Communications

Arabic language features in university students' mathematical activities in English

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In Saudi Arabia, the language of instruction at both primary and high school level is largely Arabic. When students are admitted to English-medium universities they are weak in English. As a result, they have to undergo a one-year intensive English program. In addition to learning English, students also take two pre-calculus mathematics courses. These mathematics courses are taught in English despite the weakness of the students in that language. In this communication, I use highlights from the mathematics work of students in the pre-calculus course to illustrate the difficulties they face due to conflict between Arabic and English.

As a non-Arab teacher who has been teaching mathematics in English to students who are acquiring English as a language of learning and teaching (LOLT), it took me more than a decade to appreciate the effect of language in the teaching and learning of mathematics. Although on many occasions I noticed students resorting consciously or unconsciously to Arabic (their first language), I initially could not connect this use of Arabic to anything other than lack of understanding of mathematics. The anecdotal evidence presented in this communication shows that this is not always the case. As a result, I argue that mathematics teachers working in similar situations need to have some basic knowledge of students' first language, as well as how this language is connected or may be connected with the language of instruction.

Code-switching in students' written mathematics

Code-switching is a term used in bilingual or multilingual settings to refer to switching between one language and another by teachers or students in a classroom (Setati & Adler, 2001). Despite the fact that code-switching in verbal interaction has attracted a lot of attention in research on language in mathematics classrooms, not much is known about code-switching in writing. In particular, not much is known about bilingual Arabs whose writing in their native language is in many ways different from English (Barwell, 2005). I have observed many occasions where students switch consciously or unconsciously from English to Arabic in written exams and during classwork that is supposed to be entirely in English.

For instance, in Figures 1 and 2, the students wrote most of their calculations in Arabic and then translated back into English. This process is not only cumbersome, but students often make silly mistakes. Some of these mistakes are due to conflicts between the Arabic and English languages.

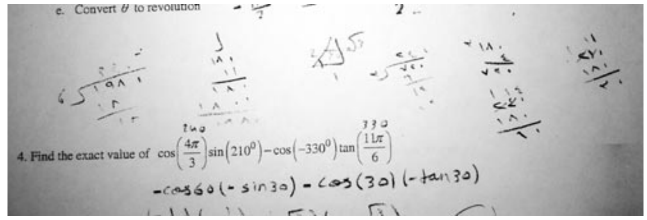

Figure 1. Code-switching in writing mathematics.

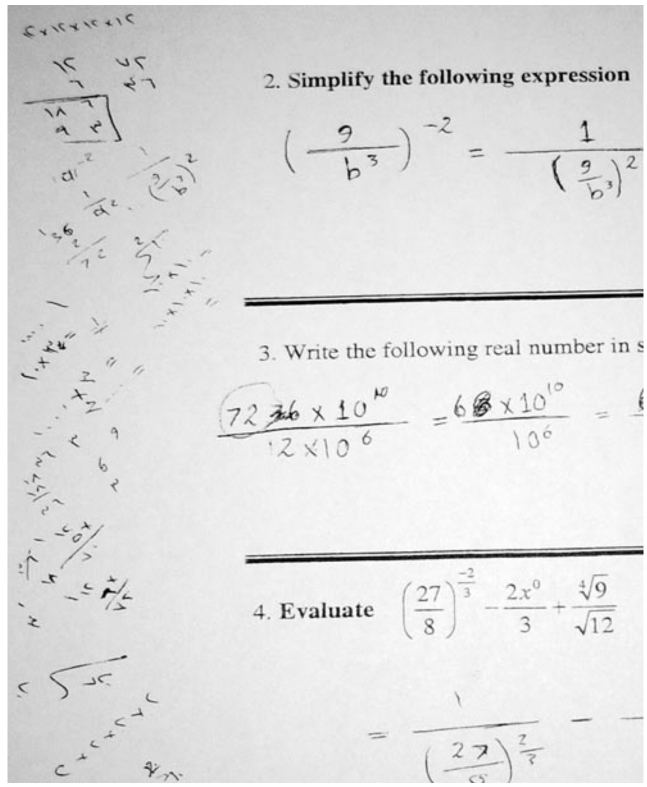

Figure 2. Dominance of Arabic (first language) while doing mathematics in English (second language).

It has been argued by Clarkson (2007) that since code-switching appears to come to the teacher or student automatically, teachers have an important role to play in utilizing this important resource in order to enhance their students' learning of mathematics. However, for the teacher in this context to perform this vital task, my anecdotal evidence suggests that they need to have a basic understanding of both Arabic and English as well as an appreciation of language factors in mathematics.

Examples of code-switching by bilingual Arabs in mathematics

One of the most difficult things for me, a monolingual teacher, to decide is if errors by a student who is acquiring English reflect a lack of mathematical understanding or a problem with English (Secada & De La Cruz, 2000). Although it has been reported in the literature that bilingual students switch languages when carrying out arithmetic computations or other general cognitive activities (see Clarkson, 2007; Moschkovich, 2007), my examples are different. My examples are from both the written and oral mathematics work of students. Here are some of the issues that can arise.

Reversed reading of numbers. Bilingual Arab students often get confused while reading numbers in English. For instance, many times a student will read 65 as 56, but I have never heard it the other way around. Arabic numerals are written in the same way as English numerals but they are read differently. In Arabic, the units digits are read first, followed by the tens digits. Therefore, the most plausible explanation is that students are reading the numbers in the Arabic way. This reversal happens frequently in class, at which time students have the chance of being corrected or of self-correction. However, when it happens in examinations, teachers are faced with a dilemma when students complain that the expression they have written is not what they mean.

Dot and zero. In Arabic, five is written like an English zero, and zero in Arabic is written as a dot. These symbols create some confusion for some students. Figure 3 is the work of an excellent student who is attempting to convert the measure of an angle from degrees to radians. Notice on the top right of the student's work, the expression is written in a mixture of Arabic and English. Particularly in the denominator, instead of dividing by 180, he appears to have divided by 18. (The number 1 could be either in Arabic or English, the number 8 is in English and the dot after the 18 is a zero in Arabic). The student is mathematically right, but there is a high chance that his work would be marked wrong by a teacher who is not familiar with the Arabic language. For instance, it is not uncommon to find students calculating 102 × 10 = 1.20. Many similar kinds of calculation errors, which are algebraically correct, but notationally wrong, are purely due to differences between Arabic and English.

Five and zero. One incident remains vivid in my mind. We were solving the radical equation:

$$\sqrt[4]{x^4 + x^2 + 2x} = x$$

After following the procedure for solving this type of equation, we found the x values as 0 or –2. As usual with radical equations, we checked the answers to avoid extraneous solutions. Checking for $x = -2$ yielded $2 = -2$, which is a clear contradiction. Surprisingly, checking for $x = 0$, which is supposed to be clear and obvious, generated a problem. A student asked why $\sqrt{0} = 0$? It took the whole class some time to understand the source of the student's problem. At last, we realized that he was thinking of 0 as five in Arabic, and so was wondering why the square root of five is equal to five. In this class of students, not many students have enough English proficiency to ask such a question, and among those who can ask the question, not many have the confidence to do so. Therefore, it was an opportunity for me to understand that the source of the student's problem is as a result of confusing a common symbol in Arabic and English but with different numerical values. It was a wonderful opportunity to clarify it. Many leave the class with so many doubts.

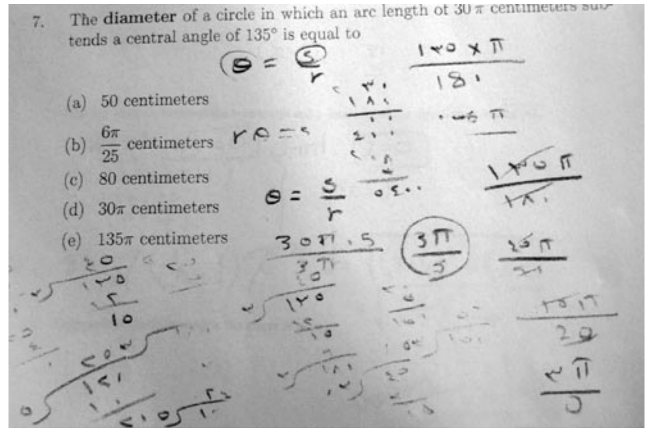

Figure 3. *Mixture of Arabic and English in one mathematical expression.*

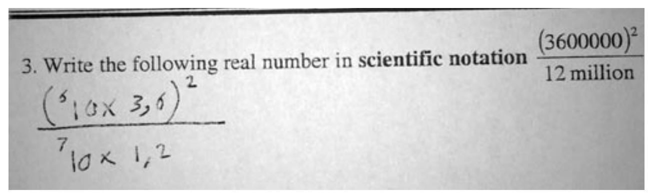

Figure 4. *Dominance of Arabic mathematical writing conventions in students' work in English.*

Comma and dot. In Arabic, the comma is used as a dot. That is to say, two point three is written as 2,3 while in English it is 2.3. Sometimes students trying to write 2.3 write 2,3. On the other hand, students may read 2.3 as two hundred and three, considering the dot as a zero. Figure 4 depicts a sample of one student's work to illustrate this issue. Notice the way the student has put his exponent of the number 10 on the left of the number (as in Arabic) instead of on the right of the number (as in English), in both the numerator and in the denominator. Also, the commas after the number 3 in the numerator and after 1 in the denominator are both supposed to be dots. But in Arabic mathematics, a comma is a dot and a dot is zero.

Left to right and right to left. Arabic writing is from right to left, while English is the opposite. I have often noticed that if there is any mathematical information to be extracted from left to right, this can generate confusion for some students. For instance, in getting information from a graph, some students make mistakes that are clearly due to the influence of their first language. In fact anything that reads from left to right or right to left can be affected, such as interval notation, intervals in which a graph is increasing, decreasing, or constant, intervals in which a graph is continuous, or the domain and range of the function. All these can generate confusion and some students write them incorrectly in English by inverting the answers. Figures 5 and 6 are examples from students' work. They are all mathematically correct, but notationally incorrect.

In the third step in the work, shown in Figure 6, the numerator of the fraction is 6 – 1. The student has read it from right to left, and obtained – 5/3, which he has also written in the Arabic way with the minus sign on the right.

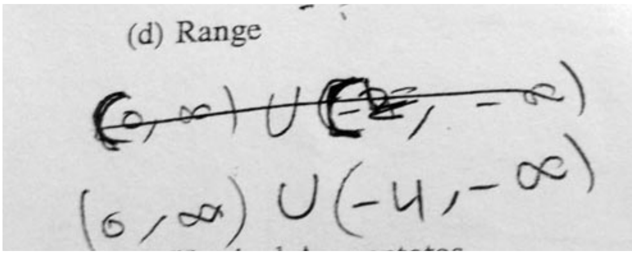

Figure 5. *Dominance of right-to-left writing (as in Arabic) in place of left-to-right (as in English).*

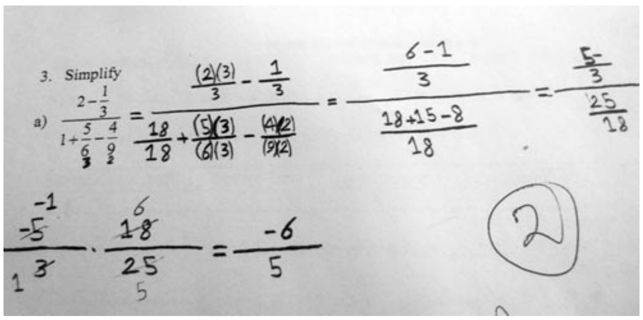

Figure 6. *Dominance of Arabic in mathematics written work and a resulting error in calculation.*

Distributive law. It is know that many students use these properties wrongly. One time I gave the following expression to students to write without absolute values:

$|x - 3| - |x + 3|; 0 < x < 1$

The answer presented by one student was:

$- (x - 3) - (x + 3)$
$= -x - 3 - x - 3$
$= -2x - 3$

The source of the confusion in this student's answer might not be obvious. At first, I thought he had made the usual mistake in applying the distributive law in which students forget to multiply all the terms in brackets by -1. No! The student insisted that he was right, and after some time he realized the minus sign in the center of the expression is to be distributed on the right bracket, and not on the left bracket, as in Arabic.

Exponential and radical notation. In Arabic, exponents are written to the left of the base. That is to say if I want to write a to the power n, it is written in Arabic as ${}^{n}a$, while in English, the exponent is written to the right as is well known.

As can be seen in Figures 4 and 7, the exponents are written before the base following Arabic convention, in place of after the base, as in English

Factorization. One time after finishing a unit on factorization, I gave the homework shown in Figure 8 to the students. After marking, I returned their papers, and one student came to my office to find out why his answer was wrong. It took us some time to realize that it was the Arabic way of factoring that persisted. It can be noticed that the student read the question from right to left, and that is how he worked his solution. As in Arabic, he considered the minus sign to be for $3x$ and $15y$ instead of for $3x$ and $10y^2$, as

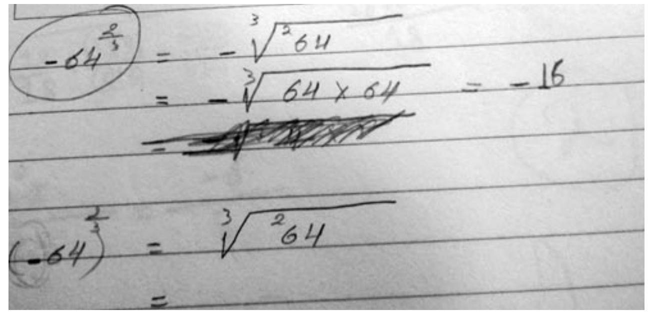

Figure 7. *Influence of Arabic written conventions during mathematics work in English.*

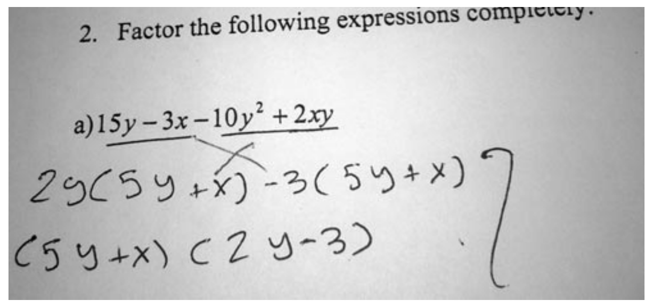

Figure 8. *Dominance of Arabic conventions in factorization and a resulting error.*

intended in the question. That is how he obtained $5x + y$ as a common factor.

Concluding remarks

Many researchers have highlighted the need to take language factors into consideration in the mathematics classroom. The anecdotal examples presented in this communication buttress this need. The examples also indicate that for successful mathematics teaching of bilingual students, especially whose writing in a first language is different from the LoLT, teachers need to be aware of the possibility of possible conflicts between the written L1 and L2, and also need to make their students aware of this possibility.

Acknowledgement

This work is supported by the Deanship of Scientific Research, KFUPM, Dhahran, Saudi Arabia, under project number IN131046.

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Sort-of-right mathematics

JEAN-FRANÇOIS MAHEUX

There's a crack in everything
That's how the light comes in
Mike Strange, Shanna Crooks &
Leonard Cohen (1992). *Anthem.*

Oxygen

In a recent issue of FLM, Weber and Mejia-Ramos (2015) offered some ideas on "relative and absolute conviction in mathematics". The article strongly connected with my current studies regarding the "imperfect" nature of (doing) mathematics, and reminded me of a video I recently watched on the Internet. The video features Seymour Papert and Robin Jettinghoff, and was recorded at the University of Maine in April 2004 [1]. Papert presents a Turtle Geometry-Logo environment, but a problem occurs with the Logo program, preventing him from showing a correct answer to the task. Papert then goes on to say:

> We did something wrong with our unit, but you see, that's great because then we go to correct the unit, and it's *sort of right,* and this is one of the things that is so wrong with mathematics as it is taught in schools: you can't be sort of right.

Developing the thought, he then talks about debugging as a fundamental mathematical process, and discusses the importance of being wrong to be able to learn from a mistake by fixing it. I got stuck, however, with the idea of *sort-of-right,* thinking that it might not (always?) be necessary to "fix" things: what if sort-of-right was right enough? Isn't mathematical certainty/conviction sort-of-right at best?

Fuel

Mathematics is often presented as the culprit of exactitude and precision, but there are many branches of mathematics in which approximate answers play an important role. We can think, for example, about inverse (or ill-posed) problems, for which different kinds of sort-of-right solutions can be produced (*e.g.,* Kabanikhin, 2008). Advocates of experimental mathematics (*e.g.,* Franklin, 1987) also argue that "insistence on instant absolute rigor is sterile" (Chaitin, 2004). So when and how is it justified to take sort-of-right as right enough? We all know how, through history, numbers like π and $\sqrt{2}$ were approximated for practical purposes: if you want to know how many one-step-long stones you need to go around a circle with a 10 step-long diameter, $3 \times 10 = 30$ certainly gives you a fully satisfying sort-of-right answer. And when we play with Fermi problems (*e.g.,* how many liters of beer are drunk everyday on earth?), there is no doubt that sort-of-right calculations are the best that can be done.

Moreover, let's not forget how mathematics itself has, as a network of ideas, developed in a similar way: from Euclid's negligence with continuity to Euler's oversight regarding polyhedrons, a mathematician's job is often one of "fixing" sort-of-right results, very well knowing, however, that his or her results are only partial, temporary, at best contributing to the winding course of progress [2]. Sort-of-right, if we want to call it that, until a better proof, a more general theorem, a new particularity, emerges as a result of what Sherry (1997) calls "the freedom mathematicians enjoy in developing new conceptual structures" (p. 414). I think we all have our favorite peculiar mathematical oddity, from Brouwer to Faltings, or from chaos theory to fuzzy logic, and so on. One of my own preferred examples is the concept of quasiperfect numbers (*i.e.,* whose divisors sums to $2n+1$), about which we know important things…except we are not sure if they exist (see Hagis & Cohen, 1982)!

But I think that fully assuming the value of sort-of-right mathematics might require a little bit more. Reading through history of mathematics with a dialectical mindset, I have recently begun conceptualizing what I call the im|perfect nature of (doing) mathematics. The idea is to examine the fruitful, mutually constitutive interplay between opposing forces in the development of mathematics as a network of ideas, and in the actual practice of doing mathematics. At the moment, I have articulated five dimensions, where im|perfection contributes to (doing) mathematics: ambiguity|clarity, traces|ideas, impressions|convictions, dispersion|discipline, limitation|expansion. The dimension called limitation|expansion, for example, refers to how what counts as mathematics is often jostled by new extensions. There is a drive towards limiting (doing) mathematics to a minimal number of concepts and ideas, but also a constant venture to create new possibilities (the successive controversial apparitions of irrational, negative, complex, and transfinite numbers illustrates this very well). Could such a line of thinking offer an appealing enough basis to both recognize, try and embrace sort-of-right mathematics?

Heat

That's a lot about mathematics; what about mathematics education? Well, to begin with, Papert's sort-of-right strongly reminds me of the concept of "good-enough" discussed in this journal by Zack and Reid (2003, 2004). Examining students' accomplishments, Zack and Reid (2004) note:

> Our investigation revealed that much of what we had marveled at was in fact incomplete, tentative, and sometimes inconsistent. Yet at the same time the students were able to continue to work on the problem, to explore, and even to explain things to each other that they seemed not to fully understand themselves. (p. 28)

Students' conceptualizations were not perfect, but good-enough to engage, and keep moving forward in relation to the tasks at hand. Thinking about it, is this not actually happening *all the time*? How far would we go if *all* students had to *perfectly* understand *everything* before moving on to the next idea? Students "make do" with sort-of-right understandings and, according to Zack and Reid, in this way, they come up with the most interesting insights and ideas.

This also reminds me of Rowland's (2000) study of the role of vagueness in (students') mathematical talk, described as a condition for mathematical activity to take place. The good-enough metaphor also has a lot in common with the constructivist replacement of truth with viability, recently taken up by Proulx (2013) to analyse students' strategies or mathematical solutions as functional (instead of optimal) within a context. In a way, this is like Papert's little program in the video: it works well as far as to show "how a program can be made to find an answer", even if it is not optimal in terms of doing so (since it does not even provide the correct figure). I am not sure, however, of how much we actually value this kind of sort-of-right mathematics. It feels more like a necessary evil (when not simply taken as something we have to get rid of), not something to celebrate.

One key idea relating to sort-of-right mathematics that seems to emerge in mathematics education literature is that of *ambiguity*. Borasi (1993), for example, worked with some students on the impossibility of writing absolutely general and unequivocal definitions (*e.g.,* of a circle); Zaslavsky, Sela *et al.* (2002) explored the various interpretations of school-made mathematical concepts ("slope" in their case), and others have focused on the ambiguity of mathematical symbols (Mamolo, 2010) or representations (Schwarz & Hershkowitz, 2001). Some might even argue that everything related to mathematical modeling, statistics and probability is in fact all about coming up with sort-of-right answers. I would be tempted to say, however, that these are great occasions to offer sort-of-right mathematical experiences to students only if we emphasize the productive, creative, or simply interesting aspect of their sort-of-rightness, which is probably rarely the case.

Chain reaction?

The idea of sort-of-right mathematics surely has a disruptive power. In the beginning of the twentieth century, the mathematical community started to struggle in what we now call the foundational crisis. But as a result, new branches of mathematics were created, new heuristics were brought forth, and richer conceptualizations developed. What if we really tried exploring the potential of sort-of-right mathematics for teaching and learning? What if we deliberately set it out, and asked students for approximate answers, good-enough solutions, roughly viable procedures, and so on? What if we tried identifying new mathematical topics to explore with students, precisely to bring forth sort-of-right mathematics? I have already mentioned a few topics, and others could certainly be found in approximation theory (*e.g.,* asymptotic comparison, or situations in which precise answers are impossible to obtain, like with the quantic), or dwell upon various "kind of proofs" and their purpose (*e.g.*, zero-knowledge proofs aiming at establishing the existence of a proof; in FLM, see Hanna, 1995), and so on.

I cannot conclude without going back to Papert's comment and react to the idea that there is something "so wrong" with mathematics as it is taught in schools. If it is okay for mathematics to be sort-of-right, maybe it should also be the case for "mathematics as it is taught in schools". What I mean is that what we generally try to do in school with mathematics is incredibly difficult, if not simply impossible. I do not understand my job as a researcher in mathematics education as finding and fixing "what is wrong" in schools. My job is to provide ideas for what could be done in, out, around, for, with, or even against mathematics educational systems. Nothing more…but nothing less.

Notes

[1] See www.youtube.com/watch?v=4iIqLc0sjjs

[2] The word "right" comes from the Proto-Indo-European root *reg-* meaning "moving in a straight line".

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Where did/do mathematical concepts come from?

ROSSI D'SOUZA

I share three anecdotes from my teaching experiences and pose some open questions addressing mathematics education research. These three experiences, in hindsight, helped me address, if not entirely answer, the said questions. I emphasize the need to recognize and present mathematics as processes carried out by people (including students), rather than final products to be learned and applied.

Exploring odd and even numbers

I teach in a school for blind children. During a mathematics class, since the topic of odd and even numbers came up, I asked, "what are odd and even numbers?" The students stated that numbers which can be evenly divided by 2 are *even*, while those that could not are *odd*. When asked about *zero*, all seemed to arrive at a consensus that it is both odd and even. They reasoned that since zero leaves no remainder when divided by 2, it is even; however, we cannot divide zero by two since we have nothing to divide, making it odd. During further discussions, one student, Faizan, expressed discomfort with the idea of including zero as odd. He argued that numbers have the property that "odd +/– odd = even", or "even +/– even = even" and "odd +/– even = odd" and "even +/– odd = odd". However, if zero were to be included as an odd number, this pattern would not work. The pattern could be maintained only if zero was categorized as even, not odd.

On continuing the discussion, the number –4 turned up. The discussions included the following dialogue:

Me: So what about –4? Is it odd or even?

Faizan: Before deciding that, we need to know where did numbers like –1, –2 come from? I mean there has to be a reason. For example, when we found numbers that could be divided by two, we called them even and those that could not, as odd. So where did these numbers like –1, –2 come from?

Me: Maybe we can look at some examples of where these negative numbers can be found and then–

Faizan: Sir, when we visited the mall, the lift had numbers –1 and –2 to indicate the upper and lower basement.

The subsequent discussion emphasized the need to conceptualize negative numbers. Faizan suddenly interrupted stating that negative numbers are very old, while malls with basements are comparatively new. He later on hypothesized that maybe during the Harappan civilization, building structures which had some equivalent of basements could have given rise to the concept of negative numbers. The discussions continued with other hypotheses and examples that led the class to concur that it makes most sense to categorize –2, –4, … as even. These numbers fit into a continuous pattern of alternating even and odd numbers, whether read backwards or forwards.

This discussion got me thinking. If a student considers 0 as an odd number, is it a misconception? If a student invents a crazy concept like "*abc*-numbers" and "*xyz*-numbers" and decides to categorize, 3.5 in the set of *abc*-numbers, does 3.5 become an *abc*-number? If another student calls 3.5 a *xyz*-number, would it be a misconception? Similarly, is calling 1 a prime number, a misconception?

A lecture on divisibility

I was asked to conduct a lecture with students selected for a mathematics competition. I decided to talk about divisibility. On beginning with asking the definition of a prime number, the students answered in unison, "...a number that can only be divided by 1 and itself." I then asked about the number 1 since it fits the definition. However, the students refused to accept 1 as a prime number (possibly because their books "say so"?). Discussions followed until we decided to redefine the concept of primes as: "numbers greater than 1 that can only be divided by 1 and itself." This discussion included the Fundamental Theorem of Arithmetic.

In *Elements Book VII,* Euclid (1908, p. 278) introduces the idea of a prime number by defining it as "[that] which is measured by a *unit* alone" thus including 1 as a prime number. Reddick *et al.* (2012) present a list of sources in which 1 is included when referring to the set of prime numbers, even though mathematicians generally do not include it. Even until 1941, the number 1 was considered a prime number by some mathematicians, but non-prime by some others. Each had a justifiable reason for their claim. For example, Lehmer (1941) introduces the nomenclature for prime numbers by stating that "the letter *p* designates a prime which may be ≥ 1, > 1, or > 2 according to the context".

So does it make sense to ask if 1 *is* a prime number? Would it have made sense to ask the same, say, during the 1930s in some exam, with the assumption that there is only one correct answer? Is the concept of *prime numbers* now "finalized"? Can we take any specific mathematical concept and confidently say that now its truth cannot or should not be questioned? Did people in the 1930s realize that the definition of prime numbers would change later? Is it possible that our present definition will change later? When would a change be acceptable? If made by a mathematician? A student? Who is a mathematician? Someone with a university or state-designated position?

Exploring a sequence

A fellow mathematics educator expressed a special affection for the numbers 4, 2, 6 and 8, in that order. On asking her why, she refused to disclose her reason. Thinking that it might involve some special day in history, I looked up possible dates that could have been represented by 4, 2, 6, 8. Nothing! I thought it could mean: 4 + 2 = 6 and 4 × 2 = 8. But that did not seem so fascinating. Another friend mentioned the Fibonacci sequence since 4 + 2 = 6; 2 + 6 = 8. I continued: 6 + 8 = 14, (focus on the units places), 14 + 8 = 22 and found a pattern. In "modulo 10", the numbers 4, 2, 6 and 8 follow a "fibonacci-ish" sequence: 4, 2, 6, 8, 4, 2, 6, 8, … I was happy and then I rested (or so I thought). But another thought came to mind: could (4, 2, 6, 8) be the shortest sequence in mod 10? I immediately realized that it cannot, since (0, 0, 0, …) is a shorter sequence (of "length" 1) whereas (4, 2, 6, 8) has "length" 4. I dismissed this "trivial sequence" (0) because of it not being "beautiful". But then I realized that the sequence (5, 5, 0, 5, 5, 0, ...) which is "equivalent" to the "reduced sequence" (5, 5, 0) is smaller than the sequence (4, 2, 6, 8). But even (5, 5, 0), albeit not as trivial as (0), was a rather simple sequence. So, putting aside (5, 5, 0) and (0), could (4, 2, 6, 8) be the smallest "mod-10 fibonacci-ish sequence"?

I thought this conjecture to be too difficult to prove, considering the many possible sequences. But then realized that I would need just two, single-digit numbers to initiate a sequence. And these two numbers would be unique to only that sequence. For example, a sequence generated by the numbers 2 and 6 (I'll represent this by [2, 6]) would be 2, 6, 8, 4, 2, 6, 8, … which is "equivalent" to (4, 2, 6, 8). I then realized that there are now fewer possible sequences, since the total number would be strictly less than the total number of 2 digits. Also, [4, 2] is equivalent to [2, 6] which is equivalent to [6, 8]. *I.e.* [4, 2] = [2, 6] = [6, 8] = [8, 4]. Similarly, [0, 5] = [5, 5] = [5, 0]. This "equivalence" reduced the number of possible sequences to much less than 100 – 4 – 3 – 1 = 100 - 7 = 93. I tried another sequence, [0, 1]. This sequence generated: 0, 1, 1, 2, 3, 5, 8, 3, 1, 4, … which just went on! Could this be an infinite sequence? Of course not! (I felt silly!) The sequence would definitely have to repeat before at least 93 numbers were written. So I continued … 5, 9, 4, 3, 7, 0, 7, 7, 4, 1, 5, 6, 1, 7, 8, 5, 3, 8, 1, 9, 0, 9, 9, 8, 7, 5, 2, 7, 9, 6, 5, 1, 6, 7, 3, 0, 3, 3, 6, 9, 5, 4, 9, 3, 2, 5, 7, 2, 9, 1, 0, 1, 1. Aha! I reached {1, 1} indicating the end of the sequence (of length 60).

Now {2, 2} was not present in the above sequence. [2, 2] generated (2, 2, 4, 6, 0, 6, 6, 2, 8, 0, 8, 8, 6, 4, 0, 4, 4, 8, 2, 0, 2, 2, …) with length 20. I tried [4, 4] and realized that it was equivalent to [2, 2] since {4, 4} was a sub-sequence of [2, 2].

I had now accounted for 1 + 3 + 4 + 20 + 60 = 88 out of the hundred "two-digits". Had I exhausted all possible sequences, the sum of the lengths of each sequence would have to add up to one hundred. I looked at a 2-number generator as a two-digit number. I saw that "13" was absent. So I tried [1, 3]. I got (1, 3, 4, 7, 1, 8, 9, 7, 6, 3, 9, 2, 1, 3, …). thereby getting a sequence of length 12 = 100 - 88. This exhausted all possibilities. So I concluded that the only possible fibonacci-ish sequences are:

(0)

(5, 5, 0)

(4, 2, 6, 8)

(1, 3, 4, 7, 1, 8, 9, 7, 6, 3, 9, 2)

(2, 2, 4, 6, 0, 6, 6, 2, 8, 0, 8, 8, 6, 4, 0, 4, 4, 8, 2, 0)

(0, 1, 1, 2, 3, 5, 8, 3, 1, 4, 5, 9, 4, 3, 7, 0, 7, 7, 4, 1, 5, 6, 1, 7, 8, 5, 3, 8, 1, 9, 0, 9, 9, 8, 7, 5, 2, 7, 9, 6, 5, 1, 6, 7, 3, 0, 3, 3, 6, 9, 5, 4, 9, 3, 2, 5, 7, 2, 9, 1)

with lengths 1, 3, 4, 12, 20, 60 respectively. This proved that (4, 2, 6, 8) was the smallest non-simple fibonacci-ish sequence in modulo 10. I think I have closure. However I had also felt a sense of closure on discovering that 4268 was a mod-10 fibonacci-ish sequence. Maybe something else will come up and I'll work on that too.

I then thought about my past excursions in mathematics. Then too I would invent concepts to communicate with peers or even myself. I jokingly thought of how unfair it would be to get students to prove or even understand claims and concepts that I invented for myself. I asked myself some silly questions like, "Is it possible for me have misconceptions about, fibonacci-ish sequences, length of a sequence, trivial sequence, reduced sequence, equivalent sequences, *etc.*?" Would it be fair to make children learn and be assessed on the concepts that I made for my own fun? Is it unfair only because these concepts have no use in the "real world"? What if I ask these questions about the mathematical concepts of "real mathematicians"? Are my ideas like length of a fibonacci-ish sequence, equivalent fibonacci-ish sequences, *etc.* mathematical concepts? If yes, then since when were they mathematical concepts? Since I invented them? Did I invent, or discover them? If I redefine fibonacci-ish sequences, will the old definition still be a part of mathematics? So what makes a fibonacci sequence (or odd and even numbers or prime numbers) a part of mathematics? Would some properties of fibonacci numbers also work for mod-10 fibonacci-ish numbers? Is this a legitimate mathematics question? Why? What if I asked "which properties, and why"? And if this is a valid mathematics problem, does that make me, the author, a mathematician? Are students mathematicians? Why not? What if students invent concepts themselves? Oh wait, they do!

Final words

We need to recognize (and teach) mathematics as an historically dynamic and ever changing process that people do. When learners engage with their life experiences and interests and even actively decide how to define concepts, they experience the joy of creating mathematics and developing a meaningful and a more holistic understanding of it.

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From the archives

Editor's note: *The following remarks are extracted (and slightly edited) from an article by Victor J. Katz (1986), published in FLM6(3).*

How many words can be formed from the letters of the Hebrew alphabet? When and where does the sun rise in Alexandria on August 16? What is the length of the circumference of a circle of radius 1? Why does $\sqrt[3]{(2+\sqrt{(-121)})}$ + $\sqrt[3]{(2-\sqrt{(-121)})}$ equal 4? Where should a 150lb man sit on a seesaw to balance his 50lb son? How should the stakes in a game of chance be split if the game is interrupted? It is the consideration of questions like these which historically has provided the impetus for the development of mathematics. The consideration of such questions with our students is one way that we can motivate and excite them.

I have found this historical approach to the topics of the mathematics curriculum to be a profitable one. I will present here some concrete examples of the use of historical materials in developing certain topics from precalculus and calculus. Some of these are ideas which can be introduced easily in the course of a standard treatment of the material. Others would require some reformulation of the curriculum. In those cases, the curriculum would benefit from the reformulation. […]

We begin with algorithms. This is the new "buzzword" in mathematics education today, with conferences, institutes, and grants designed to see how algorithmic mathematics and the "algorithmic way of thinking" can be incorporated into the freshman college curriculum But the idea of algorithms is hardly new. In fact, algorithmic mathematics is the oldest mathematics of which there are records. After all, the Babylonian texts are chiefly lists of problems and rules for their solutions, i.e. algorithms. I want to distinguish two types of algorithms here and discuss each in turn. The first type is the algorithm which produces a definite answer in a finite number of steps; often this is expressed as a formula, for example, the formula for solving quadratic equations. The other type of algorithm produces approximations to an answer; one usually stops the algorithm when one reaches a predetermined level of accuracy. As an example of this, we may cite the standard square root algorithm. Both of these types date from ancient times, but we want to take our examples here from a more modern period.

Our example of an algorithm of the first type is the cubic formula of Cardano (1545). The story of the Tartaglia-Cardano controversy is well known, and, of course, is a good story to tell a class. But we want to concentrate here on the mathematics itself. Recall that Cardano solved the equation $x^3 = px + q$ by setting $x = a + b$ where $ab = (1/3)p$ and $a^3 + b^3 = q$. When we try to solve these two latter equations for a and b, we get a quadratic equation in a^3 which leads to the solution

$$a = \sqrt[3]{(q/2 + \sqrt{[(q/2)^2 - (p/3)^3]})}$$

and

$$b = \sqrt[3]{(q/2 - \sqrt{[(q/2)^2 - (p/3)^3]})}$$

That is, we have a straightforward formula, or algorithm, giving us the solution x to the equation.

Once we have an algorithm that we have justified to some extent—and we can easily justify this one algebraically or geometrically—we need always to ask questions about it. Among the questions are first, does it always work, or are there restrictions on the input; second, does it give only one solution, or is there some ambiguity; third, can we generalize the algorithm to solve similar problems; fourth, is there a better algorithm which will solve the same problem? It is through the answers to questions such as these that mathematics progresses.

Let us consider the first question—and Cardano did exactly that. A quick glance at the formula shows that if $(q/2)^2 < (p/3)^3$, we get the square root of a negative number; and that, in 1545, did not make sense. On the other hand, if we take a concrete example where this problem occurs, such as $x^3 = 15x + 4$, the formula gives $x = \sqrt[3]{(2 + \sqrt{(-121)})} + \sqrt[3]{(2 - \sqrt{(-121)})}$, while it is obvious that a correct answer is 4. Is it somehow possible that 4 can be expressed as a sum of cube roots which include square roots of negatives? Cardano's answer to the dilemma was to find a different method to solve the equation. It was Raffaello Bombelli, however, who in 1572 gave a solution using Cardano's formula. His answer was to develop the formalism of complex numbers That is, the search for solutions to the cubic becomes the motivation for the study of complex numbers. It is pedagogically better to follow this historical development and introduce complex numbers here rather than as solutions to certain quadratic equations, because here you know there must be a solution. In the quadratic case, it is simple enough to say that solutions don't exist. Bombelli, in fact, gave a rather straight-forward discussion of the rules for operating with complex numbers, assuming that they were the same as those for real numbers with the added twist that $(\sqrt{-1})^2 = -1$. And it was only after he had justified the use of complex numbers to solve cubics that he also noted that one could use these same strange numbers to solve quadratic equations with negative discriminants.

The second question on our algorithm, is there ambiguity in the solution, also leads to interesting mathematics. After all, the equation $x^3 = 21x + 20$ has three real solutions, $x = 5$, $x = -1$, and $x = 4$; but our formula looks like it only gives one. The answer to this dilemma lies in the meaning of the symbol $\sqrt[3]{\ }$. In fact, by analogy with the rule that positive real numbers have two square roots, one guesses that they also have three cube roots. One can now search for these.

The question on generalizing the results was answered in part by Cardano himself. Secondly, his student Ludovico Ferrari found the method of solving quartics. Though it would be difficult to develop the quartic formula in a freshman class, this discussion does provide an opportunity for mentioning the outcome of the search for formulas for polynomial equations of higher degree.

Our final question, is there a better method, again leads both historically and pedagogically to new mathematics Descartes considered the solution to equations from a different point of view, building up new equations from factors of the form $x - a$. This leads to the remainder and factor theorems as well as the methods for finding integral and rational solutions to polynomial equations. Finally, since not all such equations have rational solutions, we are also led to numerical methods of approximating roots, including Newton's method. It is not necessary to introduce calculus to discuss this; after all, Newton himself presented it in a purely algebraic fashion.