A Recursive Theory of Mathematical Understanding

SUSAN PIRIE, TOM KIEREN

> "Everything said is said by an observer." [Maturana, 1980]
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> "The experiencing organism now turns into a builder of cognitive structures intended to solve such problems as the organism perceives or conceives ... among which is the never ending problem of consistent organization that we call *understanding*." [Von Glasersfeld, 1987]

Over the past 20 years or so there has been a continuing dialogue, much of it through Psychology of Mathematics Education Conferences, on what it means for a person to understand mathematics. One of the features of this dialogue has been the theoretical identification of different kinds of understanding, principally instrumental and relational understanding but also concrete, procedural, symbolic and formal understanding. Pirie [1988] has suggested that thus describing different kinds of understanding is inadequate as a means of differentiating children's performances which exhibit mathematical understanding. She claims, and illustrates from extensive taped interactions of children doing mathematics, that mathematical understanding is a complex phenomenon for the child doing it. A single category does not well describe it, nor do such categories capture understanding as a process rather than as a single acquisition. What is needed is an incisive way of viewing the whole process of gaining understanding.

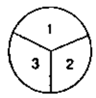
There have indeed been recent efforts to go beyond a cataloguing of kinds of understanding or thinking of mathematical understanding as a singular acquisition. Ohlsson [1988] performed a detailed mathematical and applicational analysis of fraction-related concepts. From this elaborated example, he suggested that mathematical understanding entails three things: knowledge of the mathematical construct and related theory, knowing the class of situations to which this theory can be applied, and a referential mapping between the theory and the situations. He does not however, suggest how this mapping is developed or grows. He infers but does not give a process model.

Herscovics and Bergeron [1988] give a two-tiered model of understanding and illustrate it using the understanding of number and pre-number in young children. The first tier involves three levels of physical understanding: intuitive (perceptual awareness), procedural (e.g. 1-1 correspondence) and logico-physical abstraction (e.g. physical invariance). The second tier is non-hierarchical and includes as components of understanding the use of mathematical procedures (e.g. counting) to make mathematical abstractions reflected through the use of a notational system.

Both of the above models of understanding involve levels or components which appear to have predicate quality — they define complexes of components in unique categorical terms. In that sense they give a picture of the components which might be involved in the process of understanding. Von Glasersfeld [1987], however, sees understanding as a continuing problem-solving process of *consistently organizing* one's mathematical structures.

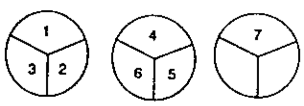
Let us consider the following example drawn from a study of 7-9 year olds working in groups doing fraction comparison tasks [Wales, 1984; Kieren & Pirie, forthcoming]. In the task children were asked to compare the amount of pizza person "A" would get if sharing 3 pizzas between 7 people with the amount person "B" would get sharing 1 pizza between 3 people. Here is a commentary by Hanne working with 2 friends (all aged 7).

Hanne: A is hard — let's skip it.
Hanne: B is easy, you "Y" it



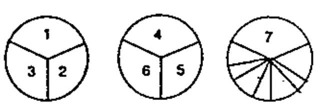
(Draws "Y" and explains her process to her friends. The numbers are ours to illustrate her explanation.)

Hanne: (I) Let's use "Ys" on A.
(Action 1, draws:



i.e., Hanne cuts each of the three pizzas into "fair shares" in order to give one third each.)

Hanne: (II) (Action 2, draws:



i.e., she cuts the remaining piece into seven smaller pieces.)

Hanne: (III) Oh, I see! A gets a third and a bite. A gets more.

What has happened here? Hanne starts by not understanding how to divide 3 among 7. It is clear from the complete tape that she can divide 1 among n for small n and in particular has *formalised* this act for $1 \div 3$ ("Y it"). In I and II we see her take this "form" from the particular environment in which it was constructed and use it, firstly to divide each separate pizza into thirds ("Y them"), and then to divide the remaining piece, now forming the "whole 1," into seven pieces (1 among n) At III she signifies her realisation that she has a successful new organization of sharing, or division, based on the combination of two previously unconnected actions. This leads us to ask: what does Hanne's understanding entail? Indeed, what does mathematical understanding itself entail? How can we depict this growing process? Our answer can be summarized as follows:

> Mathematical understanding can be characterized as levelled but non-linear It is a recursive phenomenon and recursion is seen to occur when thinking moves between levels of sophistication (as with Hanne above). Indeed each level of understanding is contained within succeeding levels. Any particular level is dependent on the forms and processes within and, further, is constrained by those without

While it is beyond the scope of this paper either to completely delineate this theory of mathematical understanding, which we call *transcendent recursion,* or to fully connect it to data on children's mathematical behaviour gathered in England and Canada, some major tenets of the theory are highlighted below and then illustrated with reference to an understanding of quadratic functions Subsequently we use the model to analyse an episode of student behaviour with a view to comprehending the growth of their understanding. We see "consistent organisation" in action.

In saying that mathematical understanding is levelled and recursive we are trying to *observe* it as a complex levelled phenomenon, defined by Vitale [1988], which is recursive if each level is in some way defined in terms of itself (self-referenced, self-similar), yet each level is not the same as the previous level (level-stepping). To this definition we have added an idea taken from Margenau's [1987] notion of growth of scientific constructs. New constructs transcend but are compatible with old ones (they are not simple extensions) What is in fact more critical is that the "new" transcendent knowing frees one from the actions of the prior knowing

The work of Maturana and Tomm [1986] and Tomm [1989] allows us a more detailed view of recursion in knowing. Maturana and Varela [1987] see knowing exhibited by effective actions as these are determined by an observer. That is, if one wishes to see if a person knows how to play the piano, the person might actually be asked to play. Then the observer determines if this action is effective. So, as we will try to show below, it also is with mathematics. Who, or rather, what is the knower? Again, following Maturana and Varela [1987], we see the human knower of mathematics as self-referencing and self-maintaining in a particular niche of behavioural possibilities.

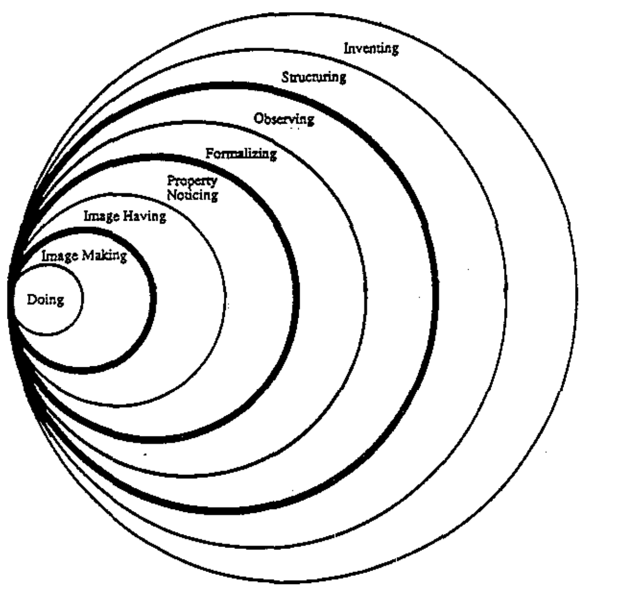
For human beings, mental states and actions are crucial elements of this niche, or sphere, of possibilities. Since a learner is self-referencing, a significant means of cognition is recursive in nature; growth is knowing, i e when effective action occurs through new thought-actions which coordinate or distinguish prior thought-actions.

In giving a more detailed analysis of this recursive growth, Tomm [1989] envisions it as occurring in a situation where one is in interaction with another person such as a fellow student or teacher He described the first recursive level as the "consensual coordination" of action. The recursion on this linguistic action is described as "languaging," an action which makes something "a thing" for someone. He sees several levels of recursion beyond this. The model presented below has the levelled feature of Tomm's description, but it also envisions growth in knowing as possibly involving internal, as well as external, interactions. The terms used for the levels in the model below reflect this vision and also relate to its use as a model of mathematical understanding.

The process of coming to know starts at a level we call "primitive doing." Action at this level may involve physical objects, figures, graphics or symbols. "Primitive" here does not imply low-level mathematics, but rather a starting place for the growth of any particular mathematical understanding

The first recursion occurs when the learner begins to form images out of this "doing." The effective actions here involve "image making." At the next level these action-tied images are replaced by a form for the images. From the mathematical point of view it is this "image having" which frees a person's mathematics from the need to take particular actions as examples This is a first level of abstraction; but it is critical to note that it is the learner who makes this abstraction by recursively building on images based in action For understanding to grow, these images cannot be imposed from outside.

Because knowing has to be *effective* action, the recursions do not stop here. The images can now be examined for specific or relevant properties. This may involve noticing distinctions, combinations, or connections between



images. This level of "property noticing" is the outermost level of unselfconscious knowing. (The word "outer" has been carefully chosen to imply that the levels of understanding wrap round each other, as illustrated in Figure 1, and contain, indeed require, the possibility of access to all previous levels. Levels of understanding do not equate with higher or lower levels of mathematics.)

The next level of transcendent recursion entails thinking consciously about the noticed properties, abstracting common qualities and discarding the origins of one's mental action. It is at this level that full mathematical definitions can occur as one becomes aware of classes of objects that one has constructed from the formation of images and the abstraction of their properties.

One is now in a position to observe one's own thought structures and organise them consistently. One is aware of being aware, and can see the consequences of one's thoughts. It is clear at this point that while this outer level of understanding is transcendent, in other words fundamentally new in some way, it has to be consistent with all previous levels of knowing.

For fuller understanding one must now be able to answer *why* the consequences of thoughts must be true. This calls for an awareness of associations and of sequence among one's previous thoughts, of their interdependence. In mathematical terms it might be setting one's thinking within an axiomatic structure.

All of these levels of recursion are referenced in a direct way to previous levels. Although new levels transcend or make one free of actions at an "inner" level, in some sense these actions on previous levels become initiating conditions which constrain one's knowing. At the highest level of recursion, as Tomm suggests, knowers act as free agents. We call this the level of inventing.* Now one can choose to initiate a sequence or structure of thought which is a recursion on the previous one in the sense that it exists as a base, but is freely, yet compatibly, created. For example, mathematicians such as Lobachewski, Bolyai, and Gauss deliberately invented non-Euclidean geometries rather than try to deduce or explain the parallel postulate as a necessary consequence of Euclidean geometry in the way that Saccheri did.

To illustrate this growth through recursion in less general terms, we turn to an example of mathematical knowing, the knowing of quadratic functions. The particular example is hypothetical but allows us to illustrate the entire chain of recursions from primitive doing to inventing. It also allows illustrations of the fact that one should not think of action in simplistic terms. Any action, and this one, plotting, in particular, may already be a recursion on previous thought or actions.

We start with the first level of recursion, image-making. The students are taking given, specific, quadratic functions, making tables of values, and then plotting them. The image-making can be seen in comments like "Does $x^2 + 2x + 1$ go like this?" or "I get 1 for -1 in $2x^2 - x - 2$." What is critical here is that these tables and graphs have an individual quality. If one were to ask such students, "What do you know about quadratic functions?," their effective action would be related to the images of specific quadratics. For them to say anything further would require their making more tables or doing more plotting; their comments about quadratics would pertain to their actions.

The next stage is reached when they have generalized these specific images. They are now able to say, "Quadratic functions? Well, they all form sort of U-shaped curves." They have now replaced their actions on quadratics with an image and can talk about it. There is an ability, but no necessity, to continue to plot individual graphs, and certainly no need to make very extensive tables of values. This stage of image-having is a recursion on the stage of image-making.

Because they can now talk about quadratics as objects, students can move to a level of noticing features or distinctions among them. Quadratics have vertices, they open up or down, they are not linear, they can have maxima or minima elsewhere than at the origin ... Effective action is not now related to a particular graph but to properties of quadratics. Students here have a generalized image.

The next recursion occurs when students realize that despite these distinctions quadratic functions can be considered as a class of things: students can talk about the set of functions $y = ax^2 + bx + c$ and see this as a consciously-made mathematical definition. They are using self-conscious thought to formalize their understanding. They now know quadratic functions as mathematical objects and not simply as graphical ones.

The next level of recursion is signalled by the students' interest in the question, "What is true for all quadratic functions?" At this level the quadratic formula or the method of completing the square are pieces of a possible theory and not simply techniques for computation as they might earlier have been. The arrival at this level, that of observing the class of quadratic functions as a mathematical object, allows us to make another point critical in our theory of understanding. Up to now, the recursions may have appeared to have a sequential or linear quality; but the development of understanding does not have this quality. Students at this new level might decide to address the question "Can quadratic equations have just one real solution?" and be prompted to drop to the level of primitive doing, to making tables and graphs of particular examples. However, the return to image-making is not to image-making as it was previously perceived. The students now know they are looking for an image which can be distinguished by certain features but which can still be characterized as belonging to the class of quadratics. In other words, while actions, images, and distinctions, formed initiating conditions for the definition of a class of quadratics, the local class of quadratics now forms an environment in which a new kind of exploratory action takes place. Theory folds back to constrain one's actions. In contrast to the original image-making, one is now acting on what may be a specific but, one hopes, representative example. Thus knowing and understanding quadratics is not a linear process: knowing folds back on itself and has the quality of being circular without contradiction. Such a folding back

*This use of "inventing" is not intended to suggest that persons do not or could not develop essentially new (for them) ideas at other levels allows one to validate or to see as consistent all the levels of quadratic knowing up to this point, and permits the recursive reconstruction of quadratic knowledge in the face of new challenges.

Let us return now to the level of quadratic knowledge we have termed "observing." As suggested, recursions at this level revolve around the question, "What is true for all quadratic functions?" Not only does this lead to theoretical consequences with respect to maxima, minima and roots, but it also provides a new focus for one's self-conscious idea of quadratic functions. For example, one may now talk about quadratics as contrasted with linear functions and generate comparative observations about these.

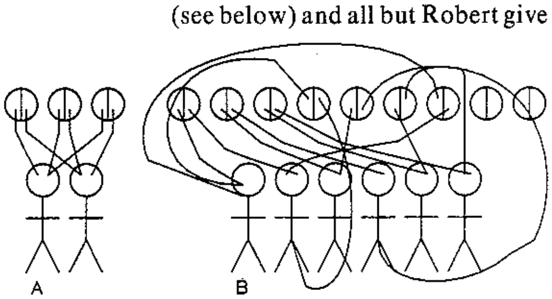
The next level of recursion is driven by the need to logically situate the consequences observed at the previous level: for example, to explain the "truth" that quadratics have two roots occurring in particular pairs. One can think of this level as the level of axiomatic-deductive proof, or the level where quadratics are situated as an exemplary system within the system of polynomial functions.

At the highest level, one would be deliberately thinking of quadratic functions in a completely new way: for example, the study of all complex functions of the form $z = kz^2 + c$, which while essentially quadratic might lead one into the field of recursive dynamic phenomena, strange attractors, fractals, and a whole new thought structure.

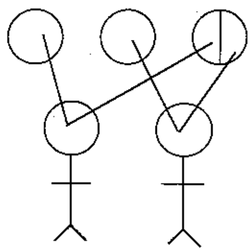
We now turn to using our model to analyze specific, actual student behaviours. That is, while we have used transcendent recursion to typify the complex-levelled, non-linear phenomena of understanding, these tenets also allow for a more informed study of such phenomena at the micro level. Consider another example drawn from the study of pupils working in groups on fraction-comparison tasks. In this task the children were asked to compare the amount person "A" would get if 3 pizzas were shared between 2 people with the amount person "B" would get if 9 pizzas were being shared between 6 people. Here is a commentary by Robert working with 2 friends.

Robert: (I) (with nods of agreement from the others)
Oh, A is easy. A gets 3 pieces. (He draws lines dividing each pizza in half, then draws lines connecting one piece from each pizza to each of the 2 people.)

Dan: (II) B is hard though.
(All three attempt to use the technique of cutting the 9 pizzas in half and then distributing these pizzas pieces, but the diagrams become hopelessly confused (see below) and all but Robert give up.)



Robert: Could I try another one?
Robert: (III) One to each
Robert: Now half to each
One and a half [pieces] each.
(Now Robert returns to the 3 for 2 example and does the following.)



Robert: (IV) One and a half each too!

All three children (at (I) above) realize that the first answer can be obtained by sharing between two persons and simply use their "forms" of halving. It would appear that they have an image for halving as an object and can use it. They are then (at (II)) faced with a dividing up situation which they cannot immediately comprehend. Like Hanne, earlier, who obtains thirds by "Y-ing," they simply call their halving action into play again — this time unsuccessfully. At (III) Robert drops back to an inner-action level of "one each" to try to cope with 9 shared among 6 and is then able to use halving to reach a solution. At (IV) he quickly applies this same pattern of action to the "3 for 2" case.

The non-linearity of effective action is visible in this example. The children seem to have an image of halving, but using this proves ineffective for them in the 9-for-6 situation, so Robert drops back to the level of image-making (sharing wholes, sharing halves). The fact that this return to the inner level is not the same inner-level activity as earlier, is seen by the way that Robert then uses his *patterns* of action in the 3-for-2 situation. He clearly has not simply gotten an "answer" for 9 from 6; he appears to have realized that he has a possible pattern for other dividing-up situations. (It is interesting to note, however, that Robert has not yet made connections among quotient situations or moved to the level of self-conscious thinking which would allow him to identify "3 pieces" with "one and a half.")

More generally one might say that the children's halving language blinded them to the dividing-up action from which it came. Robert, however, removed these blinders: for him the initiating conditions of sharing were still operative and led to a new coordination of actions within the environment of his knowledge of the existence of halving as a pattern. It was probably this knowledge of pattern which enabled him to call forth the idea that there could be a solution pattern here too, not simply a quantitative solution. Certainly we can observe the complex, levelled, but non-linear sequence of actions, as Robert tries to see his knowledge structures as consistent, as he tries to understand dividing up.

Summary and concluding remarks

The demonstration of the extended quadratic example and the analysis of two transcripts of children's mathematical behaviour have been our attempt to observe the nature of recursive levels and the inter-relationships among them We have seen that image-making and image-having can function as initiating conditions for effective action: children use inner-level knowledge to build more complex knowing. However, the actual process of knowing and understanding does not move in a linear fashion through these levels, and when one folds back to a previous level of effective action, this is not merely doing the previous action again. The outer-level knowing acts as an environmental constraint within which forms or processes at the inner levels are "called out," and these are then directly constrained by the outer level concerns.

The viewing of mathematical understanding as a dynamic process allows us to see a person's current state as containing other levels which are different, but compatible, ways of understanding the mathematics, and which allow the person to validate outer-level knowledge or which provide a basis for facing unknown but related mathematics

Considering mathematical understanding as a recursive phenomenon is not meant to replace the contemporary views of understanding suggested by Ohlsson or by Herscovics and Bergeron It is meant to provide insight into how such understanding grows and how the elements these authors describe can be integrated into a whole. As such, the theory sketched above should allow for the dynamic, levelled analysis of mathematical understanding In particular, it should allow one to see the self-similarity but transcendence in the levels, to see the process of validation of personal knowledge and to comprehend transfer as recursive reconstruction. It enables one to identify the roles of language and thought both at any level and in the growth between levels.

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