Felix Klein at Evanston: Learning, Teaching and Doing Mathematics in 1893

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In 1893 Felix Klein, then 44 and the head of the growing and successful programme in mathematics at Göttingen, was among the leading mathematics of his day. Why should he have spent the best part of the summer giving a demanding series of lectures at Evanston, Illinois, a place scarcely on the mathematical map? Why should he have taken that occasion to give an account of mathematics — what it is, how it can be done, and how it can be taught — that still merits our attention today? Part of the answer lies in the date.

The 400th anniversary of Columbus's journey to the New World was celebrated with less ambiguity than the 500th would be. Among the celebrations, the rapidly growing city of Chicago hosted the World's Columbian Exposition of 1893. It was an extravaganza. In 633 acres it contained all manner of exhibits and entertainments, from a 250-foot Ferris Wheel to 74 galleries exhibiting over 9,000 works of art. Remarkably, it even ran at a profit, and it marked the emergence of Chicago as a major American city. The mathematicians at the newly-founded University of Chicago, hoping to gain some visibility through the Mathematics Congress planned alongside the Exposition, wrote to Felix Klein urging him to attend.[1] Klein turned the invitation to his better advantage by getting himself appointed as the official representative of the Prussian Government. With his domestic status thus enhanced, he wrote to offer a two-week cycle of twelve lectures, as a sequel to the regular sessions being organised as part of the Congress. The organisers, Henry White and E.H. Moore, were delighted and it was decided to take the opportunity to put another new American University on the map: Northwestern, at Evanston, Illinois. Thus the Evanston Colloquium Lectures were established.

The Americans' choice of Klein was far from accidental. He had a broad vision of mathematics, and his lecture courses in Göttingen became so popular that a whole generation of Americans attended them, including the leaders of the mathematical community that was beginning to form in the United States.[2] Henry White, with whome Klein stayed during his trip America, was one of these. Two of Klein's former German students, Oskar Bolza and Heinrich Maschke had just joined Moore's staff at Chicago. So there was already a strong bond between the visitor and his hosts, and Klein was just the kind of high-status mathematician that would reflect well on the aspiring young community that invited him.

Klein gave his first lecture at Evanston on Monday, 28 August, and thereafter gave one lecture a day except for Sundays. He spoke in English, and there were lengthy discussions after each lecture. The texts were taken down and edited for publication as the *Evanston Colloquium Lectures* by Alexander Ziwet of Michigan, where they form a tiny volume dedicated to a landmark event in the history of American mathematics.[3] The lectures formed a prototype of the American Mathematical Society's Colloquium Lectures, a series later launched at White's suggestion. They have achieved lasting significance for a variety of reasons. Few mathematicians have ever attained the breadth of Klein's mathematical knowledge, and fewer still have shared his ability to describe a vast mathematical field without losing their audience in the details. Moreover, not many research mathematicians have paid the same kind of attention to teaching mathematics effectively as did Klein.

Although his unusual breadth and pedagogical interests were marked even as a young man[4], he had particular reasons to cultivate these qualities during the middle phase of his career. For in late 1882 his health and self-confidence had disintegrated following a friendly but ill-fated attempt to match the efforts of the brilliant young French mathematician Henri Poincaré.[5] After this, Klein was never able to work on mathematics with the same intensity, and to compensate for this he built up a school of students and co-workers who dedicated themselves to exploring the kinds of detailed ideas and arguments that Klein no longer cared to think about.

The first six lectures revolved round the idea of intuition in mathematics, particularly in Klein's chosen fields of geometry and the theory of functions. The not-so-hidden message of the first lecture was that the current school of geometers in Germany, to which Klein had belonged as a young man, placed too much emphasis on algebra in the name of rigour. Klein argued for a return to the ideas of Riemann, which he believed were more profound, and more closely tied to physics. This was not quite accurate, but it was nonetheless the case that the tide was turning finally in Riemann's direction.

Then Klein gave two lectures on the work of another mathematician who, like Riemann, was notoriously difficult to read, the Norwegian Sophus Lie. In Lie's case, however, Klein could speak with the distinct advantage of knowing that Lie was an old friend from his student days. Klein and Lie had collaborated closely together twenty years earlier. The fourth lecture was on Klein's own work on the real shape of curves and surfaces, which he connected to the many mathematical models exhibited in Chicago. The fifth was on a particular differential equation, which was connected to the subject of his famous book *Lectures on the Icosahedron* (translated into English in 1888).[6]

The sixth lecture was on spatial intuition and the relation between pure and applied mathematics, and we give some extracts below. In it, Klein described a tension between the logical development of a subject, typified by Euclid's *Elements*, and the naive intuition of the scientist, which he argued was a creative source of inspiration, notably in the creation of the calculus. The next few lectures returned to a more technical level, dealing with a variety of topics that have continued to occupy mathematicians to the present day, before he concluded with an account of how things were in Göttingen and the special needs of foreign students going there. We give an extract below. As many of his audience would have known only too well, a visitor from abroad had to be very well prepared to succeed. Some might have added that Klein himself was becoming a little too busy, and perhaps a little aloof from the details, to be the right supervisor. Indeed, two years later a younger man by the name of David Hilbert came to Göttingen and soon thereafter assumed that role.

Finally, we include a third short extract from a paper Klein wrote for a giant publication on German universities, which was also displayed at the Columbian Exposition. In it, Klein gave a brief survey of the development of mathematics in the German universities. The history is no window-dressing; Klein's interest in the history of mathematics played an integral part in his approach to mathematics. His interpretations may be partisan, but they can equally be incisive. Here the distinction he captures between the way French and German mathematicians have viewed the shifting relationships between pure mathematics, applied mathematics, and physics is astute. For better or worse, it is once again topical.

The lectures were only part of Klein's gift to his audience. Speaking far more informally than he could ever have dared to do in Prussia, Klein spent many hours with the participants after each lecture. He tried to address their concerns and felt buoyed up by their optimism. Among the people in attendance was a young woman named Mary Frances Winston, who was about to travel to Göttingen despite the fact that Klein could give no assurances that she would be allowed to study under him. In the event, and unlike her helpful but less fortunate predecessor, Christine Ladd (Franklin), Winston was finally admitted and went on to earn her doctorate under Klein in 1897. Others in the audience saw careers ahead of them in the uncluttered, if uncertain, atmosphere of America. So they too were eager not only for knowledge, but for guidance. And if they did not respond to the specific suggestions Klein threw out regarding promising directions for research, they nevertheless felt drawn by the spirit of the enterprise as a whole. Indeed, several went on to make important contributions to the creation of a powerful and independent research tradition in mathematics in America that had been practically absent prior to the Colloquium.

Extracts

Lecture VI: ON THE MATHEMATICAL CHARACTER OF SPACE-INTUITION AND THE RELATION OF PURE MATHEMATICS TO THE APPLIED SCIENCES

(September 2, 1893)

In the preceding lectures I have laid so much stress on geometrical methods that the inquiry naturally presents itself as to the real nature and limitations of geometrical intuition.

In my address before the Congress of Mathematics at Chicago I referred to the distinction between what I called the *naïve* and the *refined* intuition. It is the latter that we find in Euclid; he carefully develops his system on the basis of well-formulated axioms, is fully conscious of the necessity of exact proofs, clearly distinguishes between the commensurable and incommensurable, and so forth.

The naïve intuition, on the other hand, was especially active during the period of the genesis of the differential and integral calculus. Thus we see that Newton assumes without hesitation the existence, in every case, of a velocity in a moving point, without troubling himself with the inquiry whether there might not be continuous functions having no derivative.

At the present time we are wont to build up the infinitesimal calculus on a purely analytical basis, and this shows that we are living in a *critical* period similar to that of Euclid. It is my private conviction, although I may perhaps not be able to fully substantiate it with complete proofs, that Euclid's period also must have been preceded by a "naïve" stage of development. Several facts that have become known only quite recently point in this direction. Thus it is now known that the books that have come down to us from the time of Euclid constitute only a very small part of what was then in existence; moreover, much of the teaching was done by oral tradition. Not many of the books had that artistic finish that we admire in Euclid's "Elements"; the majority were in the form of improvised lectures, written out for the use of the students.

If we now ask how we can account for this distinction between the naïve and refined intuition, I must say that, in my opinion, the root of the matter lies in the fact that ***the naïve intuition is not exact, while the refined intuition is not properly intuition at all, but arises through the logical development from axioms considered as perfectly exact.***

To explain the meaning of the first half of this statement it is my opinion that, in our naïve intuition, when thinking of a point we do not picture to our mind an abstract mathematical point, but substitute something concrete for it. In imagining a line, we do not picture to ourselves "length without breadth", but a ***strip*** of a certain width. Now such a strip has of course ***always*** a tangent (Fig 9); i.e. we can always imagine a straight strip having a small portion (element) in common with the curved strip; similarly with respect to the osculating circle. The definitions in this case are regarded as holding only approximately, or as far as may be necessary.

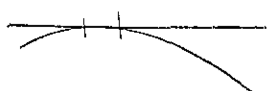

Figure 9

The "exact" mathematicians will of course say that such definitions are not definitions at all. But I maintain that in ordinary life we actually operate with such inexact definitions. Thus we speak without hesitancy of the direction and curvature of a river or a road, although the "line" in this case has certainly considerable width.

As regards the second half of my proposition, there actually are many cases where the conclusions derived by purely logical reasoning from exact definitions can no more be verified by intuition. To show this, I select examples from the theory of automorphic functions, because in more common geometrical illustrations our judgment is warped by the familiarity of the ideas.

Let any number of non-intersecting circles 1,2,3,4, ..., be given (Fig. 10), and let every circle be reflected (i.e. transformed by inversion, or reciprocal radii vectores) upon every other circle; then repeat this operation again and again, ***ad infinitum*** The question is, what will be the configuration formed by the totality of all the circles, and in particular what will be the position of the limiting points. There is no difficulty in answering these questions by purely logical reasoning; but the imagination seems to fail utterly when we try to form a mental image of the result

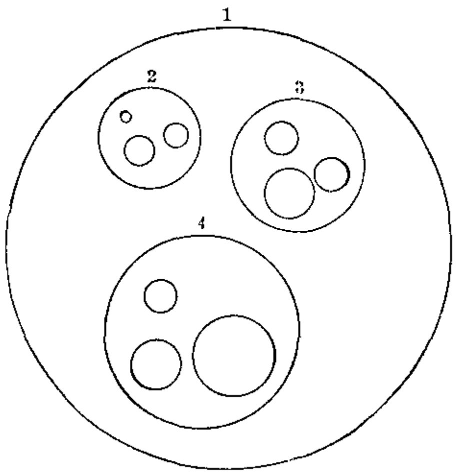

Figure 10

Again, let a series of circles be given, each circle touching the following, while the last touches the first (Fig 11). Every circle is now reflected upon every other just as in the preceding example, and the process is repeated indefinitely. The special case when the original points of contact happen to lie on a circle being excluded, it can be shown analytically that the continuous curve which is the locus of all the points of contact is not an analytic curve

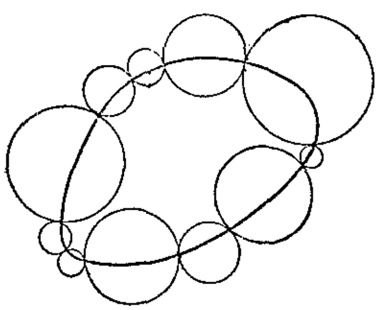

Figure 11

The points of contact form a manifoldness that is everywhere dense on the curve (in the sense of G. Cantor), although there are intermediate points between them At each of the former points there is a determinate tangent, while there is none at the intermediate points. Second derivatives do not exist at all It is easy enough to imagine a strip covering all these points; but when the width of the strip is reduced beyond a certain limit, we find undulations, and it seems impossible to clearly picture to the mind the final outcome. It is to be noticed that we have here an example of a curve with indeterminate derivatives arising out of purely geometrical considerations, while it might be supposed from the usual treatment of such curves that they can only be defined by artificial analytical series.

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What has been said above with regard to geometry ranges this science among the applied sciences. A few general remarks on these sciences and their relation to pure mathematics will here not be out of place. From the point of view of pure mathematical science I should lay particular stress on the ***heuristic value*** of the applied sciences as an aid to discovering new truths in mathematics. Thus I have shown (in my little book on Riemann's theories) that the Abelian integrals can best be understood and illustrated by considering electric currents on closed surfaces. In an analogous way, theorems concerning differential equations can be derived from the consideration of sound-vibrations; and so on.

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Now, just here a practical difficulty presents itself in the teaching of mathematics, let us say of the elements of the differential and integral calculus. The teacher is confronted with the problem of harmonizing two opposite and almost contradictory requirements. On the one hand, he has to consider the limited and as yet undeveloped intellectual grasp of his students and the fact that most of them study mathematics mainly with a view to the practical applications; on the other, his conscientiousness as a teacher and man of science would seem to compel him to detract in nowise from perfect mathematical rigour and therefore to introduce from the beginning all the refinements and niceties of modern abstract mathematics In recent years the university instruction, at least in Europe, has been tending more and more in the latter direction; and the same tendencies will necessarily manifest themselves in this country in the course of time. The second edition of the ***Cours d'analyse*** of Camille Jordan may be regarded as an example of his extreme refinement in laying the foundations of the infinitesimal calculus. To place a work of this character in the hands of a beginner must necessarily have the effect that at the beginning a large part of the subject will remain unintelligible, and that, at a later stage, the student will not have gained the power of making use of the principles in the simple cases occurring in the applied sciences.

It is my opinion that in teaching it is not only admissible, but absolutely necessary, to be less abstract at the start, to have constant regard to the applications, and to refer to the refinements only gradually as the student becomes able to understand them. This is, of course, nothing but a universal pedagogical principle to be observed in all mathematical instruction.

Lecture XII: THE STUDY OF MATHEMATICS AT GÖTTINGEN

(September 9, 1893)

As regards the former class of students, it is my opinion that in Germany (here in America, I presume, the conditions are very different) the abstractly theoretical instruction given to them has been carried too far. It is no doubt true that what the university should give the student above all other things is the scientific ideal For this reason even these students should push their mathematical studies far beyond the elementary branches they may have to teach in the future. But the ideal set before them should not be chosen so far distant, and so out of connection with their more immediate wants, as to make it difficult or impossible for them to perceive the bearing that this ideal has on their future work in practical life In other words, the idea should be such as to fill the future teacher with enthusiasm for his life-work, not such as to make him look upon this work with contempt as an unworthy drudgery.

For this reason we insist that our students of this class, in addition to their lectures on pure mathematics, should pursue a thorough course in physics, this subject forming an integral part of the curriculum of the higher schools Astronomy is also recommended as showing an important application of mathematics; and I believe that the technical branches, such as applied mechanics, resistance of materials, etc , would form a valuable aid in showing the practical bearing of mathematical science. Geometrical drawing and descriptive geometry form also a portion of the course Special exercises in the solution of problems, in lecturing, etc., are arranged in connection with the mathematical lectures, so as to bring the students into personal contact with the instructors

I wish, however, to speak here more particularly on the higher courses, as these are of more special interest to American students. Here specialization is of course necessary Each professor and docent delivers certain lectures specially designed for advanced students, in particular for those studying for the doctor's degree. Owing to the wide extent of modern mathematics, it would be out of the question to cover the whole field. These lectures are therefore not regularly repeated every year; they depend largely on the special line of research that happens at the time to engage the attention of the professor. In addition to the lectures we have the higher seminaries, whose principal object is to guide the student in original investigation and give him an opportunity for individual work.

As regards my own higher lectures, I have pursued a certain plan in selecting the subjects for different years, my general aim being *to gain, in the course of time, a complete view of the whole field of modern mathematics, with particular regard to the intuitional or* (in the highest sense of the term) *geometrical standpoint*. This general tendency you will, I trust, also find expressed in this colloquium, in which I have tried to present, within certain limits, a general programme of my individual work. To carry out this plan in Göttingen, and to bring it to the notice of my students, I have, for many years, adopted the method of having them lithographed, so as to make them more readily accessible These former lectures are at the disposal of my hearers for consultation at the mathematical reading-room of the university; those that are lithographed can be acquired by anybody, and I am much pleased to find them so well-known here in America.

As another important point, I wish to say that I have always regarded my students not merely as hearers or pupils, but as collaborators. I want them to take an active part in my own researches; and they are therefore particularly welcome if they bring with them special knowledge and new ideas, whether these be original with them, or derived from some other source, from the teachings of other mathematicians. Such men will spend their time at Göttingen most profitably to themselves.

THE DEVELOPMENT OF MATHEMATICS AT THE GERMAN UNIVERSITIES

The eighteenth century laid the firm foundation for the development of mathematics in all directions. The universities as such, however, did not take a prominent part in this work; the *academies* must here be considered of prime importance. Nor can any fixed limits of nationality be recognized. At the beginning of the period there appears in Germany no less a man than ***Leibniz***; then, follow, among the kindred Swiss, the dynasty of the ***Bernoullis*** and the incomparable ***Euler***. But the activity of these men, even in its outward manifestation, was not confined within narrow geographical bounds; to encompass it we must include the Netherlands, and in particular Russia, with Germany and Switzerland On the other hand, under Frederick the Great, the most eminent French mathematicians, Lagrange, d'Alembert, Maupertius, formed side by side with Euler and Lambert the glory of the Berlin Academy. The impulse toward a complete change in these conditions came from the French Revolution.

The influence of this great historical event on the development of science has manifested itself in two directions On the one hand it has effected a wider separation of nations with a distinct development of characteristic national qualities. Scientific ideas preserve, of course, their universality; indeed, international intercourse between scientific men has become particularly important for the progress of science; but the cultivation and development of scientific thought now progress on national bases. The other effect of the French Revolution is in the direction of educational methods The decisive event is the foundation of the École Polytechnique at Paris in 1794 That scientific research and active instruction can be directly combined, that lectures alone are not sufficient, and must be supplemented by direct personal intercourse between the lecturer and his students, that above all it is of prime importance to arouse the student's own activity,—these are the great principles that owe to this source their recognition and acceptance. The example of Paris has been the more effective in this direction as it became customary to publish in systematic form the lectures delivered at this institution; thus arose a series of admirable text-books which remain even now the foundation of mathematical study everywhere in Germany. Nevertheless, the principal idea kept in view by the founders of the Polytechnic School has never taken proper root in the German universities. This is the combination of the technical with the higher mathematical training. It is true that, primarily, this has been a distinct advantage for the unrestricted development of theoretical investigation. Our professors, finding themselves limited to a small number of students who, as future teachers and investigators, would naturally take great interest in matters of pure theory, were able to follow the bent of their individual predilections with much greater freedom than would have been possible otherwise.

But we anticipate our historical account. First of all we must characterize this position that Gauss holds in the science of this age Gauss stands in the very front of the new development: first, by the time of his activity, his publications reaching back to the year 1799, and extending throughout the entire first half of the nineteenth century; then again, by the wealth of new ideas and discoveries that he has brought forward in almost every branch of pure and applied mathematics, and which still preserve their fruitfulness; finally, by his methods, for Gauss was the first to restore that rigour of demonstration which we admire in the ancients, and which had been forced unduly into the background by the exclusive interest of the preceding period in new developments. Ant yet I prefer to rank Gauss with the great investigators of the eighteenth century, with Euler, Lagrange, etc He belongs to them by the universality of his work, in which no trace as yet appears of that specialization which has become the characteristic of our times. He belongs to them by his exclusively academic interest, by the absence of the modern teaching activity just characterized. We shall have a picture of the development of mathematics if we imagine a chain of lofty mountains as representative of the mean of the eighteenth century, terminating in a mighty outlying summit,–*Gauss*– and then a broader, hilly country of lower elevation; but teeming with new elements of life More immediately connected with Gauss we find in the following period only the astronomers and geodesists under the dominating influence of *Bessel*; while in theoretical mathematics, as it begins henceforth to be independently cultivated in our universities, a new epoch begins with the second quarter of the present century, marked by the illustrious names of *Jacobi* and *Dirichlet*.

Jacobi came originally from Berlin and returned there for the closing years of his life (died in 1851) But it is the period from 1826 to 1843, when he worked at Königsberg with *Bessel* and *Franz Neumann*, that must be regarded as the culmination of his activity There he published in 1829 his *Fundamenta nova theorice functionum ellipticarium*, in which he gave, in analytic form, a systematic exposition of his own discoveries and those of Abel in this field Then followed a prolonged residence in Paris, and finally that remarkable activity as a teacher, which still remains without a parallel in stimulating power as well as in direct results in the field of pure mathematics. An idea of this work can be derived from the lectures on dynamics, edited by Clebsch in 1866, and from the complete lits of his Königsberg lectures as compiled by Kronecker in the seventh volume of *Gesammelte Werke*. The new feature is that Jacobi lectured exclusively on those problems on which he was working himself, and made it his sole object to introduce his students into his own circle of ideas. With this end in view he founded, for instance, the first mathematical seminary. And so great was his enthusiasm that often he not only gave the most important new results of his researches in these lectures, but did not even take the time to publish them elsewhere.

Dirichlet worked first in Breslau, then for a long period (1831-1855) in Berlin, and finally for four years n Göttingen. Following Gauss, but at the same time in close connection with the contemporary French scholars, he chose mathematical physics and the theory of numbers as the central points of his scientific activity It is to be noticed that his interest is directed less towards comprehensive developments than towards simplicity of conception and questions of principle; these are also the considerations on which he insists particularly in his lectures. These lectures are characterized by perfect lucidity and a certain refined objectivity; they are at the same time particularly accessible to the beginner and suggestive in a high degree to the more advanced reader. It amy be sufficient to refer here to his lectures on the theory of numbers, edited by Dedekind; they still form the standard text-book on this subject.

With Gauss, Jacobi, Dirichlet, we have named the men who have determined the direction of the subsequent development We shall now continue our account in a different manner, arranging it according to the universities that have been most prominent from a mathematical standpoint For henceforth, besides the special achievement of individual workers, the principle of co-operation, with its dependence on local conditions, comes to have more and more influence on the advancement of our science. Setting the upper limit of our account about the year 1870, we may name the universities of *Königsberg, Berlin, Göttingen*, and *Heidelberg*.

Of Jacobi's activity at Königsberg enough has already been said. It may now be added that even after his departure the university remained a centre of mathematical instruction. *Richelot* and *Hesse* knew how to maintain the high tradition of Jacobi, the former on the analytical, the latter on the geometrical side. At the same time *Franz Neumann's* lectures on mathematical physics began to attract more and more attention. A stately procession of mathematicians has come from Königsberg; there is scarcely a university in Germany to which Königsberg has not sent a professor

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The discussion of the *Göttingen school* will here find its appropriate place. The permanent foundation on which the mathematical importance of Göttingen rests is necessarily the Gauss tradition. This found, indeed, its direct continuation on the physical side when *Wilhelm Weber* returned from Leipsic to Göttingen (1849) and for the first time established systematic exercises in those methods of exact electro-magnetic measurement that owed their origin to Gauss and himself On the mathematical side several eminent names follow in rapid succession. After Gauss's death, Dirichlet was called as his successor and transferred his great activity as a teacher to Göttingen, for only too brief a period (1855-59) By his side grew up *Riemann* (1854-66), to be followed later by *Clebsch* (1868-72).

Riemann takes root in Gauss and Dirichlet; on the other hand he fully assimilated Cauchy's ideas as to the use of complex variables Thus arose his profound creations in the theory of functions which ever since have proved a rich and permanent source of the most suggestive material. Clebsch sustains, so to speak, a complementary relation to Riemann Coming originally from Königsberg, and occupied with mathematical physics, he had found during the period of his work at Giessen (1863-68) the particular direction which he afterwards followed so successfully at Göttingen. Well acquainted with the work of Jacobi and with modern geometry, he introduced into these fields the results of the algebraic researches of the English mathematicians Cayley and Sylvester, and on the double foundation thus constructed, proceeded to build up new approaches to the problems of the entire theory of functions, and in particular to Riemann's own developments. But with this the significance of Clebsch for the development of our science is not completely characterized A man of vivid imagination who readily entered into the ideas of others, he influenced his students far beyond the limits of direct instruction; of an active and enterprising character, he founded, together with C. Neumann in Leipsic, a new periodical, the *Mathematische Annalen*, which has since been regularly continued, and is just concluding its 41st volume.

It remains now to speak of the *second Berlin school*, beginning also about the middle of the century, but still operating upon the present age. *Krummer, Kronecker, Weierstrass*, have been its leaders, the first two, as students of Dirichlet, pre-eminently engaged in developing the theory of numbers, while the last, leaning more on Jacobi and Cauchy, became, together with Riemann, the creator of the modern theory of functions. Kummer's lectures can here merely be named in passing; with their clear arrangement and exposition they have always proved especially useful to the majority of students, without being particularly notable for their specific contents. Quite different is the case of Kronecker and Weierstrass, whose lectures became in the course of time more and more the expression of their scientific individuality. To a certain extent both have thrust intuitional methods into the background and, on the other hand, have in a measure avoided the long formal developments of our science, applying themselves with so much the keener criticism to the fundamental analytical ideas. In this direction Kronecker has gone even farther than Weierstrass in trying to banish altogether the idea of the irrational number, and to reduce all developments to relations between integers alone. The tendencies thus characterized have exerted a wide-felt influence, and give a distinctive character to a large part of our present mathematical investigations

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In conclusion a few words should here be said concerning the modern development of university instruction. The principal effort has been to reduce the difficulty of mathematical study by improving the seminary arrangements and equipments. Not only have special seminary libraries been formed, but study rooms have been set aside in which these libraries are immediately accessible to the students. Collections of mathematical models and courses in drawing are calculated to disarm, in part at least, the hostility directed against the excessive abstractness of the university instruction. And while the students find everywhere inducements to specialized study, as is indeed necessary if our science is to flourish, yet the tendency has at the same time gained ground to emphasize more and more the mutual independence of the different special branches. Here the individual can accomplish but little; it seems necessary that many co-operate for the same purpose. Such considerations have led in recent years to the formation of a German mathematical association (*Deutsche Mathematiker Vereinigung*) The first annual report just issued (which contained a detailed report on the development of the theory of invariants) and a comprehensive catalogue of mathematical models and apparatus published at the same time indicate the direction that is here to be followed. With the present means of publication and the continually increasing number of new memoirs, it has become almost impossible to survey comprehensively the different branches of mathematics. Hence it is the object of the association to collect, systematize, maintain communication, in order that the work and progress of the science may not be hampered by material difficulties. Progress itself, however, remains—in mathematics even more than in other sciences—always the right and the achievement of the individual

GÖTTINGEN, January, 1893.

Notes

[1] See Karen H Parshall and David E Rowe, "Embedded in the Culture: Mathematics at the World Columbian Exposition of 1893". *Mathematical Intelligencer* 15.2, 1993, 40-45

[2] See Karen H Parshall and David E. Rowe, *The emergence of the American mathematical research community* (1876-1976): *James Joseph Sylvester, Felix Klein, Eliakim Hastings Moore* American and London Mathematical Societies Series in the History of Mathematics, Providence and London, 1993, especially Chapter 5

[3] They are reprinted in *Felix Klein, the Evanston Colloquium lectures, Erlanger Programm, and other selected works*, ed. J J. Gray and D.E. Rowe. Springer-Verlag, New York, forthcoming.

[4] See David E. Rowe, "Felix Klein's "Erlanger Antrittsrede": a Transcription with English Translation and Commentary". *Historia Mathematica*, 12, 1985, 123-141.

[5] See Jeremy J Gray, *Linear differential equations and group theory from Riemann to Poincaré*. Birkhäuser, Boston and Basel, 1986, especially Chapter 6.

[6] Recently republished with extensive modern commentary, see Peter Slodowy (Hrsg.) *Felix Klein, Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade*. Birkhäuser, Boston and Basel, 1993

It is hardly too much to say that in the traditional education so much stress has been laid upon the presentation to the child of ready-made materials (books, object lessons, teacher's talks, etc.), and the child has been so almost exclusively held to bear responsibility for reciting upon this ready-made material, that there has been only accidental occasion and motive for developing reflective attention. Next to no consideration has been paid to the fundamental necessity – leading the child to realize a problem as his own, so that he is self-induced to attend in order to find out its answer So completely have the conditions for securing this self-putting of problems been neglected that the very idea of voluntary attention has been radically perverted. It is regarded as measured by unwilling effort – as activity called out by foreign, and so repulsive, material under conditions of strain, instead of as self-initiated effort. "Voluntary" is treated as meaning the reluctant and disagreeable instead of the free, the self-directed, through personal interest, insight, and power.

John Dewey