INSIGHTS INTO THE SCHOOL MATHEMATICS TRADITION FROM SOLVING LINEAR EQUATIONS

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Theoretical developments in mathematics education offer ways to understand disjunctures between school mathematics and the discipline of mathematics as a function of institutional schooling. For example, Chevallard (1989) offers the theoretical idea of didactical transposition for this purpose:

> Bodies of knowledge are, with a few exceptions, not designed to be taught, but to be *used*. To teach a body of knowledge is thus a highly artificial enterprise. The transition from knowledge regarded as a tool to be put to use, to knowledge as something to be taught and learnt, is precisely what I have termed the *didactic transposition* of knowledge. (p. 56)

In Chevallard's telling, school mathematics originates in the discipline of mathematics. Though Chevallard emphasizes the creative activity needed to transform mathematics in the discipline into school mathematics, his orientation focuses mathematics educators on ways in which school mathematics is a pale or weak representation of the discipline. In the words of Bosch and Gascon (2006):

> The process of didactic transposition starts far away from school, in the choice of the bodies of knowledge that have to be transmitted [...] The transpositive work [...] makes teaching possible but it also imposes a lot of limitations on what can be and what cannot be done at school. It may happen that, after the transposition process, school loses the rationale of the knowledge that is to be taught, that is, the questions that motivated the creation of this knowledge [...] In this case, we obtain what Chevallard (2004) called a "monumentalistic" education, in which students are invited to contemplate bodies of knowledge the rationale of which have perished in time. (p. 53)

Critiquing Chevellard's orientation, Love and Pimm (1996) are concerned about the directionality he suggests from discipline to school and how "it renders invisible the reverse direction of influence" (p. 375). In this spirit, Popkewitz's (1987) orientation to studying what he calls the "formation of school subjects" focuses more on what schooling adds to disciplinary knowledge, that is, more on what has been gained than on what has been lost. Rather than use the imagery of transposition, he suggests that there is an "alchemy" that operates as a result of the institution of schooling. In his view:

> Pedagogy can be thought of as analogous to the medieval metallurgy that sought to transmute base metals into gold. A magical transmutation occurs as academic knowledge is moved into the space of schooling. (2004, p. 4)

In Popkewitz's imagery, alchemy transmutes the base material of knowledge in the disciplines into the gold of school subjects; the directionality has been shifted. For those who consider mathematical knowledge in the discipline as the epitome of our human capacity for reason, Popkewitz's consideration of disciplinary mathematics as a base material from which a more valuable substance (*i.e.*, school mathematics) is created can be counterintuitive. In suggesting that school subjects are more valuable than disciplinary knowledge, Popkewitz is not judging the technological power or the sophistication of school subjects as compared to disciplinary knowledge. One way to interpret his notion of school subjects as "the gold" is that he is focusing on the social value of school subjects. School subjects are the parts of disciplines that society has selected to indicate who is educated and who is not. In support of that interpretation, note that in many countries, performance with respect to school subjects is used to determine, for example, whether a person graduates from high school or not, and that high school graduation influences a person's chances of increased financial rewards in society. Thus, school subjects are "golden" in the sense of providing the high school graduate with financial rewards in society.

The notion of *instructional situation* as articulated by Herbst (2006) further illuminates Popkewitz's metaphor by suggesting another sense in which school subjects are "golden". At the heart of instructional situations is an exchange of student work for the teacher's recognition of the acquisition of knowledge. Thus, instructional exchanges are marketplaces managed by teachers; teachers recognize in the midst of students' mathematical activity those actions taken by students that "trade" as indicators of the acquisition of the knowledge that teachers are supposed to teach. In that sense, these student actions have become academic "gold".

In this article, we explore the concepts introduced above and the relationship between school mathematics and mathematics in the discipline by focusing narrowly on one particular kind of activity in school algebra: the solving of linear equations. First, we examine the nature of the solving of equations in English-language algebra text books

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from the early nineteenth century. We explore whether the solving of linear equations was "transposed" as it moved from an activity in the discipline to an activity in school and, if so, how. We note the increasing presence of a specified order of steps, which we term the canonical method for solving linear equations. Then, in examining teachers' views of correct, but non-canonical, student solutions valued by mathematics educators, we explore questions like: what is it that teachers expect students to do when solving equations? In Herbst's terms, what "trades" as evidence that students have learned to solve equations? And what do these views indicate about the possibility of asking students in schools to solve linear equations more flexibly than by simply using the canonical method? Our exploration of these questions about the solving of linear equations suggests initial understandings of the persistent gap between the school mathematics tradition and disciplinary practices.

The introduction of a method: an observation and a conjecture about solving linear equations in nineteenth century algebra text books

We examined English-language algebra texts from the eighteenth and nineteenth century as one way of exploring formal records of mathematical thought and practice and their relationship to school mathematics. Love and Pimm (1996) view such books as "text books," not yet the textbooks of institutionalized schooling that explicitly presume a teacher and learners. For Love and Pimm, text books occupy a middle ground between texts written by mathematicians for mathematicians and textbooks (see Thurston, 1994, for an argument that all mathematical texts may occupy this middle ground). Text books are texts "used for instruction, to which the accompanying teaching commentary was provided orally by the teacher" (p. 376) [1].

In looking at the presentation of the solving of linear equations in eighteenth and nineteenth century text books, we made this observation: over the course of the nineteenth century the presentation of the solving of equations in English-language algebra texts increasingly seems to include a standard method [2] for solving simple linear equation problems.

The text books we examined from the eighteenth and early nineteenth century present the solving of equations as the application of a set of rules to justify the transformation of equations (*e.g.*, Bland, 1824; Bonnycastle, 1818; Euler, 1765/1822; Ronayne, 1717; Ward, 1724). This manner of presentation follows closely in the tradition of disciplinary mathematical texts from the 1600s, such as works by Viète and Van Schooten, in laying out a set of rules for the "Reduction of Equations" that are justified by axioms of equality, and that can be applied to any linear equation. In these text books, typically the rules for transforming equations are presented in a list or in paragraphs, and then a set of example equations are solved with reference to these rules as justification for the steps of the solution (see Figures 1 and 2).

The nineteenth century algebra text and textbooks that we examined included in their presentation, in addition to axioms of equality and transformation rules, a more-or-less standard method for solving simple linear equations (*e.g.*, Davies, 1867; Peacock, 1830; Smyth, 1836; Wentworth, 1898). Thus, after the presentation of axioms and transformation rules, and their application to particular problems (see, for example, Figure 4, overleaf), these textbooks then include a paragraph or a rule that lists a set of steps, or method, for solving linear equations (see Figures 3 and 4, overleaf). With the inclusion of this presentation of a method, the solving of linear equations is now also presented to the reader as the application of a set of steps that should always be carried out in a particular order.

To reiterate, our observation is about the increasing presence of a method for solving linear equations in nineteenth century English-language algebra text books. In the earlier text books, equations are solved by applying rules that maintain the equivalence of equations and which are justified by properties of equality, while in later text books and textbooks

By Addition.

Rule 1. WHEN any Quantity is connected with the Sign —, it is added by casting away the Sign —, and transferring the Quantity to the other side of the Equation, with the Sign +.

Thus,	1	$a - b - 15 = 60$	*per Axiom* 1.
$1 + 15$	2	$a - b = 75$	
$2 + b$	3	$a = 75 + b$	

Again,	1	$aa - bc - d = cc - ba$
$1 + ba$	2	$aa + ba - bc - d = cc$
$2 + bc$	3	$aa + ba - d = cc + bc$
$3 + d$	4	$aa + ba = cc + bc + d$

By Subſtraction.

Rule 2. When any Quantity is connected with the Sign +, it is to be ſubſtracted, by caſting away the Sign +, and transferring the Quantity to the other ſide of the Equation, with the Sign —

Thus,	1	$a + b + c = d$	*per Axiom* 2.
$2 - c$	2	$a + b = d - c$	
$1 - b$	3	$a = d - c - b$	

Theſe two Rules are called Tranſpoſition of Quantities.

Figure 1. A list of rules in an eighteenth century algebra text book (Ward, 1724, p. 40).

(22.) Examples *in which the preceding Rules are applied, in the Solution of Equations.*

1. Given $4x + 36 = 5x + 34$, to find the value of x.
(17) By transposition, $36 - 34 = 5x - 4x$,
and therefore $2 = x$.

2. Given $x - 7 = \frac{x}{5} + \frac{x}{3}$, to find the value of x.

Here 15, the product of 3 and 5, being their least common multiple, every term must be multiplied by it (18. Cor. 1.), and $15x - 105 = 3x + 5x$;

$\therefore$ (17) by transposition, $15x - 3x - 5x = 105$,
or $7x = 105$;

and therefore (18. Cor. 2.) $x = \frac{105}{7} = 15$.

Figure 2. Two nineteenth century examples of solving equations by applying rules (Bland, 1824, p. 6).

these rules are synthesized into a method that involves a set order in the application of these rules. The presence of this method is consonant with Chazan and Lueke's (2009) description of the solving of equations in school. To capture the difference between the synthesized method for solving equations and solutions that use the rules more flexibly, Chazan and Lueke describe this set order of steps as the *canonical* method for solving equations taught in school (what Star & Seifert, 2006, call the *standard* method for solving equations) [3].

In articulating our observation about nineteenth century mathematical texts, we are not suggesting that textbooks beyond the nineteenth century never treat the solving of equations as the application of a set of rules to justify transformations of equations. In fact, it is not unusual for twentieth century textbooks to introduce the subject in this way before presenting a general method that will solve all linear equations (*e.g.*, Dolciani, Berman & Freilich, 1962). Our claim is about the presence of a method for solving linear equations as a class of problems. Thus, we note that it is very rare for a twentieth century text to present the solving of equations solely as the application of a set of transformational rules (as an exception, see Hart, 1940) without also synthesizing them into an overarching method.

At this time, we do not know when the canonical method was invented; we have no ways of linking it to a particular time or place or author. But we are struck by its increasing presence, and wish to draw attention to the relationship between this trend and the increasing development of institutionalized schooling in the nineteenth century. Speculatively, we suggest that the increasing prevalence of the presentation of a method for solving equations, like the move from algebra text books to algebra textbooks, may be explained by the development of schooling and the notion of didactical transposition as articulated by Chevallard. We wonder whether the increasing prevalence of a method for solving linear equations is a marker of a shift in whom the authors of certain mathematical texts were coming to think of as their modal audience, as the solving of equations shifted from a mathematical activity to an activity of school mathematics. We conjecture that the need to have *a* method for solving linear equations may have developed in parallel with the institutionalization of schooling and the move from mathematics text books to mathematics textbooks. In the remainder of this article, we explore this conjecture by looking at teachers' reactions to student work that departs from the use of the canonical method and illustrate calls for changes in the nature of equation solving in school.

Contemporary changes to school mathematics and calls to reform the solving of equations

Efforts to improve mathematics education over the last 25 years [4] often include attempts to move mathematics instruction in schools closer to the practice of mathematics in the discipline and away from the "school mathematics tradition" (Gregg, 1995). For example, mathematics educators want mathematical argumentation and justification to happen throughout the mathematics curriculum, not just in Euclidean geometry (Healy & Hoyles, 2000; NCTM, 1989; Pedemonte, 2008). Similarly, with a nod to disciplinary

744. The process above-mentioned may be at once translated into the following rule.

"Clear the equation of fractions (1): transfer the terms involving the unknown quantity to one side, and those which do not involve it to the other (2): collect the separate terms, in that member of the equation which involves the unknown quantity, into one (3): divide both members of the equation by the coefficient of the unknown quantity (4), which gives the solution required."

Figure 3. Presentation of a method for solving equations (Peacock, 1830, p. 581).

Solution of Simple Numerical Equations in x.

1. Solve $\quad 3x - 7 = 14 - 4x.$

Transpose $-4x$ to the left side and -7 to the right side,

$$3x + 4x = 14 + 7. \qquad (\S\ 53)$$

Combine, $\quad 7x = 21. \qquad (\S\ 49)$

Divide by 7, $\quad x = 3. \qquad$ **(Ax. 4)**

2. Solve the equation

$$1 - 4(x - 2) = 7x - 3(3x - 1).$$

Multiply the compound factor by the simple factor **in each side,**

$$1 - (4x - 8) = 7x - (9x - 3).$$

Remove the parenthesis in each side,

$$1 - 4x + 8 = 7x - 9x + 3. \qquad (\S\ 40)$$

Transpose, $\quad 9x - 4x - 7x = 3 - 1 - 8.$

Change the signs of all the terms,

$$4x + 7x - 9x = 1 + 8 - 3. \qquad (\S\ 55)$$

Combine, $\quad 2x = 6. \qquad (\S\ 49)$

Divide by 2, $\quad x = 3. \qquad$ **(Ax. 4)**

57. To Solve a Simple Numerical Equation in x, therefore:

Transpose all the terms that contain x to the left side, and all the other terms to the right side. Combine similar terms, and divide both sides by the coefficient of x.

Figure 4. Examples and the presentation of a solution method (Wentworth, 1898, p. 18).

practices, a focus on problem solving seeks to have students consider multiple solution strategies for the same mathematics problem (Levav-Waynberg & Leikin, 2012).

How does such rhetoric apply to the solving of linear equations? What we have called the canonical method for solving equations is well represented in contemporary algebra textbooks. For example, in Carter *et al.* (2003) in the "Concept Summary" of "Steps for Solving Equations", the canonical method is described in five steps:

1. Use the Distributive Property to remove the grouping symbols.
2. Simplify the expressions on each side of the equals sign.
3. Use the Addition and/or Subtraction Properties of Equality to get the variables on one side of the equals sign and the numbers without variables on the other side of the equations sign.
4. Simplify the expressions on each side of the equals sign.

5. Use the Multiplication or Division Property of equality to solve.
   - If the solution results in a false statement, there is no solution of the equation.
   - If the solution results in an identity, the solution is all numbers. (p. 151)

Not all contemporary presentations of this method are identical. For example, one shift in some current presentations of the canonical method for solving equations is that some books introduce students to this method by categorizing equations to solve according to the number of steps it takes to solve them: one-step, two-step, or multi-step equations. For example, Bellman *et al.*, (2004) summarize the solving of two-step equations by telling students to:

1. Use the Addition or Subtraction Property of Equality to get the term with the variable alone on one side of the equation.
2. Use the Multiplication or Division Property of Equality to write an equivalent equation in which the variable has a coefficient of 1. (p. 81)

In such a book, students only meet the full method when they are asked to solve multi-step equations.

Against the backdrop of the presence of a canonical method for solving linear equations in textbooks, calls for multiple solution strategies when solving equations involve greater flexibility in the order in which rules for transforming equations are employed. For example, some mathematics educators (Star & Rittle-Johnson, 2008; Star & Seifert, 2006) suggest that students would learn to solve equations more effectively if they were taught to employ steps for producing equivalent equations in a variety of orders, in light of the context of a particular problem (see Figure 5 for one example).

Similarly, other mathematics educators would like students to use the structure of equations in their solutions

$$20x + 5 = 5x + 65$$

$$4x + 1 = x + 13$$

$$3x = 12$$

$$x = 4$$

Figure 5. Solving equations with a flexible use of rules (a "divide first" solution).

$$4(x + 6) + 5(x + 6) = 81$$

$$9(x + 6) = 81$$

$$x + 6 = 9$$

$$x = 3$$

Figure 6. Seeing structure in an expression when solving an equation (a "structure" solution).

(Hoch & Dreyfus, 2006; Linchevski & Livneh, 1999) and to recognize when it is useful not to distribute multiplication over addition (see Figure 6 for a solution that treats an equation as linear in $x + 6$).

Finally, other mathematics educators have proposed substantive changes to the teaching and learning of school algebra, by, for example, seeking changes to what it is that an equation means in school algebra. One proposal is that for pedagogical reasons, an equation be conceptualized as a particular kind of comparison of two functions (*e.g.*, Yerushalmy & Schwartz, 1993). With such a change, solving an equation means finding the input value(s) for which the two functions produce the same output. Said in a different way, solving an equation means finding the x-coordinate of the intersection of the graphs of the functions whose expressions are on either side of the equal sign (Yerushalmy & Gilead, 1997), or it means finding the x-intercept of the function that is created by subtracting one of these expressions from the other. This last view might lead one to consider solving equations in the way illustrated in Figure 7, in addition to the canonical way.

$$9x + 12 = 22 + 4x$$

$$9x + 12 - 22 - 4x = 0$$

$$5x - 10 = 0$$

$$5(x - 2) = 0$$

$$\text{therefore}, x = 2$$

Figure 7. Finding the x-intercept of the difference function (an "all to the left" solution).

From texts and calls for reform of the solving of equations to teachers' views

We have conjectured that as schooling develops, the presentation of the solving of equations in mathematical text books shifts from a disciplinary focus on the justified and flexible applications of rules to the carrying out of what we have called the canonical method for solving equations. In the previous section, we highlighted some contemporary calls to have school mathematics maintain a stronger connection to the discipline and for the solving of equations to involve *non-canonical* alternatives to what we call the canonical method. In this section, we move from texts and calls for change to exploring teachers' views on the solving of equations. We share representative comments from study groups and an online survey as a way to begin to judge the feasibility of the calls for reform against the backdrop of modal practice.

In an initial exploration, we convened five small study groups of middle and high-school teachers from two US school districts (see Chazan, Sela & Herbst, 2012, for a report on discussions in these groups on doing word problems). In one session, the groups were shown an animation of a classroom scenario in which a student solves an equation as depicted in Figure 5. The teachers in the study groups were genuinely surprised by this solution and while they acknowledged its mathematical correctness, expressed reservations regarding the place of such solutions in class, as the following comments imply:

> It's one thing to say multiple methods are valid. It's another thing to say that they're all effective or that they're clear and simple and clean. [Laurence, HS, 21, 5]

> when I'm selecting problem sets […] I'm selecting a strategy based on what I know of all the strategies out there, that I think's gonna work best for that level of child, and if they find another way I don't inhibit them from- from using it, but I'm not necessarily presenting that method to the class. [Greg, MS, 19-21]

Reflecting on the reasons for their own reaction to the student's solution in the scenario, one teacher evoked the role of the curriculum materials and textbooks:

> I think a large part of why we haven't seen students solve equations this way is because the curriculum guides and everything that has anything to do with the way you're supposed to teach them equations always tells you, "Undo the operations. You're working backwards. You're doing the opposite of PEMDAS [A mnemonic for order of operations: Parentheses, Exponents, Multiplication, Division, Addition, and Subtraction]" [Tina, MS, 11, 47]

To further explore this phenomenon, as part of a larger ongoing project to design instruments to examine the practical rationality of mathematics teaching (Herbst, Kosko & Dimmel, 2013), we have begun to develop an online survey of teachers' views about the solving of equations [5]. One instrument we have piloted explores how algebra teachers respond to the types of solutions presented in Figures 5, 6, and 7.

Seventy-seven practicing mathematics teachers from two different states in the US have completed the pilot survey. All of the teachers, except for seven teachers who specialized in teaching more advanced school mathematics, taught a variety of classes that include the solving of linear equations. The teachers came from almost 30 different school districts (we do not have information about the nature of the textbooks and curricula in these districts, but the number of districts implies some variation among the participants).

The pilot survey presented teachers with two sample student solutions to the same linear equation (shown in Figure 8), one canonical and one non-canonical. These student solutions did not come from a particular student, but rather represent possible ways in which a student could potentially solve the equation.

A set of closed and open-ended questions asked teachers how *typical* each type of solution is, whether it is an *appropriate* solution or not, and whether it should be *introduced* or discussed in class. The questions referred to each type of solution separately and then also in comparison to each other (*e.g.,* Which of the two solutions is better for students to use?). Our goal was to explore whether teachers viewed the canonical method as what students need to master when solving equations, and teachers' views of the place of non-

Method A	**Method B**
$20x + 5 = 5x + 65$	$20x + 5 = 5x + 65$
$-5x \quad -5x$	$\frac{}{5} \quad \frac{}{5}$
$20x - 5x + 5 = 65$	$4x + 1 = x + 13$
$-5 \quad -5$	$-x \quad -x$
$20x - 5x = 65 - 5$	$4x - x + 1 = 13$
$15x = 60$	$-1 \quad -1$
$\frac{}{15} \quad \frac{}{15}$	$4x - x = 13 - 1$
$x = 4$	$3x = 12$
	$\frac{}{3} \quad \frac{}{3}$
	$x = 4$

Figure 8. An example used in the survey of two solutions to the same linear equation.

canonical solutions for solving linear equations in algebra class.

Although the survey happened a number of years after the study groups and in two states in different parts of the country, teachers' survey responses closely resembled the study group reactions to the animation. Regardless of the type of non-canonical solution, the majority of teachers referred to the non-canonical solutions in the survey as "unusual", and as something that the majority of their students would not think to do on their own, unless specifically prompted. Here are some sample responses to the open-ended questions:

> I believe that while this method works, it is not the easiest way to go about doing the problem. Thus, my students probably wouldn't do it this way even though it is correct. [DF, 2001] [6]

> This is not a method that most of my students have seen. I would encourage them to share this and have a conversation about it. However, most of my student would approach it in an "undoing the steps" sort of way. [ATTL, 2016]

> Students could see that it was correct, but they wouldn't try this independently very often. [S, 1980]

These kinds of comments were very common and consistent across all types of solutions [7]; 97% of the teachers indicated that they would hardly ever or only occasionally expect to see their students producing a non-canonical solution. Teachers' responses to closed-ended and open-ended questions suggest that they had a preference for the canonical solutions over non-canonical ones. For example, while 88% of the teachers expected more than half or even all of their students to master the canonical method, only 14% of the teachers had similar expectations of mastery for the non-canonical solutions. When presented with two solutions simultaneously and asked which one is better for solving a given equation, the overwhelming majority of the participants (94%) rated the canonical solution higher than a non-canonical one, regardless of the type of non-canonical solution used. Finally, the overwhelming majority of teachers who completed the survey (95%) indicated that students demonstrate knowledge and mastery of solving equations by producing canonical solutions.

At the same time, the nature of teachers' comments, as illustrated in the excerpts above, included appreciation of non-canonical solutions. More than 8 in 10 of the teachers thought that non-canonical solutions are mathematically appropriate, demonstrate knowledge of solving equations, and deserve full credit on a test. In their written comments, teachers indicated that dividing by a common factor as a first step, whenever there is a common factor, will lead to an equation with smaller numbers that is easier to solve. The perceived advantages of the "Structure" solution included reducing the number of solution steps, and providing good preparation for more complicated problems in the future; 1 in 5 teachers who compared the "Structure" solution to the canonical version, indicated that the former is "faster" and leads to "less possible errors" for the equations where its application is relevant. While the "All to the left" solution was generally less recognized as a useful solution, some teachers saw it as preparation for more advanced mathematical content such as factoring and solving of quadratic equations.

Overall, the teachers' responses suggest the strong expectation that all students should master what we have called the canonical method. Consider the following two teachers' responses:

> My students need absolutes, so I give them a system that works every time. Namely, parentheses, combining like terms, add or subtract, then multiply or divide. [S, 2730]
>
> I definitely teach the first [canonical] method... However, I'm willing to teach the second method [non-canonical] as an additional option when the first is mastered. [DF, 2760]

While these two teachers' responses clearly differ in tone and expressed willingness to engage with non-canonical methods, they suggest that the non-canonical solutions were considered "secondary" methods that have a place in algebra classrooms only in addition to, or after, the canonical method is introduced or mastered. We interpret responses such as these two as suggesting that teachers view the canonical method as having special status in school as compared to other correct mathematical ways of solving equations.

More generally, the survey responses point to the special status, according to teachers, of what we have called the canonical method. In Herbst's (2006) terms, this suggests that solutions that follow the canonical method "trade" as evidence that students have learned to solve equations, while correct, but non-canonical, solutions do not. In this sense, the teachers' responses provide evidence of the alchemical process described by Popkewitz; the canonical method has become the gold of school subject matter. If the canonical method is indeed the result of such an alchemical process, then there is some support for our earlier conjecture that the trend toward the presentation of a method for solving linear equations in nineteenth century text books might be related to the growth of institutionalized schooling.

Understanding potential implications of the teachers' views

We conclude this article by considering calls for flexibility in the deployment of rules for transforming equations when solving linear equations as instances of a more general desire to move school mathematics closer to the practice of mathematics in the discipline. Herbst's notions of instructional situations and instructional exchanges suggest that teachers need ways to manage the work of recognizing the acquisition of knowledge in the buzz of student activity. It is this need that may contribute to the reification of the canonical method. The canonical method's presence in mathematical textbooks seems to be driven by a need to standardize student work in ways that will make it easier for teachers to assess student work and for students to know what will count as successful solving [8]. Teachers' expressions of preference for solutions that are instances of the canonical method, might thus be viewed not just as the product of their individual views and beliefs, but rather as the result of what happens in schooling when people take on the position of teacher and assume responsibility for ensuring that students learn the mathematics that the teacher is teaching.

Going back to where we began this article, the didactical transposition suggests that particular parts of mathematics as a discipline, like the solving of linear equations, will be identified as the mathematics to be taught to students. The alchemy of school subjects suggests that bits of knowledge, say the canonical method, are plucked from the discipline and reshaped to become the "gold" that teachers are responsible for teaching as the cultural knowledge that has been deemed important. Instructional exchanges help us understand that given teachers' needs to support their students in learning this content and their need to be able to transact easily the exchange that is at the heart of the solving of equations in school, the canonical method becomes a useful pedagogical tool. This tool allows teachers, in the midst of classroom activity, to assess whether or not students have learned to solve equations. Thus, the centrality of the canonical method for solving equations in school algebra can be understood as a result of ways the didactical transposition and the alchemy of school subjects shape the instructional exchanges that teachers manage.

Our initial study group and survey responses suggest that calls to have students solve linear equations more flexibly are unlikely to receive a positive response unless that sort of flexibility can become "gold" in the same way that the canonical method did [9]. What remains to be seen is whether the flexibility of multiple solutions can be transmuted by Popkewitz's (2004) alchemy and become the basis for new instructional exchanges that teachers can feasibly be asked to manage.

Notes

[1] The institutional context of such teaching may well have varied by country; these books may have been used both for individual study and tutoring as well as for state or church sponsored schooling of groups.

[2] We use the term method, rather than algorithm, following Chazan and Lueke (2009). There are slight differences in the steps of the methods presented (*e.g.*, whether denominators are cleared first or not; whether transposition is included as one step or as two steps).

[3] In Chazan and Lueke's (2009) description, in school, when the canonical method is being taught, following a set order of steps has perhaps unintended consequences; the focus of student activity is on what specifically is being done to a particular equation (*e.g.*, subtracting $5x$ from both sides) to create equivalent equations; the general justifications for why these actions to both sides of an equation are appropriate (the propositions or rules) are not as prominent.

[4] In the US, for example, see NCTM (1989, 2000) or National Governors Association Center for Best Practices/ Council of Chief State School Officers (2010).
[5] A detailed report is in preparation by O. Buchbinder, D. Chazan & A. Mason-Singh.
[6] The information in the parentheses indicates the participant's ID number and the type of non-canonical solution addressed: "Divide First" (DF), "All to the Left" (ATTL), or "Structure" (S).
[7] In their open-ended responses, the teachers in our sample wrote about methods not solutions. In our interpretation, they saw through the presented particulars to a more general process.
[8] This need for tools that will aid teachers in managing an instructional exchange is akin to Herbst's (2002) understanding of reasons for the development of the two column proof format for writing geometrical proofs.
[9] From our perspective, the Standards for Mathematical Practice in the US Common Core State Standards might be viewed as just such an attempt to make disciplinary practices into school subject gold.

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