amazement, strengthen the conceptual network and thus put the students in the *mode of being*. The focus is on the thing itself, not an external goal.

A tool for reflection

The transfer of the metaphor of *having* and *being* is not about finding a binary association between aspects of learning mathematics and the two modes. Rather, these terms can serve as a lens for reflecting on one's own behavior as a teacher in the classroom. Ultimately, it can be questioned in every situation whether the *mode of being* or the *mode of having* is dominant. Is it about the acquisition of mathematical procedures, about a performance-oriented acquisition of skills that sees the students in a competitive struggle? Or do you focus on doing mathematics yourself, immersing yourself in a mathematical problem and feeling joy solving it, a non-judgmental learning environment, disciplined thinking in the literal sense, establishing relationships within and outside of mathematics?

This reflection should not and cannot have the aim of classifying one's own teaching or attitude on a scale between *having* and *being* using objective criteria. Rather, it is about finding a good way for the learning process between *having* and *being*, creating a humane learning environment and becoming aware of the possibilities that we as teachers can use by adapting our attitude.

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When a few is the right number

SEAN CHORNEY, CANAN GÜNEŞ, NATHALIE SINCLAIR, SHEENA TAN

In his recent article in issue **42**(3), Colin Foster raises the question of how many representations (or models) are required or desirable for learning mathematical concepts. Based in part on the problem of the representational dilemma (Rau, 2017), he argues for prioritising the number line as a singular coherent representation for multiplication. We read this with great interest since we have been working for the past five years on a multi-touch application called *TouchTimes* (Jackiw & Sinclair, 2019), which provides novel and multiple representations of multiplication. We were thus provoked by several aspects of Foster's argument, which we address below. In so doing, we highlight some of the ontological and epistemological assumptions underlying Foster's argument and show how different assumptions about the focus of learning and the nature of concepts would lead to different conclusions. We note, as well, that there is an underlying axiological assumption being made by Foster, which we will make clear at the end of this communication.

Are representations separate from or part of the concept?

As part of Foster's argument for proposing one representation, he raises the issue of the representation dilemma, which refers "to the problem that for students to be able to make use of a representation in order to learn some mathematics they need a certain level of familiarity with that representation" (p. 21). The very formulation of this dilemma assumes that familiarity with a representation *precedes* the learning from the representation. This ontological view is based on the assumption that a representation is separate from a concept. Maffia (2023), in his communication piece in **43**(1) responding to Foster, cites his grounding in a semiotic approach which also assumes that the representation is separate from the object (multiplication in this case). The separation of representation and concept is a view that has been articulated many times with respect to student learning using digital tools, namely, that students first need to learn how to use the tool before they can learn the concept. In a historical, material view of mathematics, the tool and the concept are not ontologically distinct—the very idea of a circle, for example, is intertwined with the strings and compasses and cans that are used to produce circles. From these perspectives, learning to see a representation, to work with it, is part and parcel of developing mathematical conceptual awareness.

We align with Maffia's point that being introduced to many representations is how one learns multiplication; however, where Maffia suggests it is to be able to differentiate between the representation and the object, we see the learning of multiplication as emerging by sewing together experiences from multiple representations. It is Vergnaud (2009) who suggests a concept is not an "explicit object of thought" (p. 93) but comes from a "variety of situations which requires children's meeting and being faced with contrasting situations" (p. 86).

Going back to Rau (2017)'s notion of representation dilemma, four representational competencies (visual understanding, visual fluency, connectional understanding and connectional fluency) were proposed to be important for students to develop in order to learn mathematics with visual representations and to navigate the representation dilemma.

	x	x	1
x	x^2	x^2	x
4	$4x$	$4x$	4

Figure 1: The area model that represents (2x + 1) × (x + 4)

These may be one way of making sense of the term 'familiarity'. But, interestingly, in three of these four competencies, one of the design principles was the need to provide students with a variety of representations that they could compare across to discern the key features from the extraneous ones. This seems to suggest that experience with multiple representations is necessary for students to gain familiarity with using visual representations for learning mathematics. We will return to this point.

What is involved in multiplication?

Foster claims that some representations, such as the area model, may confuse students because they represent the same number by using different entities. For example, in Figure 1, the number 4 is represented by two different entities—length of a line segment and area of a rectangle. He suggests that students should engage with a "coherent representation" in which all numbers in the multiplication are represented by the same entity. However, drawing on Davydov (1992), who describes multiplication as an indirect quantification concerning two types of unit counts, we argue that using a single 'coherent' representation like the number line might work for some operations such as addition, where all the quantities are measured with the same type of unit count, but not for multiplication. Instead, the area model in Figure 1 may provide another form of coherence in relation to multiplicative thinking, where it can be highlighted that the two factors have their unit count as length (4 unit and 1 unit) which generates the product with a different unit count as area (4 units^2).

Before moving on to the next section, however, we want to add a further point to the discussion of the area model, this time based on the work of Maffia and Marrioti (2018). In their article in **38**(3), they focus on the didactic advantages of two models—the repeated sum model and the array model—and argue that "different models of multiplication can serve in different ways to justify algorithms, showing the 'why' of arithmetical properties" (p. 31). Of interest to us here is their treatment of the array model, which provides a better intuitive explanation of the commutative property of multiplication than the repeated sum model. Their argument highlights the idea that learning multiplication does not just involve learning what it *is* (a change in units, repeated addition, *etc*.) but also how it *behaves* (properties such as commutativity, associativity, *etc*.).

Is there a single concept of multiplication?

Davis and Renert (2014) draw on the notion of *realisation*, which refers to "all manner of associations a learner might draw on and connect in efforts to make sense of a mathematical concept" (p. 58). They extend Sfard's notion and describe the understanding of a mathematical concept as emerging from finding metalevel connections among multiple realisations. The focus on connections resonates with Maffia and Mariotti (2018)'s assertion that it is "not just that presenting different models to pupils is relevant, but also that the operation of putting in relation the models has the potential to be particularly productive" (p. 35).

One can believe that there are single formal definitions of mathematical concepts, while still acknowledging that there are different ways of thinking about that concept. One can also make the ontological claim that the concepts themselves are plural and that, in the case of multiplication, the single word is somewhat misleading since it merely names a multiplicity—we might therefore speak of 'multiplications'. Rather than seeing multiplication as emerging from one model, metaphor, or representation, learning multiplication involves multiple interpretations of various situations followed by weaving those experiences into structures. For this reason, Davis and Renert suggest that multiplication be understood as having an open definition, one that is continually being adjusted and developed.

Realisations are not fixed, since they are the material bases of concepts. They are therefore a function of the technological infrastructure. New drawings, tools and actions can introduce new realisations. For this reason, realisations are part of an evolving system, which is why Davis and Renert warn that if a realisation is fixed it may overtake or rule out other interpretive possibilities. In the case of *TouchTimes*, for example, there are dynamic realisations that are not found in static representations of multiplication. A static screenshot of 3 × 4 (shown in Figure 2a) contains a visual representation of multiplication as unitising (a change in units), going from 3 pips as 3 units, to 3 pips/pod as 1 unit, to 4 pods of 3 pips/pod as 4 units. The placement of the thumb on the left side provides an additional realisation of multiplication as spreading, which increases the number of pips by 1 across each and every pod (shown in Figure 2b). If such a realisation is helpful for students to think multiplicatively, then it might be short-sighted

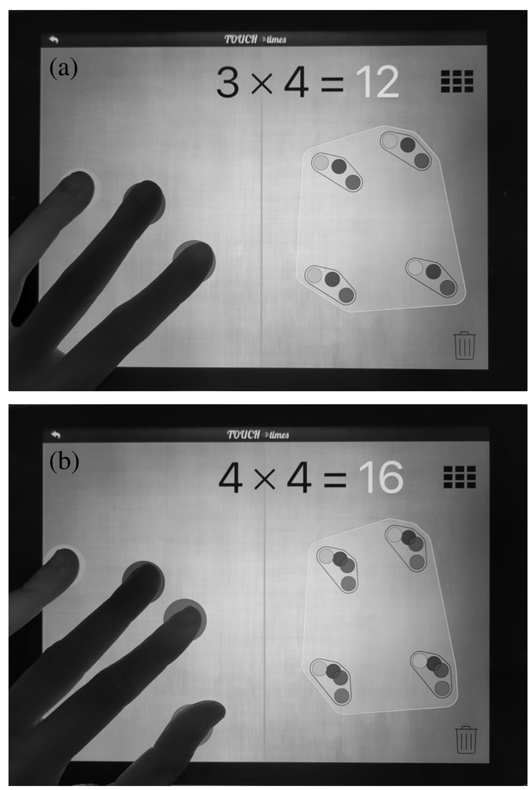

Figure 2: (a) Static screenshot of 3 × 4 in TouchTimes; (b) Placing the thumb on the left side results in the spreading of a new pip across each and every pod.

to restrict students to a single paper-based representation. We would argue that what is necessary is for researchers and teachers to be aware of the effective realisations that any representation offers and work towards developing a small suite of representations that invite students into the plurality of multiplications while addressing the most significant properties, contexts and behaviours of multiplications.

From many to one to a few

We are sympathetic to Foster's argument that mashing a bunch of representations together and hoping students will figure out multiplication is problematic. We would argue that a small set of well-chosen representations might provide a good balance that acknowledges the multiplicity of the very idea of any concept (including multiplication), that supports the noticing of relations that pertain across contexts, that recognise the materiality of concepts and that therefore remains flexible on the digital potential of well-designed tools. We suggest that more research attention be paid to creating coherence across representations. Maffia and Maracci (2019) have started doing this by exploring the semiotic interferences that arise when students move between different artefacts (often physical and virtual).

Regarding the axiological dimension of Foster's argument, we note that in gravitating towards the single representation of the number line, there is an obvious appeal of simplicity and unity, which may well facilitate the development of curriculum materials and tasks. Our own predilection for multiplicity carries different aesthetic tendencies, which might tend towards eclecticism and variety. However, it is the associated ethical implications we think need to be clearly articulated. If we imagine that learners come with diverse prior experiences, capacities for learning and aesthetic preferences, then should we not be meeting them with diverse representations? We would not suggest that the number line would be intimidating for students, but it may well be less intuitive, persuasive and relevant to them.

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Glocaliszation in mathematics education

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We were waiting for a colleague in a hotel lobby in Barcelona soon after the PME conference in Alicante. There was light music in the background and the following conversation took place:

Arindam	Oh wow! I know this song, I have heard this in India in Hindi and also in Brazil in Portuguese with exactly the same beats and melody.
Danyal	Yes, it is everywhere! Even in conservative places.
Arindam	Is it that Macarena was initially a local song which became popular and turned global and spread to other places? And thereafter many other places then customised this song in their languages and also gave a local flavour of their practices to it? Like global turning into a local and yet retaining its 'global' form?
Danyal	Yes, precisely so!
Danyal & *Arindam*	[*together and spontaneous*] That's Glocalisation!

This was how we got to the word 'glocalisation' [2]. When a dominant global voice gets the flavour of a local practice, that is when 'glocalisation' occurs. Much like how the Macarena tune has been glocalised, many other things, objects and practices have been glocalised as well, not only products or goods and but also languages such as the one which we are using to write this short communication—it is one such glocalised form of the English language. English is a language of power, and in many academic contexts it is the lingua-franca [3] through which we write, publish and interact. One or two centuries ago French was the language of power, and the lingua franca in which diplomacy and science was carried out. Recall that the editors of the first journal in mathematics education chose French as the sole language of publication (Reid, 2023). After the Second World War, the United States became the dominant military power in the West, as well as giving rise to new forms of popular music like jazz and rock that influenced the world over and yet acquire local flavours. Gangnam Style, for example, is a pop song glocalised to Korea.

The term 'glocalisation' has previously been used while referring to the combination of local (emic) and global (etic) approaches in ethnomodelling research contributing to a holistic understanding of mathematics (Orey & Rosa, 2021). We argue here that the modern trends of standardisation, for example, standardisation of the representation of mathematical ideas, concepts and symbolisation, processes of foreign economic trading, modern outlets of fashion and design, home décor, food outlets, modern cultural forms