Linking Reality and Symbolism: a Primary Function of Mathematics Education

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To understand a mathematical concept, as opposed to the blind memorization of rules without reason [Skemp, 1971], it is crucial that one link abstract mathematical symbolism with representations from the everyday world whenever this is possible Yet what proportion of the population has developed such representations?

Having discussed the teaching of mathematics with a large number of prospective and practising elementary and middle school teachers over the last several years it has become evident that few are able to provide real world examples for many rudimentary mathematical abstractions. From comments made, the lack of understanding very often stems from the adults' experiences of mathematics during childhood, and the resulting perception of mathematics as a collection of isolated rules to be memorized

A question which often provides some indications as to how a person has learned mathematics is 10 ÷ ½. When answering this question a few contribute the answer 5 (their answer being sensible for the representation they are using — 10 shared in half) but most give the correct answer, 20. However, when asked to explain why the answer is 20, the vast majority are unable to provide a real life example and resort to rules which have been memorized during their school years. As one student teacher wryly explained:

> Ours was not to reason why
> Just invert and multiply

When these same individuals are asked, "How many people can attend a party if there are 10 pizzas and everyone receives half a pizza?" the correct answer of 20 is always quickly contributed. However, the teachers do not necessarily recognize the connection between the verbal question and its equivalent mathematical representation of 10 ÷ ½ Being able to provide real world representations for mathematical symbolism and also provide mathematical models for real world situations are important aspects of a person's mathematical development. Only when one travels comfortably in either direction between the real and symbolic worlds can the term *mathematically literate* be appropriately applied.

Similarly, having representations of the two forms of division, partition and quotition, is important. Adults and children are often capable of providing real world examples which demonstrate partition (8 metres, cut into 4 pieces. How long is each piece?), but few can provide examples which represent quotition (8 metres, cut into pieces, each 4 metres long. How many pieces?). The latter representation is a prerequisite for understanding division by rational numbers between one and zero For 10 ÷ ½ one has to consider "how many halves in ten?" for the symbolism to have meaning The failure to link reality and symbolism is in no way unique to Canada or Canadian teachers [see Ball, 1990; Civil, 1990; Graeber, Tirosh & Glover, 1989; Fischer, 1988; Post, Harel, Behr & Lesh, 1988]

It should not be surprising then that pupils in our classrooms have similar difficulties linking reality and symbolism. The vignettes of classroom activities provided in this article are offered to illustrate that the pupils in our classrooms are extremely knowledgeable when discussing mathematics in a real world context, yet frequently fail to understand the same basic concepts if they are presented only in a symbolic form In effect, the pupils fail to construct links between the abstractions of school mathematics and its real world context. Without such links, mathematical abstractions remain mysterious, and pupils frequently fail to grasp the importance and power of mathematical models for solving real world problems

The first case involved a group of eleven and twelve year old pupils who were discussing the answer to the question 2 ÷ 8. During the discussion various learned rules emerged which led to the incorrect answer of 4:

> — You always divide the big number by the smaller one.
> — 2 ÷ 8 can be read as 2 shared into 8 — treating the ÷ sign as being synonymous with the $\overline{)\ }$ sign.

One pupil knew intuitively that the answer for 2 ÷ 8 was not the same as the answer for 8 ÷ 2, but was unable to convince any of his peers of this fact because he had no concrete representation to illustrate his belief and was unsure of what the answer for 2 ÷ 8 might be The teacher attempted to clarify the situation by asking if there was any difference between two apples being shared between eight children and eight apples being shared between two children. The pupils thought this to be an absurd question. Of course there was a difference, and they all knew what that difference was What was missing for the pupils, however, was the *link* between the symbolic and the real world As one girl explained succinctly:

> "Everyone knows that if you share two apples between eight children, then everyone will get a quarter of an apple. But, if you have 2 ÷ 8 on a test, then the answer would be 4."

Implicit understanding between the real world and the mathematical world does not occur automatically for many of our pupils. Helping pupils make connections between mathematical symbolism and reality is one of the primary functions of mathematics education. It is the teacher's responsibility to facilitate such links and through discussion allow pupils the opportunity to explain the meaning of mathematical symbolism. When questions are within the limits of pupils' conceptual frameworks, discussion permits the sharing of different interpretations of a concept and the clarification of new ideas.

> "A clash of convictions among children can readily cause an awareness of different points of view. Other children at similar cognitive levels can often help the child more than the adult can to move out of his egocentricity." [Kamii, 1973, p. 200].

Such interactive dialogue can allow for beneficial interchange. However, teachers must also be aware that when discussions are not within the conceptual grasp of their pupils they lead to "little more than the sharing of ignorance, prejudice, platitudes, and vague generalities" [Ausubel, 1968, p. 468]. Although little will be gained from the latter situation, well orchestrated discussions can play a decisive role in clarifying a pupils' conceptual understanding. Cloutier and Goldschmid [1978] found that peer interaction among eleven year old pupils studying proportionality was powerful enough to bring about "significant, durable and generalizable improvement" [p. 139], whereas pupils exposed to the same material without discussion did not improve. Pupils need the chance to say what they mean, and mean what they say [Cockcroft, 1982, p. 72], if they are to recognize connections between their everyday world and their mathematical world. Naturally, not all mathematical formulations can be represented in a real world context, and history has shown us that the constraints of real life representations must often be transcended if mathematics is to advance. However, it is often by linking symbolism to real world phenomena that pupils are able to construct their own mathematical understanding and recognize mathematics as an integrated body of knowledge rather than a collection of arbitrary, unrelated facts.

A second case took place in a classroom in England. A research group was searching for a real world context in which directed quantity, a pre-requisite of directed number, could be introduced to a class of twelve and thirteen year old pupils [Bell, Swan, Onslow, Pratt & Purdy, 1986]. Since the negative numers have characteristics similar to the ordinal numbers, it was decided to use the context of "Pop Charts," and have popular records of the day moving up and down the charts. Prior to starting the activity one of the pupils asked a most interesting question: "Are these real pop charts or are they maths pop charts? " When the teacher asked, "What do you mean?" the child replied, "Well, in real pop charts when a record goes down the number gets bigger, doesn't it? But that could never happen with maths pop charts, could it?" The reply indicates how comfortable she was with the real world context which, to her, was logical and made complete sense. However, at this point in her mathematical development the real world context was in conflict with, and disparate from her mathematical world, which was the world of whole numbers and a few positive fractions. If the pupil is unable to link her real world framework to her mathematical world it is likely that operations with integers will be little more than the rote memorization of blindly accepted rules. It is the responsibility of educators to provide opportunities which encourage pupils to construct bridges between symbolism and reality and enable the pupils to cross these bridges in both directions [Reys, Suydam & Lindquist, 1984].

One final illustration examines in more detail the difficulties associated with the multiplication and division of rational numbers discussed earlier. The following questions were included in a test given to a group of 207 grade 7, 8, 9 and 10 (age 12 - 16 years) Canadian pupils (see Figure 1). Half of the pupils from each class wrote test A, the other half wrote test B.

TEST A

1. A relay race is run in ½ km stages. Each runner runs one stage. How many runners are needed to run a total distance of 10 km?
   a) 5 c) 20
   b) 15 d) None of the given answers.

9. $10 \div \frac{1}{2} =$
   a) 5 c) 20
   b) 15 d) None of the given answers

TEST B

1. It takes ½ hour to mark a test. How long will it take to mark 10 tests?
   a) 5 hours c) 20 hours
   b) 15 hours d) None of the given answers

6. $10 \times \frac{1}{2} =$
   a) 5 c) 20
   b) 15 d) None of the given answers

Figure 1
Questions included in the two tests given to split halves of the same classes.

It is often assumed that pupils have difficulty reading mathematical verbal problems and hence perform poorly because of this deficiency. The results from these four questions (see Tables 1 & 2) suggest that context assists understanding and pupils can be more successful solving a verbal question than its symbolic equivalent expression presented in isolation.

It is disconcerting to discover that the results for the two division questions are similar across the grade levels. In fact even more of the older pupils are attracted by the incorrect distractor 5 (see Table 1).

The incorrect answer of 15 for the symbolic question 10 x ½, seems to be the result of younger pupils wanting the answer to be larger than 10 [Hart, 1981], and therefore selecting ten plus one half of ten. Pupils were more successful when answering the verbal questions than answering the equivalent symbolic questions, which indicates a failure to connect the symbolic mathematics with a real world example. Once again it appears that being able to express symbolic questions in a real world context is imperative for the learning of mathematics.

Response percentages to Questions #1 & #9 Test A

Pupil Response	Question number	GRADE 7 n = 28	8 n = 23	9G n = 11	9A n = 13	10G n = 9	10A[a] n = 18
c 20	1	71	74	64	77	56	83
	9	21	39	9	46	22	38
a 5	1	14	22	36	23	33	11
	9	36	35	82	46	56	56
b 15	12	11	4	0	0	0	0
	9	14	9	0	0	0	0
d None	1	4	0	0	0	11	6
	9	25	13	0	8	22	6
Omit	1	0	0	0	0	0	0
	9	4	4	9	0	0	0

[a]In the two Ontario High Schools in which the test was given. approximately 24% of the pupils took general level mathematics (G), and 73% took advanced level mathematics (A) One grade nine and ten class from each ability group was given the test

Table 1

Response percentages to Questions #1 & #6 Test B

Pupil Response	Question number	GRADE 7 n = 26	8 n = 24	9G n = 10	9A n = 20	10G n = 12	10A n = 13
a 5	1	96	88	60	95	84	92
	6	42	33	40	70	67	84
b 15	1	4	0	10	5	0	8
	6	31	25	20	20	0	8
c 20	1	0	8	10	0	8	0
	6	12	0	0	0	0	8
d None	1	0	4	20	0	8	0
	6	15	42	40	10	33	0
Omit	1	0	0	0	0	0	0
	6	0	0	0	0	0	0

Table 2

Further studies of pre- and post-school populations offer additional evidence to support this contention Martin Hughes [1986] has described how very young children understand early number concepts when presented in a real world context, but have little or no understanding of the formal mathematics sometimes demanded of them in today's classrooms The following conversation is between Martin Hughes and Amanda, who is 3 years and 11 months:

MH: How many is two and one? (Long pause No response) Well how many bricks is two bricks and one brick?
A: Three
MH: Okay So how many is two and one?
A: (Pause) Four? (Hesitantly)
MH: How many is one brick and one more brick?
A: Two bricks
MH: So how many is one and one?
A: One, maybe

[Hughes, 1986, p 46]

Amanda's mathematical development was advanced enough for her to be able to combine real world objects but she was unable to recognize the link between the real world questions Hughes was asking about bricks and the typical symbolic questions asked of young children during their early years of schooling Unless Amanda is provided with opportunities to make the connections and allowed to discuss her thoughts with her peers and her teacher, concrete materials will be of little value in the development of her understanding of symbolic addition

Sylvia Scribner [1984] reported on how dairy employees often invented their own mathematical strategies, rather than using traditional school algorithms, for solving everyday problems. When finding the cost of various quantities of milk, experienced dairy employees used a standard measuring unit, the case. All cases were the same size, but because of the non-proportional nature of the smaller containers placed in each case, the amount of milk per case varied. A case held either 4 gallons, 9 half gallons, 16 quarts, 32 pints or 48 half pints. The experienced employees used multiple scaling factors of the case, adding or subtracting smaller sub-units:

Order	*Solution strategy*
17 quarts	Driver took the case price and added a unit price
31 pints	Driver took the case price and subtracted a unit price
48 gallons	Driver took ten times the case price and added two case prices.
98 half pints	Driver took two times the case price and added two times the unit price

[Scribner, 1984 p. 30]

The calculations used by the drivers were more efficient than the conventional school algorithms used by the less experienced workers. Being a flexible problem solver, who is able to adapt mathematical knowledge to a real world situation, saves both time and effort.

Many adults fail to see the relevance of school mathematics to their daily occupations; consequently, positive reactions to mathematics education are scarce In the BBC production *Twice five plus the wings of a bird* [Campbell-Jones, 1986], Mary Harris, from the Maths in Work Project, interviewed different workers about their views of mathematics Although scaffolders used considerable geometrical problem solving to erect their scaffold, they saw their task as being related to safety regulations, not mathematics. When Harris indicated that the Pythagorean theorem was apparently being applied to the construction of a particular layout, they replied, "That's not Pythagoras, luv, that's 3,4,5 " The workers made no connection between the content of their geometry classes and the *mathematics* they were using Adults at the London College of Fashion were equally amazed when they discovered that they had to take a mathematics class as part of their course The practical applications of geometry, ratio and area were far removed from what they held school mathematics to be ". maths was the stuff that you had to learn by heart at school, because there is no point to it otherwise" [Campbell-Jones, 1986]. Unfortunately these adults failed to recognize the link between symbolic school mathematics and the mathematics that was part of their daily experiences. By providing pupils with the opportunity to build mathematical frameworks for themselves from real world experiences which they perceive as being logical and sensible, teachers can assist pupils in constructing a framework for mathematics that is more than the rote memorization of rules without meaning

There is an increasing body of knowledge describing how pupils construct their own understanding of mathematics [Hiebert, 1986; Lampert, 1986; National Council of Teachers of Mathematics, 1989], and the influence of people such as Piaget and Dienes has led to an increase, over the last thirty years, in the number of concrete manipulatives used in our elementary schools. However, one question at issue is whether or not concrete materials are being used effectively. I would suggest that although we want to observe active involvement and discussion in our classrooms —

concrete materials ≠ active learning.

Concrete materials might well invoke mental activity but such materials are not necessarily going to induce it. More importantly they may not help pupils link the symbolic code of mathematics to the reality of their everyday world, because concrete materials can in themselves be quite abstract representations (e.g. Zoltan Dienes' base ten blocks) This statement is not meant to imply that we should no longer use concrete materials in our classrooms, but rather stresses the importance of using the materials appropriately; that is, helping learners to construct representations which make sense to them Seymour Papert made the statement that "Anything is easy if you can assimilate it to your collection of models. If you can't anything can be painfully difficult" [Papert, 1980, p vii]. He expresses how, at an early age, he fell in love with gears, and it was this involvement with gears which provided him with an understanding of ratio and a framework for making sense of mathematical abstractions The concrete materials found in our classrooms need to be used as concrete representations which facilitate discussion and assist the understanding of mathematical symbolism. Manipulatives should be used to complement the real world phenomena which makes sense to the learner and thereby enrich a person's schema for representing abstract mathematical symbolism The NCTM's *Curriculum and Evaluation Standards for School Mathematics* [1989] is a most welcome resource for teachers who are searching for strategies and contexts to assist their pupils' understanding of mathematical symbolism and appreciate the significance of linking it with reality.

The works of Magdalene Lampert [1986] and Constance Kamii [1984 and 1989] illustrate what is possible in classrooms when children are provided with the opportunity to construct understanding in a stimulating environment Both of these educators have the pedagogical content knowledge [Shulman, 1986] necessary to assist children in the development of their conceptual understanding. In the classroom Lampert and Kamii have been able to assist pupils in making connections between what they already know and what they are about to learn The rich background of the two educators enables them to provide suitable metaphors, analogies, and pertinent examples to promote and strengthen the linking of reality with symbolism However, the majority of our elementary and middle school teachers lack the appropriate knowledge base and only receive a limited number of hours of mathematics education in their preservice courses. Most often the emphasis is on decontextualized pedagogy rather than content

To change the way children learn mathematics in our schools, we must first of all enable our teachers to construct new mathematical frameworks for themselves so that they will be able to link the abstract mathematics which they have to teach with real world representations to which they and their pupils can relate. As Marta Civil recounted, some teachers' recollections are that "You only do math in math" [1990, p 7], and Deborah Ball [1990, p. 136] found that almost one third of the prospective teachers in her study thought it impossible to represent 1¾ ÷ ½ in real world terms. I am convinced that most, if not all, teachers are capable of understanding the mathematics which they teach, given the appropriate opportunity. What is not needed, however, is the requirement for future teachers to endure several more advanced level mathematics courses to be learned by rote It is unlikely that memorizing rules to answer advanced level mathematics questions will enhance teachers' understanding of the elementary concepts that they have to teach

Pupils have the potential to understand mathematics if given opportunities to develop connections between symbolic mathematics and appropriate real world representations When pupils cannot provide a real world example for an abstract mathematical concept, teachers should be wary of the pupils' true understanding and explore the concept further. If, as teachers, we cannot provide a representation from the real world to illuminate mathematical abstractions, it is our responsibility to search out such representations whenever they exist, and use them to assist understanding. When one cannot provide a representation for mathematical symbolism then, generally, one does not truly understand that which the symbolism portrays

References

Ausubel D.P [1968] *Educational psychology: a cognitive view* New York: Holt, Rinehart & Winston

Ball, D L [1990] Prospective elementary and secondary teachers' understanding of division *Journal for Research in Mathematics Education* 21(2), 132-144

Bell, A W., Swan, M, Onslow, B A. Pratt, K., & Purdy. D [1986] *Diagnostic teaching: teaching for long term learning* Shell Centre for Mathematical Education. University of Nottingham, Nottingham, England

Campbell-Jones S (Producer) [1986] Twice five plus the wings of a bird (Videotape). London, UK: BBC Enterprises

Civil, M [1990] "You only do math in math": a look at four prospective teachers' views about mathematics *For the Learning of Mathematics* 10(1), 7-9

Cloutier, R, & Goldschmid, M L [1978] Training proportionality through peer interaction. *Instructional Science* 7 127-142

Cockcroft, W H [1982] *Mathematics counts* Report of the Committee of Inquiry into the Teaching of Mathematics in Schools. London: H M S.O.

Fisher, L C [1988] Strategies used by secondary mathematics teachers to solve proportional problems *Journal for Research in Mathematics Education*. 19(2), 157-168

Graeber, A., Tirosh, D. & Glover, R. [1989] Preservice teachers' misconceptions in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education* 20(1), 95-102
Hart, K. [1981] *Children's understanding of mathematics*. London: Murray
Hiebert, J. [1986] *Conceptual and procedural knowledge: the case of mathematics*. Hillsdale, NJ: Lawrence Erlbaum
Hughes, M. [1986] *Children and number*. New York: Blackwell
Kamii, C. [1973] Pedagogical principles derived from Piaget's theory: relevance for educational practice. In M. Schwebel & J. Raph (eds.) *Piaget in the classroom* (pp. 199-215). London: Routledge & Kegan Paul
Kamii, C. [1984] *Young children reinvent arithmetic*. New York: Teachers College Press
Kamii, C. [1989] *Young children continue to reinvent arithmetic*. New York: Teachers College Press
Lampert, M. [1986] Knowing, doing and teaching multiplication. *Cognition and Instruction* 3(4), 305-342
National Council of Teachers of Mathematics, Commission on Standards for School Mathematics. [1989] *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author
Papert, S. [1980] *Mindstorms*. New York: Basic Books
Post, T., Harel, G., Behr, M., & Lesh, R. [1988] Intermediate teachers; knowledge of rational number concepts. In E. Fennema, T. Carpenter & S. Lamon (eds.) *Integrating research on teaching and learning mathematics* (pp. 194-219). Madison, WI: Wisconsin Centre for Education Research, University of Wisconsin
Reys, R. M., Suydam, M.N., & Lindquist, M. M. [1984] *Helping children learn mathematics*. Englewood Cliffs, NJ: Prentice Hall
Scribner, S. [1984] Studying working intelligence. In B. Rogoff & J. Lave (eds.) *Everyday cognition: its development in social context* (pp. 9-40). Cambridge, MA: Harvard University Press
Shulman, L. S. [1986] Those who understand: knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14
Skemp, R. R. [1971] *The psychology of learning mathematics*. London: Penguin

Here is an accomplishment, my lord the king, which will improve both the wisdom and the memory of the Egyptians. I have discovered a sure receipt for memory and wisdom. Theuth, my paragon of inventors, replied the king, the discoverer of an art is not the best judge of the good or harm which will accrue to those who practise it. So it is in this case; you who are the father of writing, have out of fondness for your offspring attributed to it quite the opposite of its real function. Those who acquire it will cease to exercise their memory and become forgetful; they will rely on writing to bring things to their remembrance by external signs instead of on their own internal resources. What you have discovered is a receipt for recollection, not for memory. And as for wisdom, your pupils will have the reputation for it without the reality: they will receive a quantity of information without proper instruction, and in consequence be thought very knowledgeable when they are for the most part quite ignorant. And because they are filled with the conceit of wisdom instead of real wisdom they will be a burden to society.

Plato, *Phaedrus*