervision) had a strong mathematics component." (http://pi.math.cornell.edu/~henderson/VITA-Jan2018.pdf)

[3] He also published a seminal undergraduate geometry book, entitled *Experiencing Geometry on Plane and Sphere,* in 1996, with a second edition in 2001 with a different title (*Experiencing Geometry in Euclidean, Spherical, and Hyperbolic Spaces*), and a third edition (*Experiencing Geometry: Euclidean and Non-Euclidean with History,* jointly authored with Daina Taimina) in 2004. Apparently, he was working on a fourth edition at the time of his death.

[4] I had taken a mathematics education course in the mathematics department from David Tall in 1972 at the University of Warwick, where I was a mathematics undergraduate, one of the very few such courses being taught by mathematician-mathematics-educators at that time in the UK.

[5] In a curious coincidence, my first article published in FLM was also in 1(3), as was one by Stephen Brown.

[6] Though I now wonder, given its on-going interaction between mathematicians and mathematics educators, whether it might be better termed 'CEMSG'.

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The torpedo's shock

ROBERT ELY

A certain mode of teacher questioning arises in the teaching of mathematics, through which the student can experience what Plato in the *Meno* calls the 'torpedo's shock'.

Piaget's idea of equilibration provides a lens for interpreting this type of questioning. Here I make some observations about this Piagetian interpretation in light of two earlier accounts of this kind of questioning, those of Socrates and of Søren Kierkegaard, concluding with an illustrative episode from one of my classes.

Socrates and the torpedo's shock

Meno says that Socrates' questioning is capable of having a rather shocking effect on those he questioned:

> And if I may venture to make a jest upon you, you seem to me both in your appearance [1] and in your power over others to be very like the flat torpedo fish [2] , who torpifies those who come near him and touch him, as you have now torpified me, I think. (Meno 80a)

This condition of torpor one experienced when zapped by Socrates' questioning was called *aporia* (perplexity or impasse); for this reason Plato's early dialogues are sometimes called 'aporetic' dialogues. Of these dialogues, the *Meno* is the one where Socratic questioning is illustrated in the context of mathematics, which is why it is of interest to me here. In this dialogue, Socrates is trying to show Meno where knowledge comes from, that it does not enter a learner from outside, but already lies within him/her, needing only be "recollected." To illustrate this, Socrates questions

Meno's slave boy about the areas of squares. The boy has clearly never learned the particular geometry fact (in this life), but by being questioned he comes to figure it out through his own reasoning without being told it explicitly.

Socrates' questioning of the slave boy happens in two distinct phases. In the first phase, he ascertains that the boy believes, erroneously, that you can double a square's area by doubling the length of each side. He then questions the boy until the boy realizes that this new square would actually have quadruple, not double, the area. The boy had certainty initially; now he doubts. But this torpedo's shock is progress, according to Socrates:

> He did not know at first, and he does not know now, what is the side of a figure of eight feet: but then he thought that he knew, and answered confidently as if he knew, and had no difficulty; now he has a difficulty, and neither knows nor fancies that he knows. […] If we have made him doubt, and given him the 'torpedo's shock,' have we done him any harm? (Meno 84a-b)

Once the boy doubts the false belief, Socrates embarks on the second phase of his questioning, in which the boy comes to learn the correct fact. I am focusing just on the first type of questioning here.

Kierkegaard and the torpedo's shock

In his first book, *The Concept of Irony,* Søren Kierkegaard distinguishes between two types of questioning, using Socrates to exemplify these types:

> The intention in asking questions can be twofold. That is, one can ask with the intention of receiving an answer containing the desired fullness, and hence the more one asks, the deeper and more significant becomes the answer; or one can ask without any interest in the answer except to suck out the apparent content by means of the question and thereby to leave an emptiness behind. The first method presupposes, of course, that there is a plenitude; the second that there is an emptiness. The first is the *speculative* method [3]; the second the *ironic*. Socrates in particular practices the latter method. When the sophists, in good company, had befogged themselves in their own eloquence, it was Socrates' joy to introduce, in the most polite and modest way of the world, a slight draft that in a short time expelled all these poetic vapors. (Kierkegaard 2013, p. 36, emphasis mine)

In Socrates' first phase of questioning the slave boy, he uses the ironic method. He sucks out the apparent content, leaving a vacuum. This reflects Kierkegaard's idea that ironic questioning is purely negative, not as in 'bad' but as in 'seeking to negate'. It never posits anything new, but seeks only to undermine that which has been posited.

Kierkegaard's use of the word *ironic* may sound odd to us. We more commonly use the word irony to refer to when someone says the opposite of what they mean (verbal irony), or when an event happens that is contrary to what is expected (situational irony), or when the audience understands the significance of something that a character does not (dramatic irony). But the original Greek word *eirōneía* meant something like feigned ignorance, which is more in keeping with Kierkegaard's usage. In all of these various meanings, there seems to be one general idea in common: the phenomenon is not the essence. Thus, suppose Person A believes that Person B thinks they know a thing's essence but actually "knows" only a phenomenon, or appearance, of the thing. Person A can then question Person B ironically, seeking to cause them to perceive the mismatch between the phenomenon and the essence. Kierkegaard sees Socrates' irony as being complete ('infinite absolute negativity'), because Socrates never himself lands on a sure positive statement that *now* he finally sees the essence of a thing. He treats everything as phenomena, never essence, and thus he always questions it to undermine it, delivering torpedo's shocks. Thus, Socrates always claims ignorance.

Piaget and the torpedo's shock

Piaget's theory of equilibration provides a different lens for interpreting the torpedo's shock. The questioning Kierkegaard calls ironic seeks to cause disequilibrium, to induce a perturbation for the learner. The questioner may not have particular new knowledge in mind for the learner, but seeks rather to impel the learner to experience the insufficiency of their current knowledge. For instance, the questions might cause the learner to consider an object or context that (the questioner imagines) will invoke one of their schemas, but which the schema will not assimilate. Or perhaps the questions provoke the learner to derive implications of one of their conceptions that (the questioner imagines) will contradict implications of another of their conceptions—Piaget calls such a conflict "a temporal *decalage* between accommodations in different domains" (Piaget 2001, p. 314).

Just because a questioner induces a perturbation for a learner doesn't mean they can predict how the learner will resolve the cognitive conflict. Piaget notes that the resulting accommodation is unpredictable "because it is done exogenously, as a function of hitherto unknown properties of the [unassimilated] objects" (p. 314). Simon (2011) points out that this is precisely what makes teaching through this form of questioning an unreliable way of getting the learner to learn a particular thing or develop a particular way of thinking.

Kierkegaard says that ironic questioning only negates, but the idea of equilibration gives a slightly different perspective. Ironic questioning may seek to negate the learner's erroneous *belief*, the phenomenon, but the learner's perturbed *schema* will never end up negated or deleted, although it may become more limited in scope or in some other way differentiated (Piaget 2001). Differentiation here means if the learner develops a new distinction, consciously or unconsciously, between types of situations where the schema applies and types where it does not. An old belief is now doubted or negated, but the schema remains, and a new distinction between two different types of situations has now been created.

To clarify: the slave boy initially believed if you double a square's side, you double the square's area, and through ironic questioning he came to doubt this belief. But suppose this belief came from a schema of actions developed in the context of other situations. Perhaps, it resulted from his experiences where when you double something's length you double how much space it takes up also. That schema, now perturbed, would not be jettisoned entirely, but the boy

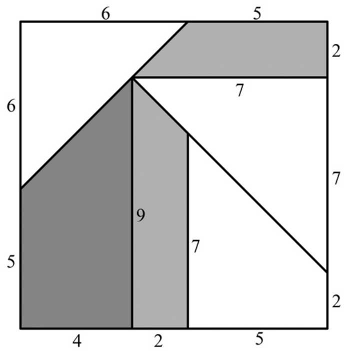

Figure 1. A scaling puzzle. (Brousseau, Brousseau & Warfield, 2008)

might end up differentiating situations where it applies from the new situation where it does not.

Torpedo's shock in one of my classes

I once gave a class of pre-service elementary teachers a problem based on a well-known task used by Guy Brousseau (Brousseau, Brousseau & Warfield, 2008). I put them in groups of three and gave each group a copy of the puzzle in Figure 1. On their copy the side lengths were not marked, but they were exactly the sizes (in cms) as appear in Figure 1, and the students were given rulers. The task was to make a larger version of the same puzzle, one where the segment that is currently 4 cm becomes 7 cm in the new puzzle. Each student in each group was to pick two of the six pieces, measure them, then determine what their new dimensions ought to be in the big puzzle, then cut out two larger pieces with those new dimensions. When all three students in the group had done this, they were to combine their pieces to make a full new enlarged puzzle.

Most of the groups had at least one person who knew to find a common scale factor by which each old measurement could be *multiplied* to determine the new measurement. But in one group, all three students decided to *add* 3 cm to each measured length. Of course when they tried to put their new big puzzle together, the pieces did not fit. They began to accuse each other of measuring wrong or cutting badly. Two of the students started repeating their measuring, calculating, and cutting procedure.

While they did this, I questioned the third student, something like this:

Me	Using your method, how long is the left side of your new puzzle?
Student	*[calculates in her head*] 17 cm.
Me	And how long is the right side of the new puzzle?
Student	*[calculates in her head]* 20 cm. Oh, wait, that means it can't be a square. *[To her groupmates]* Guys, I think our measuring was fine but there must be something wrong with adding 3.

The students worked more but never figured out what the right approach was. Eventually they gave up and asked another group for help.

I intended the problem to perturb students' additive thinking, so they could develop multiplicative thinking for scaling, just as it was used in Brousseau's class. My students in this one group had a pretty similar experience to his 5th grade students. Brousseau *et al.* report:

> By the end of the class, they are all convinced that their plan of action was at fault, and they are all ready to change it so they can make the puzzle work. But not one group is bored or discouraged. At the end of the session they are all eager to find 'the right way'. (2008, p. 156)

They experienced the 'shock' but did not come away torpified!

In my case it took questioning to provide the torpedo's shock; the task alone did not shake my students' belief that their method was correct in theory. My (ironic) questioning occasioned them to see that their procedure could not result in a square. Their erroneous belief was negated, leaving Kierkegaard's "emptiness" behind. The questioning induced a disequilibrium which led not to the negation of their schemas but rather their differentiation. They came to know that there are some situations where adding the same amount to all lengths does not make all parts grow in the 'same' manner. Yet, they still had no idea what *does* work for this situation, nor did they seem to know when other situations should be viewed as being like it. For that, they needed to develop a new multiplicative schema, as Brousseau's students did over the course of several further lessons.

Notes

[1] Wow, Meno—ouch!
[2] Electric eel.
[3] Kierkegaard picks this word because he equates such questioning with speculative philosophy, which he sees as seeking to derive or systematize truths into an ever-growing framework (and which he dislikes). One might aptly call this method of questioning "constructive" instead.

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