MATHEMATICAL CONVENTIONS: REVISITING ARBITRARY AND NECESSARY

IGOR' KONTOROVICH, RINA ZAZKIS

In his analysis of the mathematics curriculum, Hewitt (1999, 2001a, b) distinguished between *necessary* and *arbitrary*. Necessary refers to "things which [some] students can work out for themselves and know to be correct" (1999, p. 4). For example, with appropriate mathematical knowledge a learner can determine that the sum of the interior angles in a quadrilateral is 360°. Arbitrary, on the other hand, refers to a piece of mathematical knowledge that "someone could only come to know it to be true by being informed of it by some external means—whether by a teacher, a book, the internet, etc." (1999, p. 3). The idea can be exemplified with the name of a concept, such as 'quadrilateral'. Names are social and cultural agreements within the mathematics community, and thus a newcomer has no way to be sure whether her guess for the concept's name is the one that is accepted in the community. Moreover, even familiarity with the linguistic pattern of 'octagon', 'heptagon' and 'hexagon' will not help the newcomer to guess the conventional name of a geometrical shape with four sides.

Hewitt uses 'arbitrary' to stress that for a learner who is exposed today to the choices of the mathematics community made many years ago, these choices may seem random, or resulting from a personal whim, rather than from any reason. However, arbitrariness from a newcomer's perspective does not necessarily indicate arbitrariness from the mathematical perspective. Moreover, even when a learner cannot come to know the arbitrary by herself, she may be capable of coming up with a reason for the choice after being informed about it. For example, there is a good reason for naming a four-sided polygon with some variations of 'quad'. The choice is arbitrary in a sense that it could have been called quadrat, quadrant, quadragon, etc. In this article we use 'arbitrary' as described by Hewitt, pointing to a possibility of a different choice, rather than implying a lack of reason behind it.

In the case of necessary, the education community has widely acknowledged the importance of students (re)inventing mathematics (*e.g.,* Freudenthal, 1991). In the case of arbitrary, research has accumulated varied evidence showing that "getting students to accept and adopt names and conventions is not always easy" (Hewitt, 1999, p. 3). For example, Long (2011) discussed a classroom episode in which a student questioned why the symbol of an acute angle '$\angle$' is used for denoting *any* angle, including straight, obtuse and reflex.

We infer from these examples that students can experience an intellectual need in explanations behind the arbitrary (Harel, 2013), the need which if not fulfilled, can hinder acceptance and adoption. Accordingly, teachers should be prepared for mediating conventions appearing in school curriculum in a way that reaches the students and not just their memory. Furthermore, while Hewitt's distinction between the arbitrary and necessary addresses the school curriculum, the boundaries between the two can be blurred even for teachers. Indeed, in the study of Levenson (2012) three high-school teachers approached the convention $a^0=1$ as a theorem that can be proved, for instance with a geometrical sequence or rules of exponents. Accordingly, we maintain that discussions around conventions can serve as a framework in which teacher knowledge of mathematics can develop.

Our paper can be considered as a belated response to Hewitt, who argued that "mathematics does not lie with the arbitrary, but is found in what is necessary" (1999, p. 5) and that students need to accept the arbitrary rather than question it. The aim of this paper is to demonstrate that *unpacking*—that is, offering plausible explanations and arguments for some choices of the mathematical community—can lead learners to active engagement with mathematical concepts. Hence, mathematics can also be found in what is arbitrary.

Mathematical conventions and convention-unpacking tasks

By *conventions* we refer to the choices of the mathematics community regarding definitions of concepts, their names, and symbols. Table 1 contains four examples of such conventions and ideas that can be employed for their unpacking. For instance, in the case of the first convention the explanation can stem from considering a *group structure*. Indeed, rational numbers (without a zero) under the operation of multiplication, and bijective functions (with an appropriate choice of domains) under the operation of composition are groups. Accordingly, a reciprocal of a fraction and an inverse of a function are code names for inverse of an element in a group structure denoted by the symbol '$^{-1}$'.

In convention-unpacking tasks we engage teachers in providing plausible explanations for the choices of mathematical conventions. Following Harel and Sowder (2007), we distinguish between *ascertaining* and *persuading*. In ascertaining, teachers pursue the reasons for establishing a particular convention that they themselves will perceive as convincing. Persuading, on the other hand, is the practice of convincing others using these reasons. These processes are interrelated because one of the criteria for an ascertaining explanation is its anticipated power to persuade others. Accordingly, we integrate phases of ascertaining and persuading into our convention-unpacking tasks.

Conventions	Intended ideas
1. The same symbol of ‘ $^{-1}$’ is used for denoting reciprocals and inverse functions	The notion of inverse element in a group and other algebraic structures.
2. The same symbol of ‘| |’ is used for denoting a modulus of complex numbers and determinants of matrices	a. Complex numbers are isomorphic to the subset of matrices 2 × 2: $$a + bi \leftrightarrow \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$ b. Notion of multiplication-preserving homomorphism $\lvert xy\rvert = \lvert x\rvert\lvert y\rvert$.
3. a^0 is defined to be 1 for any non-zero a	a. Exponential rules for multiplication and division. b. Definition of ab for natural a and b as the number of functions between the set with b elements to the set with a elements. c. Continuity of the function $f(x)=a^x$. d. Convergence of the functions $$f_n(x) = x^{\frac{1}{n}} \text{ for } n\to\infty.$$ e. The empty product convention. (See Kontorovich, 2016 for additional ideas).
4. The function is called *odd* or *even* if for any x in the domain $f(x) = -f(-x)$ or $f(x) = f(-x)$, respectively	a. The oddness/evenness of monomial functions $f(x) = ax^n$ corresponds with the oddness/evenness of n. b. McLaurin series of odd/even functions correspondingly contain only odd/even powers of the variable. (See Sinitsky, Zazkis & Leikin, 2011 for details.)

Table 1: Conventions and useful ideas for their unpacking.

In the (−1) Task:

> Have you noticed that superscript (−1) is used to indicate reciprocals (5^{-1}) and inverse functions ($f^{-1}(x)$)? Why do you think this is the case? Offer an explanation for establishing this convention and be prepared to persuade your peers.

the instruction “Offer an explanation for establishing this convention and be prepared to persuade your peers” addresses ascertaining in the first clause and persuading in the second.

In the next two subsections we analyze convention-unpacking tasks through the lenses of focus of attention (Mason, 2008) and uncertainty (Zaslavsky, 2005).

Convention-unpacking tasks through the lens of focus of attention

Mason (2008) argues that learning new mathematics is deeply related to what is in the learners’ focus of attention and how it is attended to. While the focus of attention can shift, Mason proposes five structures of attention: *holding wholes*, when a whole structure is in the focus; *discerning details,* when a particular element of the whole is attended; *recognizing relationships,* when connections between the discerned details are attended; *perceiving properties,* when the discernment of details is driven towards generalization of their property or connections between them; and *reasoning on the basis of perceived properties,* when details are discerned as a result of an a priori established connection or property.

One of the implications of Mason’s theory is that directing learners’ focus of attention is necessary for sense-making and internalization of ideas intended by a teacher or task designer. In the task under consideration, the intended ideas include the mathematics that can be employed for unpacking a particular convention, towards which the learners’ attention is directed. As is shown below, learners’ structures of attention play an important role in seeking unpacking ideas.

Convention-unpacking tasks through the lens of uncertainty

Zaslavsky (2005) maintains that tasks that make learners’ uncertain of the mathematical validity of a claim, problem-solving method, conclusion or outcome can facilitate the learning of mathematics. Zaslavsky follows the many scholars who claim that when an individual is experiencing uncertainty (dissonance, in terms of Festinger; cognitive conflict or disequilibrium, in terms of Piaget) she will be encouraged to modify something in her ways of acting and thinking in attempt to escape from this situation.

Convention-unpacking tasks are expected to elicit uncertainties related to the existence of a plausible explanation and competing explanations. The uncertainties related to the existence of an explanation are expected to be particularly intense among learners who are not familiar with the tasks. The lack of experience in reasoning about conventions can evoke an initial reaction that conventions are chosen arbitrary, which is the case for some conventions, but not for others. Also this perspective is in tension with a common perception of mathematics as a logical discipline where every decision can be sustained. In some conventions (such as 1, 2 and 4 in Table 1) a common symbol and the similarities in the names of the concepts hint that there exists a more convincing explanation than a random choice.

Explanations for the choice of a particular convention can compete with each other in being perceived as more ascertaining and persuasive by suggesting different arguments (see conventions 2, 3 and 4 in Table 1 for example). Accordingly, the request to come up with an ascertaining and persuading explanation can elicit uncertainty in choosing among alternatives. Some examples of uncertainties that can emerge when learners engage with a convention-unpacking task are presented in the next section.

Teachers' responses to the (−1) Task

In this section, we present snapshots of teachers' engagement with a convention-unpacking task. We frequently use such tasks in teacher education courses and professional development workshops (Zazkis & Kontorovich, 2016). We invite teachers to respond in writing and to provide explanations with which they are satisfied. We allocate at least a week for the task completion, so there is no pressure of time at the ascertaining phase. The persuading phase occurs in the classroom where teachers share and discuss their responses.

This section contains abbreviated responses of four prospective secondary school mathematics teachers who worked on the (−1) Task in a problem-solving course during the last term of their teacher education program. The course participants were accustomed to working on challenging tasks, which could potentially lead them to new mathematical insights. In class discussions persuading peers and clarifying mathematical ideas were strongly encouraged. Particular attention in the course was drawn to connections between undergraduate and school mathematics.

We chose to present the responses of the particular teachers as they exemplify the variety of teachers' approaches to the task. Specifically, the response of Sophia (the names given are pseudonyms) is an example of a learner whose mathematical knowledge was sufficient for providing the intended explanation at the ascertaining phase. Ezra's work illustrates how the uncertainty evoked by the task can intensify the need for discussion. Attending to Tanya's explanation exemplifies how a group discussion can lead to identification of gaps in a learner's knowledge. In contrast, we present the response of Anton to illustrate that the convention-unpacking task does not lead all learners to complete internalization of ideas intended by task designers. Our comments on teachers' responses are shaped by the question "How did teachers' mathematical knowledge evolve whilst resolving the uncertainty elicited by the (−1) Task?"

Sophia: ascertained by mathematical structure

In reflecting on the task, Sophia (as well as all other teachers) indicated that while being quite familiar with reciprocals and inverse functions, she had never noted that these concepts are denoted by the same symbol, and consequently, did not consider any connections between them. Sophia said that after reading the task, she sought explanation in advanced topics that she studied when majoring in mathematics. She recalled topics in which a superscript (−1) appeared. Eventually, she recalled concepts from an abstract algebra course, which assisted her in formulating the following explanation:

> Multiplicative inverses in a group/ring/field are consistent with function inverses in the sense that applying the inverse operation/function twice returns the original element.

Sophia shared her response in the classroom and it was willingly accepted by some of her peers.

Comments: Sophia interpreted the symbol of '$^{-1}$' as a hint for a relationship between reciprocals and inverse functions. For *recognizing the relationship,* she activated her advanced mathematical knowledge in abstract algebra and browsed the course topics in a search for the symbol. This suggests that her attention seemed to be structured by *holding wholes*. Eventually, she *discerned* the overarching concept of "inverse elements" that appeared in the topics of "group/ring/field". Her explanation is based on a property of an inverse element that can be represented symbolically as $(a^{-1})^{-1} = a$. The explanation demonstrates Sophia's realization of the conceptual relationship between reciprocals and inverse functions, a realization that was not obvious for her at the beginning.

We do not know whether the persuading phase contributed to expanding Sophia's knowledge. However, in what follows we show how Sophia's response contributed to the classroom discussion.

Ezra: help seeking and persuaded by peers

Ezra indicated that while initially attending to the (−1) Task, he could not come up with any connection that explained the common symbol. At some point, he decided to look for ideas on the Internet. Ezra started with an etymology dictionary in attempt to find the source of the word 'inverse'. He explained that he learned this technique in a linguistics course. Ezra found that 'inverse' comes from the Latin '*inversus*', which is "turned upside down" or 'overturned'. He focused on the procedures for finding reciprocals and inverse functions and came up with the 'reversing' explanation: for finding a reciprocal of a fraction, the numerator and denominator should be reversed; for finding an inverse function of $y = f(x)$, the variables exchange their names and then the created equation should be solved for y. In the obtained equation x's are on the left side and y's are on the right side, so he called it "a reversed equation". However, Ezra addressed this explanation as being personally unsatisfactory and commented:

> I couldn't come up with something mathematical that can really explain this fact [using the same symbol]. So I decided to search for an explanation in some mathematical websites.

In his search, Ezra encountered the paper of Even (1992), which discussed the 'undoing' approach. He said that he liked this approach and formulated it in the following way:

> An inverse is something that will return you to the starting point. Let's say I pushed the wrong button on a calculator and multiplied by 5. For correcting this, I need to divide by 5, which is multiplying by 1 over 5. The same goes with functions: applying an inverse function undoes the effect of the original function.

In the discussion Ezra shared with his classmates the deficiencies that Even (1992) found for this 'undoing' approach

[1]. He also noted that the convention-unpacking task revealed to him "a bigger picture" of connections between reciprocals and inverse functions. He added that the explanation that he offered is "not mathematical enough" as it lacked more precise mathematical details; and thus, he readily considered alternatives provided by his classmates. Responding to Sophia's explanation, Ezra noted that it was very abstract and that it was good that he had found the 'undoing' approach beforehand, as it helped him to make sense of the situation.

Comments: In the ascertaining phase, Ezra reached out to web-outlets for help and came up with the 'reversing' and 'undoing' explanations. The former was inspired by the linguistic meaning of the word 'inverse' and it was based on *discerned similarities* in the procedures for determining a reciprocal of a fraction and an inverse of a function. However, Ezra considered this explanation as not being sufficiently ascertaining. We suggest that when developing this explanation, his attention was captured in the *reasoning-on-the-basis-of-perceived-property* structure, which in this case was the property of procedural connections between determining reciprocals and inverse functions.

Ezra's latter 'undoing' explanation was formulated based on the mathematics education paper and it can be symbolically represented as $a^{-1} \times a = I$. Although Ezra perceived this explanation as ascertaining, he anticipated that it would not be fully persuasive for his classmates as it was "not sufficiently mathematical". Apparently, the deficiencies of this explanation identified by Even (1992) intensified Ezra's perception. However, the 'undoing' approach turned to be useful for internalizing Sophia's response. Namely, the 'undoing' and "group/ring/field" explanations completed each other by filling "the bigger picture" with mathematical details.

Tanya: ascertained with error and persuaded by peers

Tanya indicated that the (–1) Task evoked for her an association of the formula for the derivative of an inverse function, which she (erroneously) remembered as

$$[f^{-1}(x)]' = -\frac{1}{f'(x)}$$

Her explanation was:

> If you derive $f^{-1}(x)$, then –1 comes forward, the inner derivative is f' and the remaining superscript –1 goes to the denominator and becomes a positive one. In this way, we get
>
> $$[f^{-1}(x)]' = -\frac{1}{f'(x)}$$
>
> This is similar to the slopes of perpendicular tangent lines with the multiplication of –1.

In the classroom discussion Tanya's peers rejected her explanation and argued that if she considers the superscript (–1) as an exponent, then according to the chain rule it should be

$$[f^{-1}(x)]' = -\frac{f'(x)}{f^2(x)}$$

Then the class engaged in reconstructing the correct formula for the derivative of an inverse function. When the formula

$$[f^{-1}(x)]' = \frac{1}{f'\left(f^{-1}(x)\right)}$$

was obtained, Tanya modified her initial idea and said that using superscript (–1) for reciprocals and inverse functions can be explained with the fact that, "The derivatives of a function and its inverse at the same point are reciprocals".

Tanya's peers referred to the modified Tanya's explanation as acceptable, but they considered the connection between reciprocals and inverse functions through the notion of derivative as rather "farfetched". They preferred Sophia's explanation that highlights a direct link between the two structures.

Comments: At the ascertaining phase, Tanya's attention seemed to get captured in the structure of *reasoning on the basis* of derivatives. Indeed, with no attention to the details, she seemed to remember that there exists a formula with a reciprocal structure that contains derivatives, functions and their inverses. In addition, her initial explanation indicates that she also *recognized a relationship* with perpendicular lines, the multiplication of slopes which equals –1. Since a derivative of a function at a point determines the slope of its tangent line (at the same point), Tanya's approach is consistent with an assumption, or a commitment to a recalled relationship involving reciprocal slopes, leading to the conclusion that the tangent lines of a function and its inverse are perpendicular. Considering the superscript (–1) as an exponent, she could have derived an inverse function, which shifted (–1) forward. Apparently, Tanya ignored the remaining details that did not support her formula:

$$[f^{-1}(x)]' = -\frac{1}{f'(x)}$$

After the correct formula was reconstructed in the classroom, Tanya modified the previously indicated relations between a function and an inverse function, and *recognized the relationship* stating that the slopes of their tangent lines are reciprocals.

Anton: arbitrary and resistant to peers' persuasion

Anton indicated that the task surprised him, as he could not think of any explanation for using the same symbol for reciprocals and inverse functions. In his submitted work he wrote that "a symbol is the only similarity between the two", and also the equation:

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

Anton also stressed the difference in the placement of ' $^{-1}$' for a reciprocal of a function and an inverse function. In his words:

> Some students' confusions with inverses and reciprocals can be resolved in seconds, just if they are pointed at the position of '–1'. Look at the placement of '–1': For inverse functions, '–1' would be located in between f which is the name of the function, and (x). For the case of reciprocal functions, your '–1' would be placed outside of the function notation $f(x)$, which is $[f(x)]^{-1}$. I learned this trick from my math teacher and loved it.

In the classroom discussion Anton vigorously rejected Sophia's approach to reciprocals and inverse functions as inverse elements in a group structure. He claimed that referring to reciprocals as inverses was inappropriate. He recalled the procedures for finding a reciprocal of a function

$$(i.e., \frac{1}{f(x)})$$

and an inverse of a function, and said that the apparent differences between the procedures confirm his explanation and there was no connection between the ideas.

The other teachers engaged in persuading Anton of the reasonableness of Sophia's explanation and clarified the concept of inverse element. Anton replied: "Ahh… OK, so now I get your point." However, in the continuation of the discussion, when teachers referred to reciprocals and inverse functions as "inverses", he corrected them and insisted on using the terms of "multiplicative inverse" and "inverse function" instead.

Comments: Anton struggled with the task on three occasions: First, similarly to Sophia, Tanya and Ezra, Anton considered the common symbol of ' $^{-1}$' as a hint for existence of a connection between a reciprocal of a function and an inverse function. However, during the ascertaining phase, he was convinced of the lack of such connections. This assertion entailed uncertainty as the task required him to explain something that was nonexistent from his perspective. Noticing that

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

and the position of the superscript (–1)—which is different for reciprocals and inverse functions—can be considered as a resolution of the uncertainty: the ideas complement each other by stressing the differences between the concepts and he aimed to persuade others in the lack of connections.

Second, at the persuading phase Anton's explanation, indicating the lack of connections between reciprocals and inverse functions, competed with Sophia's explanation, which highlighted a connection. Although Anton rejected Sophia's approach, he shifted his focus of attention to the difference in the procedures for determining a reciprocal of a function and inverse of a function. Accordingly, at this phase Anton's focus of attention was still *based on the perceived property,* which was the lack of connections.

Third, even after an elaborated persuasion by his peers in the connection between the concepts, Anton refused to address the concepts with the same term of 'inverse' and insisted on distinguishing between them by name. Note that compared to the first occasion, where he addressed the concepts as 'reciprocals' and "inverse functions", at this occasion he used the word 'inverse' in both names ("multiplicative inverse" and "inverse function"). We interpret this change in terminology as a reflection of Anton's internalization of some connections.

Concluding remarks

We agree with Hewitt (1999) who wrote that mathematics is full of conventions, consideration of many of which probably does not lead to deeper mathematical understanding. However, in this paper we exemplified mathematical conventions, considering which may lead the learner to non-trivial mathematical ideas and connections that are rarely attended. We also presented empirical examples of learners engaging with some of these ideas and connections via ascertaining and persuading. In this section we offer suggestions for using convention-unpacking tasks in research and practice.

In the presented snapshots four teachers actively engaged with mathematical concepts as a result of an episodic encounter with a convention-unpacking task. Systematic variation of every aspect of this setting can turn into a fascinating research venue. Possible outcomes of these venues can contain characterization of the relations between the considered conventions and activated mathematical knowledge, as well as recognition of the roles of learners' mathematical backgrounds and social interactions. In addition, it is interesting to explore the impact of task formulations on learners' responses. For instance, the (–1) Task can be reformulated without mentioning reciprocals:

> Suggest a convincing explanation for the decision to use the symbol of ' $^{-1}$' for denoting inverse functions. Be prepared to persuade your peers.

The uncertainty that convention-unpacking tasks are expected to elicit among learners who are not familiar with this activity, can expose some of the classroom socio-mathematical norms, which become tangible when there is a deviation from them (Yackel & Cobb, 1996). Specifically, learners' responses provide evidence for the existence of explanations that are perceived as convincing, and other which are addressed as "farfetched" or invalid (see also Kontorovich, 2016, for exploration of characteristics of accepted and not accepted explanations for the convention $a^0=1$). Accordingly, convention-unpacking tasks can provide a unique setting for exploration and cultivation of argumentation norms.

Mathematical conventions are usually presented as part of an introduction to a topic, and they are taken for granted afterwards. Students, who are newcomers to mathematics, can experience an intellectual need to understand why a particular convention was established (*e.g.,* Rabin *et al.*, 2013). The presented cases of Sophia, Ezra, Tanya and Anton show that providing convincing explanations can be challenging even for mathematics majors. We suggest that this is because they are used to perceiving these conventions as unquestionable norms in the mathematics community. Accordingly, working on convention-unpacking tasks can prepare teachers for contingent events where a particular convention is questioned by a student or a peer (see Rowland & Turner, 2007, for contingency component of teacher knowledge).

Not all explanations for convention choices are both self-ascertaining to a teacher and within reach of school students (see the response of Sophia). Not all explanations which are appropriate for school students are persuasive from a teacher's perspective (see the response of Ezra). As well, not all explanations that are useful for students (from a teacher's perspective) are mathematically consistent (see the responses of Anton and Tanya). Accordingly, enriching convention-unpacking tasks with a pedagogical layer offers the potential to contribute to teachers' pedagogical skills, as in this reformulation of the (–1) Task:

> You are a high-school mathematics teacher. In a lesson on inverse functions one of your students asked you to

explain why the same symbol of ' $^{-1}$' is used for reciprocals (5^{-1}) and inverse functions ($f^{-1}(x)$).

Suggest a convincing explanation for the attentive student and be prepared to discuss it with your colleagues.

This layer can also turn the tasks into a powerful tool for exploring and enhancing the intertwining components of teacher knowledge (Zazkis & Kontorovich, 2016).

Explanations for the choice of some conventions can stem from advanced mathematics, which is usually acquired in tertiary education (see Table 1 for examples). Accordingly, unpacking provides teachers with opportunities to put their advanced mathematics knowledge to use. According to Zazkis and Leikin (2010), such opportunities are relatively rare in teachers' practice.

We are pleasantly surprised by cases in which teachers' discussions lead to considering new conventions. For instance, driven by the convention in the (–1) Task, one of the teachers elicited a convention regarding iterated functions. To recall, when f is a function from a set X to itself, the n-th iteration of f is defined as $f^n \equiv f \circ f^{n-1}$ for any natural $n \geq 2$. The teacher posed a question, "Why are iterated functions denoted with the symbol of power?" Another teacher recognized connections between this question and the (–1) Task and asked: "So if $sin^2(x) = sin(sin(x))$ and $sin^{-1}(x) = arcsin(x)$ what does $sin^{-2}(x)$ mean?" Such questions inspire us to continue the exploration on how conventions can be used in learning and teaching mathematics.

Notes

[1] Even (1992) argued that while 'undoing' is helpful in understanding the concept of inverse functions, one should not be limited to this approach as it may result in mathematical difficulties. For instance, she found that perceiving the root functions as "undoing" of power functions hindered about one third of the teachers who participated in her study from detecting an inverse to exponential functions. An additional misconception that might emerge is that all functions have inverses.

References

Even, R. (1992) The inverse function: prospective teachers' use of "undoing". *International Journal of Mathematical Education in Science and Technology,* **23**(4), 557-562.

Festinger, L. (1957) *A Theory of Cognitive Dissonance.* Stanford, CA: Stanford University Press.

Freudenthal, H. (1991) *Revisiting Mathematics Education: China Lectures.* Dordrecht, NL: Kluwer.

Harel, G. (2013) Intellectual need in the mathematical practice: a theoretical perspective. In K. Leatham (Ed.), *Vital Directions for Mathematics Education Research.* New York: Springer Science+Business Media.

Harel, G. & Sowder, L. (2007) Toward comprehensive perspectives on the learning and teaching of proof. *Second Handbook of Research on Mathematics Teaching and Learning,* **2**, 805-842.

Hewitt, D. (1999) Arbitrary and necessary part 1: a way of viewing the mathematics curriculum. *For the Learning of Mathematics,* **19**(3), 1-9.

Hewitt, D. (2001a) Arbitrary and necessary part 2: assisting memory. *For the Learning of Mathematics,* **21**(1), 44-51.

Hewitt, D. (2001b) Arbitrary and necessary part 3: educating awareness. *For the Learning of Mathematics,* **21**(2), 37-49.

Kontorovich, I. (2016) We all know that a^0=1, but can you explain why? *Canadian Journal of Science, Mathematics and Technology Education,* **16**(3), 237-243.

Levenson, E. (2012) Teachers' knowledge of the nature of definitions: the case of the zero exponent. *The Journal of Mathematical Behavior,* **31**, 209-219.

Long, G. S. (2011) Labelling angles: care, indifference and mathematical symbols. *For the Learning of Mathematics,* **31**(3), 2-7.

Mason, J. (2008) Being mathematical with and in front of learners: attention, awareness, and attitude as sources of differences between teacher educators, teachers and learners. In Jaworski, B. & Wood, T. (Eds.) *The Handbook of Mathematics Teacher Education, Vol. 4: The Mathematics Teacher Educator as a Developing Professional* (pp. 31-56). Rotterdam: Sense Publishers.

Piaget, J. (1985) *The Equilibration of Cognitive Structures: The Central Problem of Intellectual Development.* Chicago, IL: University of Chicago Press.

Rabin, J. M., Fuller, E. & Harel, G. (2013) Double negative: the necessity principle, commognitive conflict, and negative number operations. *Journal of Mathematical Behavior,* **32**(3), 649-659.

Rowland, T. & Turner, F. (2007) Developing and using the 'Knowledge Quartet': a framework for the observation of mathematics teaching. *The Mathematics Educator,* **10**(1), 107-123.

Sinitsky, I., Zazkis, R. & Leikin, R. (2011) Odd + Odd = Odd: is it possible? *Mathematics Teaching*, **225**, 30-34.

Yackel, E. & Cobb, P. (1996) Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education,* **27**(4) 458-477.

Zaslavsky, O. (2005) Seizing the opportunity to create uncertainty in learning mathematics. *Educational Studies in Mathematics,* **60**(3), 297-321.

Zazkis, R. & Kontorovich, I. (2016) A curious case of superscript (–1): prospective secondary mathematics teachers explain. *Journal of Mathematical Behavior,* **43**, 98-110.

Zazkis, R. & Leikin, R. (2010) Advanced mathematical knowledge in teaching practice: perceptions of secondary mathematics teachers. *Mathematical Thinking and Learning,* **12**(4), 263-281.

By relieving the brain of all unnecessary work, a good notation sets it free to concentrate on more advanced problems, and in effect increases the mental power of the race.

Alfred North Whitehead (1911)
An Introduction to Mathematics,
New York: Henry Holt. p. 58