Calculations in the Style of Kepler

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Freudenthal (in [1981]) writes: "As a young assistant I built into my 'Analysis' course historical relations—there are plenty indeed. This then was to my students the signal to put down their pens and to have a rest. To examination questions as to when logarithms were invented I could expect all periods from fifth century B.C. to the twentieth A.D."

In the 1980s a new fashion started in mathematics textbooks in Czecho-Slovakia. Here and there several lines of historical information would be inserted, mostly stating the years in which there were important turning-points in the development of the discipline and the names of some mathematicians with brief biographical notes. When referring to these in class a teacher would often be asked, "Will we have to know this for the exam or not?"

The two situations, though distant from each other in time and place, are close from another point of view: the historical material was not integrated with the subject matter and the pupils felt it to be extraneous. The goal of this article is to describe a teaching experiment which endeavoured to make the history of mathematics "... a suitable and attractive vehicle through which many mathematical topics can be learned, relearned (remediation) and enriched, without the feeling of *déja vu*." [Arcavi, Bruckheimer & Ben-Zvi, 1982, 1987]

Experimental teaching

The experiment was carried out in a class of 16-year old children with higher mathematical ability. We set the context by telling a short story about Kepler's life [Bero, 1989], stressing his computational abilities and the enormous quantity of astronomical calculations he carried out [Edwards, 1979]. Kepler displays his extraordinary computational abilities in his book, *Nova stereometria doliorum vinariorum* [Kepler, 1987], which contains 87 evaluations of the volumes of solids of revolution. Two examples from this book were presented to the pupils.

Example 1: Volume of a torus

A torus is generated by revolving a circle $k(S, b)$ around an axis lying in the plane of the torus at a distance a from its center S. We dissect the torus into thin slanting cylinders by means of planes containing the axis of revolution. We pile these slices on top of each other to form a cylinder with radius b and height $2\pi a$. Then the volume of the torus is equal to the volume of the cylinder, which is

$$V = 2\pi a . \pi b^2 = 2\pi^2 ab^2.$$

This is the method Kepler used in *Nova stereometria*. He dissected a given solid into an "infinite" number of very small particles and after "suitably" reshaping these he composed another solid whose volume he was able to calculate.

Example 2: Area of a circle

Kepler wrote: "I divide the circumference into as many parts as there are points on the circumference, that is an infinite number. We consider each of them to be a base of an isosceles triangle with altitude r." Kepler's further considerations we condense slightly: If we unroll the circumference onto the line segment KL then the (infinitely small) arc AB will fall onto the line segment $A'B'$ and the sector ABS will be transformed into the triangle $A'B'S$ having the same area.

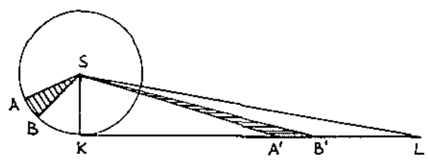

Figure 1

All these triangles together will form the right-angled triangle KLS whose area will equal that of the circle, that is

$$A = (KL) . \frac{(KS)}{2} = \frac{1}{2} . 2\pi r . r = \pi r^2 .$$

When we presented these computations to the pupils we observed an unexpectedly strong aesthetic response. We seized the opportunity and challenged the pupils to try to evaluate other volumes, plane and surface areas, in a similar way.

What the students produced

Over the next few days the pupils brought many nice evaluations, correct and incorrect. We present some of them here. (Two of the following examples are in Bero & Szelepcseny [1985].)

Robert

Evaluation of the area of the region bounded by the curve $y = \cos x$ and the portion of the x-axis between $-\pi/2$ and $\pi/2$.

• Considering a sphere with radius 1, I conceived it as composed of an infinite number of thin rings lying in planes perpendicular to the diameter AB of the sphere. Geographically the points A and B are the south and north poles and the rings correspond to the parallels of latitude.

The length of each ring is $2\pi\cos x$. I straightened the rings (circles) into bands (segments) and put them together in such a way that one end was lying on the segment with endpoints $(-\pi/2, 0)$, $(\pi/2, 0)$ (in a chosen system of coordinates).

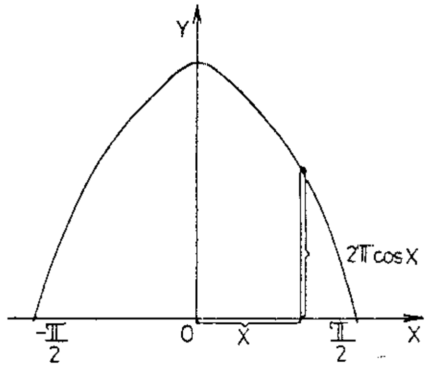

Figure 2

Now we can see that the other endpoints create the graph of the function $y = 2\pi\cos x$. From the way the shape has been created it clearly follows that this region must have the same area as the sphere of radius 1, that is, 4π. Because the area of the region bounded by the curve $y = \cos x$ must be 2π times smaller, it equals 2.

Robert later admitted that he originally wanted to evaluate the surface area of the sphere and only later turned his calculation around. It is remarkable that he took for granted that the area of the region under the function $y = 2\pi\cos x$ *is* 2π *times larger than the area under* $y = \cos x$. *Cavalieri would certainly have been pleased to see that.*

Darina

Evaluation of the volume of a cylinder

• We dissect the cylinder into a large number of circles perpendicular to the height of the cylinder. I am able to reshape each of the circles into a right-angled triangle with the lengths of its sides r and $2\pi r$. This triangle has an area of πr^2. We reshaped all the circles in this way and put them on top of each other up to a height of v.

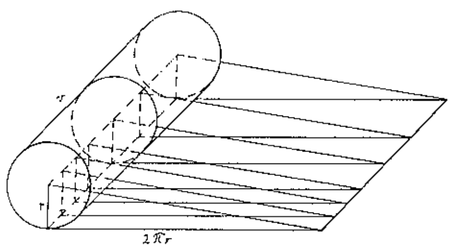

Figure 3

It follows that the triangle will be there v times and therefore the volume of the cylinder will be $V = \pi r^2 v$.

Darina's solution has two interesting features. First, she says "a large number of circles", not an infinite number, and second, "the triangle will be there v times". Would she know what, say, "$\sqrt{2}$ *times" would mean?*

Andrej

Evaluation of the volume of a cone

• We dissect the cone into infinitely thin triangles and put them together to make a solid figure *ABCDEF* in such a way that the triangles *SKL* will form the base *ABCS*, the triangles *VKL* form the face *BCFE*, and the segments *VS* form the face *ADFE* of the solid.

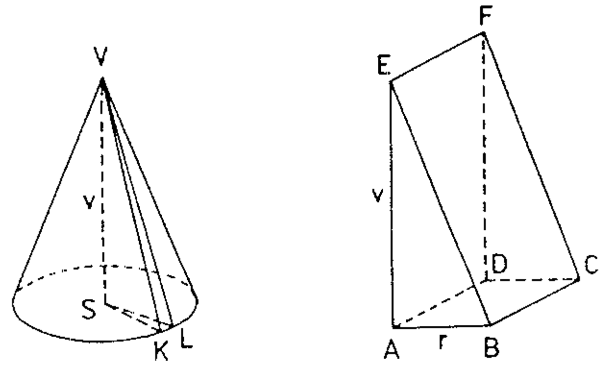

Figure 4

Thus, $AB = r$, $AE = v$, $BC = 2\pi r/3$, and computing the volume of *ABCDE* we obtain $V = (AB).(BC).(AE)/2 = \pi^2 v/3$.

In this example we can see the difference between Kepler and ourselves. Using "infinitely thin" segments VS. *Andrej filled up roughly an equal area to that obtained by using the "infinitely narrow" triangles* KLV. *He did not have the experience Kepler had: finding that not all that is "terribly small" is equally small. Andrej gets the right result because he makes the appropriate choice for the length of the segment BC.*

Erika

Evaluation of the surface area of a sphere

• We divide the surface of the sphere into infinitely thin strips. Putting them together we get a parallelogram *ABCD* for which $AC = 2\pi r$ and $BC = \pi r$. Its area, and hence the surface area of the sphere, is $\pi^2 r^2$.

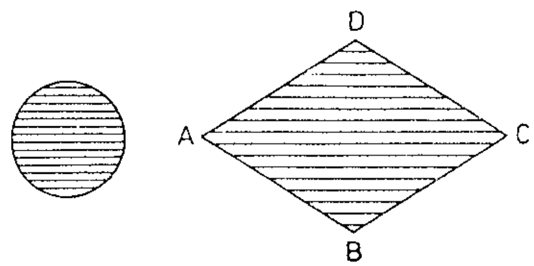

Figure 5

Erika was of course trying to get a figure from the dissected sphere whose area she would be able to calculate. It is interesting to see how the idea of linearity emerged here.

Vlado

Evaluation of the volume of a sphere

• We divide the sphere into an infinite number of pyramids having their vertices at the center and their bases on the surface of the sphere. each of these pyramids has the volume $pr/3$, where p is the area of the base of the pyramid. Adding up all these volumes we must get $Sr/3$, where S is the surface area of the sphere. It follows that $V = 4\pi r^3/3$.

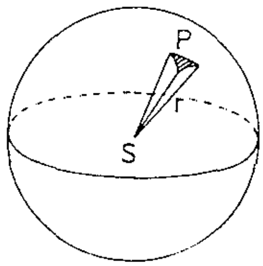

Figure 6

The problem with this solution is its reliance on the surface area of the sphere, which is more or less equivalent to knowing the sought-for volume. Robert found a beautiful way round the difficulty.

Robert

Evaluation of the volume of a sphere

- I can create the volume of a sphere by revolving a semicircle. I inscribe in it rectangles with bases Δx and heights $y_0, y_1, \ldots, y_n$ respectively.

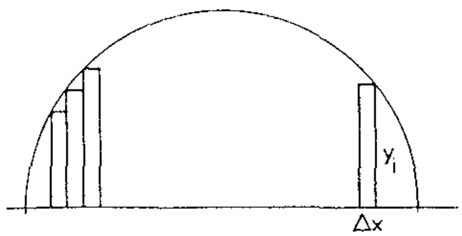

Figure 7

The revolution of these rectangles generates cylinders with volumes $\pi y_0^2 \Delta x, \ldots, \pi y_n^2 \Delta x$. We can clearly see that the sum of the volumes of all these cylinders will be fairly close to the volume of the sphere:

$$V \cong \pi \Delta x (y_0^2 + y_1^2 + \ldots + y_n^2) \qquad (*)$$

First of all I tried to compute the sum of the right hand side, but failed. Then I got the idea that the formula for the volume of a sphere that I am looking for looks just like the expression for the area of a segment of a parabola, $S = 4ab/3$. (The pupils had met this at this time.)

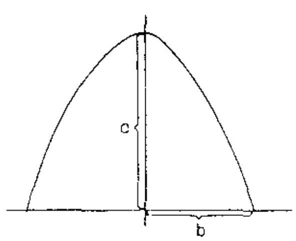

Figure 8

The unusual coefficient, 4/3, common to both made me suspicious. How are these two situations related? I tried to overlap Figure 7 and 8 to make a single figure

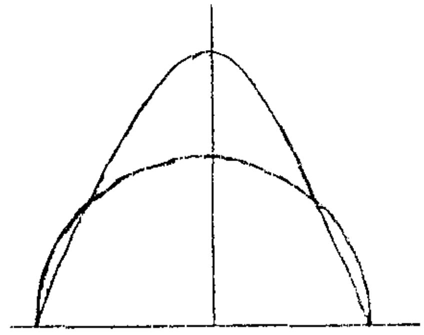

Figure 9

I placed the coordinates on the plane in such a way that the parabola had the equation $y - r^2 - x^2$ and the circumference had the equation $y = \sqrt{(r^2 - x^2)},\ x \in [-r, r]$. Now one can determine the area $S = (4/3) r^2 . r = (4/3) r^3$ by using "the method of strips". I inscribed in the parabola rectangles with widths Δx and heights $z_0, z_1, \ldots, z_n$, respectively.

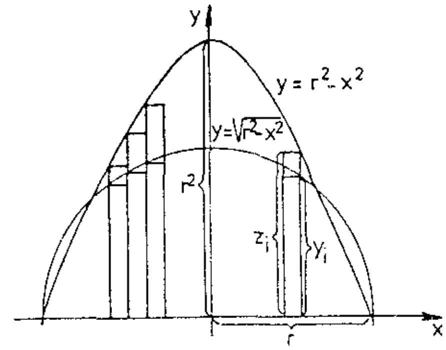

Figure 10

Clearly

$$S \cong \Delta x (z_1 + z_2 + \ldots z_n).$$

From the previous figure we can see that

$$z_t = r^2 - x_t^2 = y_t^2.$$

Thus

$$S \cong \Delta x (y_0^2 + y_1^2 + \ldots + y_n^2) \qquad (**)$$

Comparing (*) and (**) I have got that

$$V = \pi S = (4/3) \pi r^3.$$

Robert's approach is so nice it needs no further comment.

What the teacher produced

Inspired by the pupils, I tried to find similar evaluations of other integrals. Here is one result: an evaluation of the definite integral of the function $y = x \sin x$ between the limits 0 and t, $t \leq \pi$.

- In order to use Kepler's method we need to find an appropriate surface, region, or solid, connected with the function $y = x \sin x$. It is of course easy to find a plane figure: the region bounded by the curve $y = x \sin x$ and the x-axis.

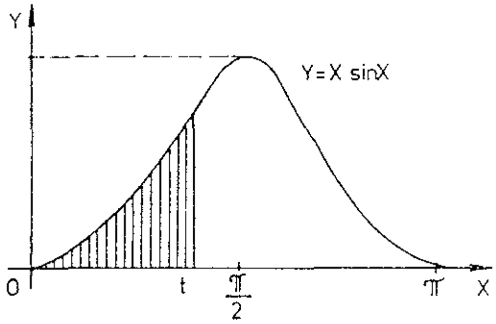

Figure 11

But it is not clear how this region can be used to fulfill our aim. Let us therefore look for a geometrical interpretation in three-dimensional space of the sought-for integral.

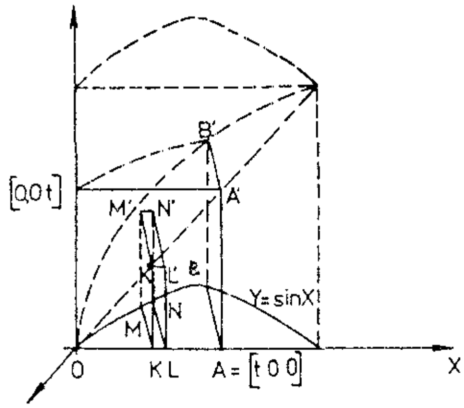

Figure 12

A semi-cylinder has been drawn, its base being a plane figure bounded by the curve y - sin x and the x-axis. Then the product x.sin $x.dx$ represents the volume of an elementary rectangular prism. Putting these elementary rectangular prisms together on the interval [0, t] yields a solid and our task is transformed — we need to calculate the volume of the solid *OABA'B'*

Suppose we partition the interval [0,t] into n subintervals of width t/n. Then the volume of the kth elementary rectangular prism is

$$V = (kt/n)\,(t/n)\sin(kt/n).$$

We can add up all these volumes:

$$V_0 + V_1 \ldots V_{n-1} = \sum_{k=0}^{n-1} V = \sum_{k=0}^{n-1} (kt/n)\,(t/n).\sin(kt/n)$$

To evaluate this we can use the following formula, which is not very nice but quite useful:

$$\sum_{k=1}^{n} k.\sin k_\alpha = \frac{1}{2\sin\frac{\alpha}{2}}\left[\frac{\sin n\alpha}{2\sin\frac{\alpha}{2}} - n.\cos\frac{2n+1}{2}\alpha\right]$$

Then we get

$$\sum_{k=0}^{n-1} V = \frac{t^2}{n^2}\,\frac{1}{2\sin\frac{t}{2n}}\left[\frac{\sin\frac{n-1}{n}t}{2\sin\frac{t}{2n}} - (n-1).\cos(2n-1)\frac{t}{2n}\right] =$$

$$= \frac{t}{2n}\,\frac{1}{\sin\frac{t}{2n}}\left[\frac{t}{2n}\,\frac{1}{\sin\frac{t}{2n}}.\sin\left[1-\frac{1}{n}\right]t - \left[1-\frac{1}{n}\right]t.\cos\left[1-\frac{1}{2n}\right]t\right]$$

Obviously the last sum is an approximation to the volume of the solid OABA'B' and it will get more precise as n gets larger. So let n tend to infinity. Then

$$\frac{t}{2n}\,\frac{1}{\sin(t/2n)} \quad \text{and} \quad 1-\frac{1}{n} \quad \text{and} \quad 1-\frac{1}{2n}$$

will all approach 1. It follows that the volume of *OABA'B'* equals sin t - $t$cos t.

Conclusion

One can make many objections to the evaluations presented here. Here and there they seem to rely on magic rather than twentieth century mathematics. Many steps could be taken with greater precision, by using the notion of limit, for example, but we did not go this far. We think, however, that anyone who considers himself to be a teacher of mathematics will clearly see the point of these computations. And to those who ask us why we continue to evaluate integrals in these "doubtful" ways, we offer at least one argument: Beethoven's setting of Schiller's "Ode to joy" has been played by many orchestras but we are always glad to hear it again

Mathematics is an art too. Very often in our haste to reach new results and attain perfection we cheat ourselves of the aesthetic experience it can provide.

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