New Maths may Profit from Old Methods

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1. "A place to stand"

Mathematics is alive. New parts of mathematics are being developed, most of the existing parts survive in their struggle for life, but every now and then a part loses everyone's attention and passes into oblivion. This dynamical character of the mathematical discipline can be used to bring dynamics into mathematics teaching, as a way to make mathematics lessons lively. It is clearly not *the only* way, and in the hands of a teacher who is not interested in the history of mathematics herself or himself, it may even have a dire effect. My central theme is the following observation which I made in my lessons:

> Mathematics education has a strong static, routine-like side: it teaches standard solutions to all kinds of problems. A look at "old methods" can help teacher and students evaluate theirstandards, to step away for a while from "just doing mathematics" to thinking and speaking about what they are doing, and then to step back to doing, but now doing it more deliberately. Speaking like Archimedes: If you want to see what you are doing, old methods "give you a place to stand".

Such a general claim cannot go without proof. As arguments I shall present some of my classroom activities: four stories by a mathematics teacher, three examples of the application of history to mathematics lessons at secondary school, and one example from a history course for mathematics teachers.

The first example concerns a first year class in a Dutch secondary school (grade 7), with pupils aged about 11 years, the next two examples feature classes three and four (grades 9 and 10) of the same school, and in the last example we meet a group of teachers doing a course in the history of mathematics as part of a programme for earning a higher degree. Finally I shall state some conclusions; but let us now first change the scenery to a geometry lesson in the first year of secondary school.

2. Draw an angle and construct its bisector

When we see grade 7, the first class of secondary school, at work on geometry we are immediately confronted with the standards set by the local mathematical community, in this case by the Dutch curriculum reformers about 25 years ago. In 1968, that strange year of great movements, Euclid's *Elements* ceased to be the spine of the Dutch geometry curriculum. Its place was taken by reflection, rotation and translation, a pale shadow of Klein's *Erlanger Programm*. In the scheme that is used in my school there was no longer room for the classical constructions with ruler and compass, apart from the construction of a triangle when the three sides are given.

So, let's enter Class 1, at the point in the curriculum where the pupils have learned an intuitive definition of angle, they know how to measure an angle with a protractor and they can draw an angle of a given size. For them the problem

> "Draw an angle and construct its bisector."

has a standard solution: draw the angle, use the protractor to measure its size, and draw an angle, which has the same vertex and first leg as the first angle, and which measures a number of degrees that is half the number in the first angle. At the end of the lesson in which they have carried out some of these constructions I tell them that in the past children learned to construct the bisector in a different manner, and I give them a sheet with some exercises to take home (reproduced here in Figure 1; the Dutch original contained the drawing and text fragment in facsimile from the 1629-edition, a transcription in modern characters, and the four Exercises that I had added as guidelines).

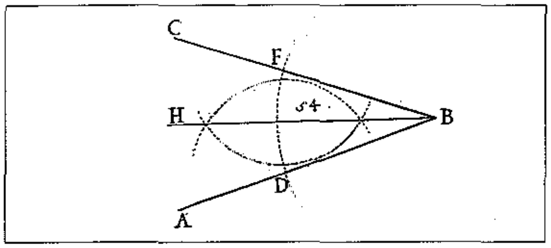
The next lesson is devoted to "Geometry according to Marolois" and has a variety of ingredients. One is inspection of an original copy of Marolois' *GEOMETRIE Ofte MEET-CONST*. This book, which is more than 3 1/2 centuries old, circulates through the class. It evokes amazement throughout, with its printing in gothic type and its beautiful engravings. As another ingredient I sketch briefly the life of Samuel Marolois. He was born in 1572 shortly after his father had emigrated to the liberal Dutch Republic because of religious repression in France. Marolois taught mathematics, gave technical advice to the government, and wrote a number of books, which because of their illustrations kept their fame long after his death in 1620.

As the main ingredient, we discuss with each other the work that pupils have done at home. Marolois' construction of the bisector receives a warm welcome. Pupils think that it is easier to carry out, and more accurate. They had already noticed the differences in their answers when they all had measured the same angle in their books, so the protractor had caused some uncertainty. "Why don't we construct the bisector that way", they ask me. I tell them that constructions with ruler and compass were abandoned when, in 1968, the curriculum was reformed, but that everyone did this construction in the manner of Marolois up to 1968 and that it was taught already way before Marolois, by a Greek geometer called Euclid, who lived in Egypt 2000 years before Marolois. This astonishes them, and it seems to make the new construction even more attractive. The only problem now, I tell them, is that for the time being we have to believe that the construction really gives the bisector, since Marolois does not prove this, and we have not yet proceeded so far that we can do the proof now, but it will come! It is a practical solution; at the

Geometry according to Marolois

1 The bisector

'to cut the angle *B* in two equal parts '



Marolois, *GEOMETRIE Ofte MEET-CONST* (Amsterdam 1629), Figure 54

De 54. Figuyre.

De hoeck B in twee ghelijcke deelen te ſnijden.

VAnt punt B als Centrum laet gemaeckt worden de boghe F D; daer na van de Centrums F ende D, met deſelve wijte ghemaeckt twee bogen/ malcander door ſnijdende in H, ende getrocken B H; die ſal den hoeck recht in't midden ſnijden.

Marolois, *GEOMETRIE Ofte MEET-CONST* (Amsterdam 1629), p. 7

Translation of the text in the box:

"The 54th Figure. To cut the angle *B* in two equal parts. With point *B* as centre let the arc *FD* be drawn; if then with centres *F* and *D* two arcs are made with the same width, intersecting each other in *H*, and if *BH* is drawn, this line will cut the angle straight in the middle."

Exercises

1. Rewrite the text paragraph in your own words
2. Draw an angle *B* of 70°, and draw the bisector of this angle in the manner of Marolois.
3. Draw an angle *C* of 180°. Try to draw the bisector of this angle in the manner of Marolois.
4. Draw a line l with a point *D* on it. Draw with ruler and compass a line through *D* perpendicular to l

Figure 1
English version of the Marolois-worksheet.

moment that is the most important argument for them. And I say that I am quite happy for them to use the method now. From later work it appears that they do!

We now change the scenery to a class which has proceeded two years further, grade 9 doing algebra.

3. Solve $x^2 + 4x = 60$

When we meet Class 3 (grade 9), they are about to finish a chapter on quadratic equations. The year before these pupils learned how to solve the equation which is central in this story:

$$x^2 + 4x = 60$$

Most of them have mastered the standard solution (and all of them should have mastered it), which runs: write the equation in the form $x^2 + 4x - 60 = 0$, then factorize this to $(x + 10)(x - 6) = 0$, and since a product is zero if at least one factor is zero you have $x = 6$ or $x = -10$. Ready, completely standard.

In the previous lessons they have also learned to master the case in which the factorization does not readily work since the discriminant is not the square of a rational number. They know that you have to bring such an equation into the form $(x + p)^2 = q^2$, after which you find solutions $x = -p +/- q$, and at the end this series of steps culminates in the treatment of the general quadratic equation $ax^2 + bx + c = 0$. They have been taught that its solution is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, provided the discriminant $b^2 - 4ac$ is not negative, for in the latter case there are no solutions.

So here is the standard for solving quadratic equations: try to factorize, and if you do not succeed quickly, then use the so called *abc*-formula.

After this series of lessons, full of algebraic manipulations, I thought: it is time for some history. As a starting point I took one of the well-known Dutch algebra textbooks from the first half of the17th century: *ALGEBRA Ofte NIEUWE STEL-REGEL, Waer door alles ghevonden wordt inde WIS-KONST, wat vindtbaer is* by Johan Stampioen (The Hague, 1639). A translation of the title causes a problem typical of Dutch mathematical terminology. Of course "ALGEBRA" can do without translation, "Ofte" means "or", "NIEUWE" means "new", but the word "STEL-REGEL" has no English equivalent. It was coined by Simon Stevin, the great propagator of decimal fractions, who tried hard to replace foreign mathematical terms (that is, almost all of them) by Dutch neologisms. For "algebra" he proposed "stelregel" or "stelkonst", which remained in use until the beginning of the twentieth century, but has now lost the struggle to "algebra". Stevin replaced "mathematica" by "wiskonst" (which appears also in Stampioen's title) or "wiskunde", which is still the Dutch word for mathematics. So the title of Stampioen's book is something like: "Algebra or new algebra, by which everything is found in mathematics which is findable".

As the title shows, Stampioen (1610 - c.1689) had a high opinion of himself. Some people agreed. The father of Christiaan Huygens, for example, asked Stampioen for advice about the mathematical training of young Christiaan. Others entered into furious debates with Stampioen (for details see the biography in the *Dictionary of Scientific Biography* vol. 12, pp. 610-611)

Back to Class 3. I showed them Stampioen's book, and handed out a paper that I had based on it. It contained

- — a facsimile of part of the section in Stampioen's *ALGEBRA* that deals with quadratic equations;
- — an explanation of some strange 17th century terminology; and
- — some questions to guide the pupils when reading Stampioen's text.

To give an idea what this handout looked like I reproduce its beginning (Figure 2), which contains two pages of Stampioen's book and the Dutch glossary to the text. The Dutch handout is accompanied by an English translation of a large part of it, which follows on page 42.

ALGEBRA

Ofte

NIEUWE STEL-REGEL,

Waer door alles ghevonden wordt inde WIS-KONST, wat vindtbaer is.

Noyt voor desen bekendt.

Gevonden, ende beschreven

Door JOHAN STAMPIOEN d'Jonghe,

Mathematicus.

Residerende in 's GRAVEN-HAGE.

Matth. 10.

Want daer en is niet bedeckt, het welck niet en sal ontdeckt worden; ende verborgen, 't welck niet en sal geweten worden.

'S GRAVEN-HAGE,

Ghedruckt ten Huyse vanden Autheur in *Sphera Mundi.* 1639.

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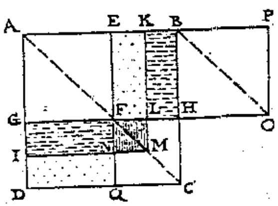
1 ② — 6 ① ghelyck — 8
½) — 3
— 3
+ 9
— 8
√ 1
1 ofte — 1
— 3 — 3
1 ① ghelyck + 4 ofte + 2.

comt + 1. diens vier-cant-wortel is 1 ofte — 1. daer noch van ghetrocken — 3 de helft van 't ① ghetall, rest de waerde des 1 ①. 4. ende 2.

'T BEWIJS.

Om desen handel te bewysen, soo laet daer toe bereydt zijn het vier-cant A B C D. dat gesneden is met de linien E Q. ende G H. even-wydigh teghen te zyden. Dit soo zijnde, soo nemen wy dat de zyde des vier-cants A E F G als A E ofte E F doet 1 ①, soo doet sijn inhoudt 1 ②. daer by ghedaen een recht-hoeck, diens inhoudt is 4 ①. die wy stellen te zijn E B H F, daer van E B doet 4. (het ① ghetall) soo is E B H F 4 ①. Comt dan voor den gantschen recht-hoeck A B H G 1 ② + 4 ①. die ghelyck ghesevdt wordt 60. Neemt nu de helft van E B, comt E K ofte K B 2. diens vier-cant is 4 voor F L M N, 't selfde ghedaen tot den winckel-haeck K L F N I A die noch ghelyck is 't rechthoeck A B H G (om oorsaecke dat K H, daer af ghenomen is, ende gevoeght aen G F, wordende alsoo F I.) comt het vier-cant A K M I 64. diens vier-cant-wortel is 8 voor A K. daer van ghetrocken E K 2. rest A E 6. de waerde des 1 ①.



Maer

Vierkantsvergelijkingen in een zeventiende-eeuws Algebra-boek

De volgende vreemde woorden en notaties komen voor:

<u>Stel-regel</u> door Simon Stevin ingevoerd Nederlands woord voor algebra

p. 43 <u>Stel-regelsch</u> is het bijv naamw hierbij, en betekent dus algebraïsch

<u>Teerling</u>: kubus Een <u>Teerling-Stel-Regelsche verghelycking</u> is een vergelijking waarin de derde macht (= kubus = <u>teerling</u>) van de onbekende voorkomt (zoals $x^3 + 3x^2 + 4x = 8$)

p. 43/44 ② staat voor de onbekende tot de tweede macht. Wij schrijven tegenwoordig x^2.

1 ② + 4 ① ghelyck 60 staat dus voor $x^2 + 4x = 60$

<u>ledighe ghetallen</u>: constanten, dwz getallen die niet met x vermenigvuldigd zijn, zoals hierboven 60

p. 44 <u>Generalen Regel</u>: algemene regel

<u>ware</u> en <u>ghedichte</u> wortel: bij Stampioen kan $\sqrt{64}$ zowel 8 zijn (de <u>ware</u> wortel) als −8 (de <u>ghedichte</u> = denkbeeldige wortel)

Figure 2

Start of fragment from Stampioen's *ALGEBRA, pp. 43-46*

1. Chapter.

On the resolution of quadratic algebraic equations.

The equation is said to belong to the algebra of quadratics when 2 + 1 is equated to bare numbers, in which equations three sorts of cases can be noted. Namely:

1 ② + ①	equal to	⓪	First
1 ② - ①		⓪	Second
1 ② - ①		- ⓪	Second

For which three cases we make a general rule. By which each of those is mathematically resolved, as follows:

General Rule

Take half of the ① number, to its square add the bare number, of the sum extract the square root, from the outcome subtract half of the ① number (continuously paying good attention to the signs + and –) and there comes always the value of 1①.

Example on the first case.

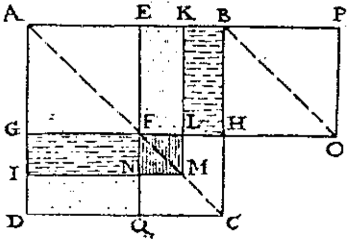
Let 1② + 4① be equal to 60. Half of the ① number is +2. Its square +4. Thereto added the bare 60, comes 64. Of this the square root is 8, the true one, and -8, the fictitious one. From it subtracted +2, the 1/2 of the ① number, remains 6, for the true and -10 for for the fictitious, which both satisfy as value of the ① number.

Take notice:

Since the square-root of 64 is both 8 and -8 (for each of them, when squared, gives 64), therefore this equation has two sorts of values for the ① number, as above, which can be checked as follows:

1①	equal to 6	1①	equal to -10
1② + 4①	equal to 60	1② + 4①	equal to 60
36 24		100 -40	
60		60	

[examples "on the second case" and "on the third and last case" left out; J.v.M.]



THE PROOF

In order to prove this thing, so let therefore be prepared the square *ABCD*. Which is cut by the lines *EQ* and *GH* parallel to the sides. This being so, let us suppose that the side *AE* or *EF* of the square *AEFG* does 1①, so its area does 1②. If we add a rectangle of which the area is 4①, which we suppose to be *EBHF*, of which *EB* does 4 (the ① number), so *EBHF* IS 4①. Then comes for the whole rectangle *ABHG* 1② + 4①, which is said to be equal to 60. Now take half of *EB*, that is *KB* 2, of which the square *FLMN* is 4. If the same [square] is added to gnomon *KLFNIA*, which is equal to rectangle *ABHG* (for reason that *KH* [i.e. rectangle *KBHL*] has been taken off and added to *GF*, therefore becoming *FI* [i.e. rectangle *GFNI*]), there comes the square *AKMI* 64. Its square-root is 8 for *AK*. If *EK* 2 is subtracted from it, there remains *AE* 6, the value of 1①.

End of fragment from Stampioen's *ALGEBRA*

In order to guide my pupils in reading this 17th century text I had prepared the following questions:

1. Stampioen distinguishes three types of equations (three "Cases"). For each type he presents an example. First solve the equations in your own way, and then see how Stampioen does it
2. Does Stampioen's "General Rule" agree with your own way of solving quadratic equations?
3. For the first "case" 1② + 4① = 60 Stampioen on p. 46 gives a "Proof" in which he uses a geometrical diagram. He literally completes a square. Describe his procedure in your own words

The pupils first studied these questions at home, and then, at school, discussed them in pairs. A plenary discussion followed, and the sound was the sound of surprise. The accessibility of the geometrical proof, as compared to its formal algebraic counterpart, was the main reason for this surprise. On the other hand the weakness of the geometrical proof was also recognised. Since it represents numbers by line segments it does not allow the unknown to be negative (in practice Stampioen applied without worrying the "Rule", which he had proved only for positive numbers, to find negative solutions)

Another surprise was caused by the notation and the rhetorical phrasing of the "General Rule". Both were strange to these pupils who are growing up with the standards of modern symbolical algebra. I told them that our way of writing equations was introduced by Descartes (in a book that he wrote and published in the Netherlands) in 1637, i.e. two years before Stampioen published his *Algebra*, and we spoke about the advantages of Descartes' notation over earlier systems and over the system of Stampioen.

The realization that Stampioen's Rule is equivalent to the *abc*-formula, since it leads to the well-known formula when applied to 1②+$\frac{b}{a}$① = $-\frac{c}{a}$,, gave rise to a new view on a thing like $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. Apparently this static, almost graphical, structure of letters and other symbols could also be seen as a list of actions that have to be carried out:

"Take half of the ①-number . ..". And, said Janneke, one of the pupils, since you must carry out these actions, why not remember them directly? We concluded that you could remember the "Rule" if you would also remember that you had to transform $ax^2 + bx + c = 0$ to $x^2 + \frac{b}{a}x = -\frac{c}{a}$ first (Janneke indeed decided to do so), but that an alternative was to read and remember $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$ rather as a list of actions than as a structure of symbols

Furthermore I told them that Stampioen had not found his "Rule" and geometrical proof himself. He owed much of his work to earlier colleagues, especially to the Arab mathematician Al-Khwarizmi who, about A D. 825, wrote a book *Hisab al-jabr wa'l-muqabala* in which he gave rules with geometrical proofs for the solution of quadratic equations. In the Latin translation of the book (c 1140) "al-jabr" appeared untranslated as "algebra", which became the common name of the theory of equations So this one word "algebra" served as a pivot for a snapshot history of mathematics

So far this account of history in an algebra lesson in Class 3. The next section returns to the same pupils, but one year later, in Class 4.

4. Estimate $\log_{10} 5$

What would pupils at the end of the fourth year answer to "Give the first three decimals of $\log_{10} 5$ or $^{10}\log 5$, as it is written in Dutch schools and in the rest of this section? I expect that most of them will be able to produce the "right" answer 0.699. Some of them will even be able to "prove" this by keying 10 ^{699} into their calculator to check whether the outcome is 5 or almost 5 (I do not say that I expect this result, it is more a kind of hope) They have grown up with the axiom "What the calculator says is true"; further questions are superfluous and almost unethical And whoever substitutes in this axiom "table" for "calculator" will realize that this situation is as almost as old as the logarithm itself The validity of the answer is based on trust in an external device, not on independent reasoning. For practical purposes I also use and teach the calculating machine, of course, for do not think that I am going to calculate by hand the moment when my bank account (which started in year 0 with 1000 guilders, or according to the more realistic textbooks, in the year 1953 with 25 guilders) will pass 1,000,000 guilders. But mathematics should go further than trust in a device.

The last time that I taught the logarithm in my fourth class, I decided to go a bit further, and to find out, with my class and with the assistance of Leonhard Euler, how logarithms can be calculated

The idea of consulting Euler was based on several reasons In the first place Euler was one of the first mathematicians to define the logarithm as the inverse of an exponential function, just as we do today. Next, it is always a pleasure and very rewarding to read Euler, since his writings are clear, open, and rich, and he has a practical approach to mathematics without neglecting its principles. And last but not least, the relevant text "On exponential and logarithmic quantities" (Chapter 6 in Euler's famous *Introductio in analysin infinitorum* of 1748) is readily available, both in English translation [*Introduction to Analysis of the Infinite*, translated by John D. Blanton, New York etc.: Springer, 1988] and in a facsimile reprint of the 1748 Latin edition [Brussels: Culture et Civilisation, 1967].

Euler's approach to the logarithm looks very modern He takes an arbitrary positive constant a and says: "given a positive value for y, we would like to give a value for z, such that $a^z = y$. This value of z, insofar as it is viewed as a function of y, is called the LOGARITHM of y" (quoted from [Euler, 1988], p. 78). From this definition Euler derives the well-known rules for calculations (§ 104), like

$$\log vw = \log v + \log w, \text{ and}$$
$$\log v^n = n \log v,$$

and he notices the monotonic property of the logarithm (§ 106). An application is: the logarithm of the geometrical mean ($\sqrt{AB}$) of two numbers A and B is the arithmetical mean of the logarithms $\frac{1}{2}(\log A + \log B)$. He is then able to show how a logarithm can be approximated. His example is $^{10}\log 5$. Since $1 < 5 < 10$ and $x \rightarrow {}^{10}\log x$ is monotonically increasing Euler finds the following boundaries:

$$0 = {}^{10}\log 1 < {}^{10}\log 5 < {}^{10}\log 10 = 1$$

Then Euler takes $\sqrt{1 \times 10} \approx 3\ 162277$, which is the geometrical mean between 1 and 10. Its logarithm is $\frac{1}{2}({}^{10}\log 1 + {}^{10}\log 10) = 0\ 5000000$, and since $\sqrt{10} < 5 < 10$ Euler has

$$0\ 5000000 = {}^{10}\log \sqrt{10} < {}^{10}\log 5 < {}^{10}\log 10 = 1$$

Euler repeats this procedure Now the geometrical mean is taken between $\sqrt{10}$ and 10, which is $\sqrt{\sqrt{10} \times 10} \approx 5,63413$. Its logarithm is $\frac{1}{2}({}^{10}\log \sqrt{10} + {}^{10}\log 10) = 0\ 7500000$, and since $\sqrt{10} < 5 < \sqrt{\sqrt{10} \times 10}$ Euler finds

$$0\ 5000000 < {}^{10}\log 5 < 0\ 7500000$$

And in this way Euler proceeds. At each step the interval in which $^{10}\log 5$ lies is bisected, and after a total of 22 root extractions in total he arrives at $^{10}\log 5 \approx 0.698970$. (He displays the results of his calculations on page 76 of Volume 1 of the *Introductio*.) In this manner, Euler comments, Briggs and Vlacq calculated their tables, but later on methods were found by which logarithms can be calculated much quicker. One of these methods uses the series expansion of $\log (1 - x)/(1 + x)$ and is discussed by Euler in a later chapter of the *Introductio*.

When I saw this passage in Euler I liked it as an example of good teaching. Euler does not want to use a table before he has explained how it was (or could have been) constructed, and I decided not to use the calculator before I had explained to my class how you can calculate a logarithm yourself. Besides an understanding of the basic principle

$$\log \sqrt{a \times b} = \frac{1}{2}(\log a + \log b)$$

the calculation requires that you know an algorithm by which you can approximate square roots. And happily the pupils knew such an algorithm from two years before. So I decided to introduce Euler to my class. I told them about his enormous creativity and about his efforts to standardise what other mathematicians had developed in the two centuries before him, as he did with the logarithm of Napier and Briggs. I told them about the importance of the logarithm, and of tables of logarithms in particular, as a means to speed up multiplication, division, and root extraction, and said how Euler thought about these tables: you must use them, but not without knowing how they were composed. You should base your trust in them on your own independent judgement.

And there we went. We discussed the passage in the *Introductio* where Euler derives the rule $\sqrt{a \times b} = \frac{1}{2}(\log a + \log b)$. With some explanation the Latin text, which the pupils had already studied at home, was clear. Then we checked the steps in Euler's calculation up to $\sqrt{\sqrt{10} \times 10} \approx 5.62$, that is up to finding:

$$0.5000000 < {}^{10}\log 5 < 0.7500000$$

and then we believed that his subsequent calculations, which were also in their working material, were right. And so Euler guided us to: ${}^{10}\log 5 \approx 0.698970$

This procedure provoked several comments:

- Clever guy, this Euler, and honest too, since he writes in such a manner that you can check what he says. Of course, you will never need to calculate a logarithm by hand, but Euler's presentation makes that you can trust this man.
- How strange that it is in the mathematics hours that we really use our knowledge of Latin.
- How beautiful that old book is ..... (indeed: if you have never seen it, try to find a library which has a copy of the 1748 edition: it is rewarding)

How essential is history in such a lesson? Can't you teach the whole thing without mentioning Euler? We shall return to these questions in the final section.

5. Calculate the area of the 13-14-15 triangle

The last example comes from the Faculty of Education of Utrecht Polytechnic, which offers a 2 1/2 years part-time programme for teachers who already have the lower of the two degrees in teaching and who want to earn the higher degree. As well as mathematical and educational subjects this programme includes a 60-hour course in the history of mathematics. Students are expected to increase their knowledge of the history of mathematics, and they are also set to develop practical skills, which include reading and presenting original sources and applying historical material in their lessons.

The task "Reading and presenting an original mathematical text" is part of the course since it combines two skills that are fundamental for any historian. Without the study of original sources history degenerates to retelling second-hand stories, and only in presenting the results of the source study to others does one "write" history. For this task the students work together in pairs. They choose a text from a collection of about 20 very diverse original mathematical documents which I have gathered together for them. The collection contains short texts (e.g. Proposition 1 of Archimedes' *Measurement of a circle*) and long texts (e.g. 20 pages from a Dutch 18th century textbook for surveyors), texts from many different periods and cultures, influential texts (e.g. sections from Euclid's *Elements*, Descartes' *Géométrie*, Euler's *Introductio in analysin infinitorum* and Gauss' *Disquisitiones Arithmeticae*) and forgotten texts (e.g. a "try to discover the mistake" pamphlet printed in Vienna, 1850), and the texts cover a wide range of mathematical subjects (algebra, classical construction geometry, coordinate geometry, quadrature, early calculus, exponential and logarithmic functions, number theory and several other subjects). The pairs are asked to give a talk of half an hour which concentrates on the contents of the text, but which also gives some information about the text (where and how does it fit into the history as a whole, why and by whom was it written, in what sense was it important, ...). Through a plenary discussion, in which I also participate (and sometimes rather substantially), the students add the results of the interpretation of the text to their knowledge. In this sense the group is partly autodidactical.

The problem that I want to discuss here stems from such a presentation text. The whole text is a selection of four pages from a Dutch manuscript on practical geometry (Leiden University Library, Dept. of Western Manuscripts,

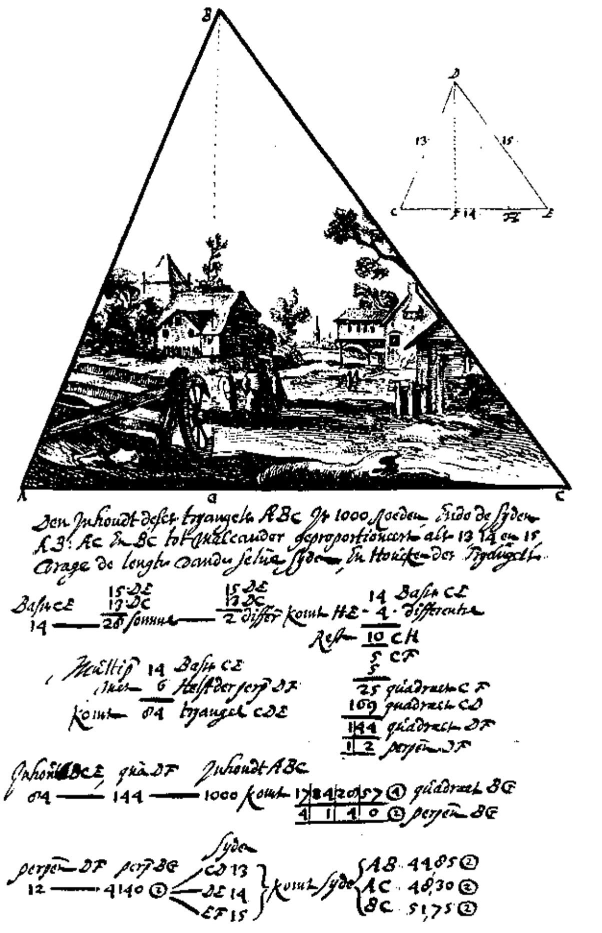


Figure 3

BPL 1013), written about 1623 by Frans van Schooten, who taught at the Leiden Engineering School and whose courses in the vernacular were influential in The Netherlands. One of the sons of Van Schooten, another Frans van Schooten, succeeded his father in 1646 and earned himself some reputation as the great propagator of Descartes' geometry.

One of the problems in Van Schooten's text, represented here with the facsimile in Figure 3, and with an English translation of the text, concerns a 13-14-15 triangle of area 1000 The sides and angles of the triangle are asked for. In the translation abbreviations have been preserved: differ for difference, squa for square and perp or perpen for perpendicular

Start of translation of the Dutch text in Figure 3

"The content of this triangle *ABC* is 1000 ares and the sides *AB*: *AC* and *BC* are proportioned to each other as 13 14 and 15, One asks the length of these sides and the angles of the Triangle

	15 *DE*	15 *DE*	
Basis *CE*	13 *DC*	13 *DC*	14 Basis *CE*
14 ———	28 sum ———	2 differ̄ comes *HE* –	4 difference
		Rest	10 *CH*
			5 *CF*
Multip̄	14 Basis *CE*		5
by	6 Half of the perp̄ *DF*		25 square *CF*
comes	84 triangle *CDE*		169 square *CD*
			1|44 square *DF*
			1|2 perpen̂ *DF*

Content *DCE*	squā *DF*	Content *ABC*		
84 ———	144 ———	1000 comes	17| 14| 28| 57	④ square *BG*
			4| 1| 4| 0	② perpen̂ *BG*

		sides		
perpen̂ *DF*	perp̄ *BG*	*CD* 13		*AB* 44 85②
12 ———	4140②	*DE* 14	comes side	*AC* 48,30②
		EF 15		*BC* 51 75②

End of translation

How would we solve this problem nowadays? Students presenting this text said that they would first calculate the area of a triangle of sides 13, 14 and 15 (in Van Schooten's drawing in the top right corner) using their standard method, i.e. applying the law of cosines It yields cos C = 5/13, and consequently CF = 5 and DF = 12, where DF is the perpendicular in ΔDEF.

There were at least two different standard methods in the past. One, which was standard at least in medieval Islamic mathematics, was to first calculate the area of the 13-14-15 triangle with the help of Heron's rule : $\sqrt{s(s-a)(s-b)(s-c)}$ (in which s is the semi-perimeter of the triangle). From A and the base CE = 14 the height DF would follow immediately.

Van Schooten used yet another method for calculating the height DF, and my students had to find out how and why it worked One of them presented the following explanation. In the little triangle CED in the top right corner there is a point H on CE, and HE is calculated from the proportionality

$$\frac{CE}{DE+DC} = \frac{DE-DC}{HE}$$

Everything in this proportionality is known, apart from HE, which therefore can be determined: HE = 4 (by the way, in the text the proportionality is expressed in the "rule of three" language, which my students knew from an earlier lesson). But how then is this point H defined? The clue is in the calculation which follows directly after HE was found. The next steps are $CH = CE - HE = 14 - 4 = 10$ and CF = 5 There is no indication how CF = 5 relates to CH = 10, but the little triangle clearly suggests that H is drawn so as to make CF and FH equal, that is: an isosceles triangle ΔCDH is constructed within ΔCDF (see Figure 4).

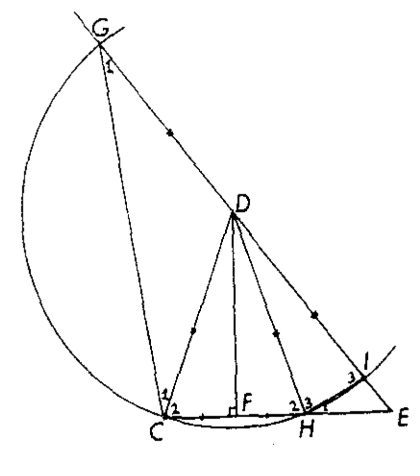


Figure 4: $\frac{CE}{DE+DC} = \frac{DE-DC}{HE}$

Furthermore the terms $DE + DC$ and $DE - DC$ in the proportionality have a natural interpretation in the diagram. For, construct the circle with centre D and radius DC First, it will pass through H since $DC = DH$. Secondly, it will intersect the side DE in a point I such that $EI = DE - DC$. And finally it will intersect ED prolonged in a point G such that $EG = DE + DC$. And, behold, there are similar triangles

$\Delta CEG \sim \Delta IEH$

from which the proportionality follows directly One detail is still to be verified, of course (viz. $\angle G_1 = \angle H_1$), but that goes smoothly. It is even quicker, I remarked in the discussion, to use the theorem that for any line through E that intersects the circle in points P_1 and P_2 (which may coincide in P, in which case the line EP is tangent to the circle) the product $P_1 \times P_2$ is constant. The proportionality is now straightforward, since the theorem yields $EH \times EC = EI \times EG$ where $EI = DE - DC$ and $EG = DE + DC$. Apart from some students who were in secondary school before 1968 (late vocations) no one knew the theorem.

What does history provide in this case?

Students report that they find it a difficult but very interesting puzzle, first to find out what the handwritten text says and then what Van Schooten meant and why it worked. So, at any rate history may provide an interesting puzzle. Furthermore such a problem makes students aware that methods and standards are changing. Most teachers know, at least theoretically, that there may be many different solutions to one and the same problem, but seldom do they have to analyze this phenomenon so consciously. In this sense history stresses the dynamical character of mathematics, and it trains teachers to be open-minded. History shows that mathematics can be presented in different ways and designs. Van Schooten had an artist illustrating his text. After many church towers and mountain peaks this unknown artist (maybe Van Schooten's brother Joris, who was a well-known painter, but about 1620 there were other successful artists living at Leiden, one of whom was Rembrandt) could not rein in his creative energy when the text changed from calculating heights and distances to more theoretical problems. That is also a thing that history can provide: amazing examples, mathematics with a different outlook.

6. Conclusion

To guide the evaluation of the material discussed above I shall turn the title of this article into a question: "May new maths profit from old methods?"

In my opinion all four examples present arguments for a positive answer. Excitement, enthusiastic pupils and students, a fresh look at everyday things, a stimulus for a discussion about standards, cross-curricular work, puzzles: if old methods have these effects I tend to say that they are profitable for new maths. Often present-day methods are handier and quicker, but exactly that can also be very disappointing for pupils. Their own solutions to problems are not always the handiest and quickest, and the knowledge that there is no royal road to mathematical truth may comfort them.

Of course, a question that starts with "May ..." easily admits a positive answer. What then about a sharpened version of the question as posed by Hans Freudenthal. In 1977, in a lecture ["Should a mathematics teacher know something about the history of mathematics", *For the Learning of Mathematics* Vol. 2 No. 1 (July 1981) pp. 30—33] he asked: "Can we instil into "inhuman" mathematics more humanity by convincing the learner that mathematics has been conceived by men ..." and added doubtfully "or wouldn't it be a shorter way, a stronger proof to have some mathematics they are really concerned with recreated by the students themselves?" Provocatively Freudenthal went on to say that for language students knowledge of historical grammar "could be useful to students who aspire to more profound linguistic understanding. But Sanscrit as a precondition for studying modern languages was abolished quite a time ago at European universities ......".

This way of stating the question suggests that there is a royal road to learning mathematics, and already the diversity of huge ICME congresses shows that such a road does not exist. Active learners, who achieve their learning in a good balance between efforts and results, is what teachers want (at least I hope that I am not the only one to say so), and if these learners bring happy faces into the classroom, even better. Having students recreate mathematics they are really concerned with can work out perfectly well. Doing games or concentrating on the applications are other ways to make mathematics attractive. And old methods may add flavour as well, as I hope to have shown. History is not a precondition for learning mathematics, it is a valuable help, and, just as with the "doing games" approach, it may be effective in the hands of one teacher and completely unsuitable for the colleague next door.

Behind any door, though, one should consider it as a possibility. And it is time that educational research be done in order to find out what the effects may be of using history in teaching mathematics. It is a good start that, at ICME-8 in Seville, the International Commission on Mathematical Instruction (ICMI) decided that "The role of history in teaching and learning mathematics" will be the theme for the next ICMI Study. The Discussion Document for this study is now available in a number of journals and also via the Web, at http://www.math.rug.nl/indvHPs/Maanen.html#dd, and in French at http://www-leibniz.imag.fr/DDM/ICMI.html