More than Formal Proof

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The hallmark of the mathematics curriculum adopted in the sixties was an emphasis on formal proof. Among the manifestations of this emphasis were an axiomatic presentation of elementary algebra and increased classroom attention to the precise formulation of mathematical notions and to the structure of a deductive system. The curriculum change was so profound that it was dubbed a "revolution".

In this paper I discuss the origins of this emphasis on formal proof and consider its limitations as a focus for the secondary-school mathematics curriculum, in light of these aspects of mathematical practice which complement and go beyond formal proof.

Origins of the emphasis

The curriculum revolution of the sixties was predicated upon a number of beliefs, one of which was that formal proof is the most important characteristic of modern mathematics This view was no doubt due in large part to the impressive work done during the first half of the century in clarifying the very foundations of mathematics, work which had demonstrated the enormous power of formal systems constructed step by step from a base of definitions, axioms and rules of inference

The brilliant mathematicians who were so united in a desire to lay a firm foundation for mathematics were by no means united in their approach to this task But though the schools of thought which came to be known as logicism, formalism and intuitionism differed greatly in their philosophical accounts of mathematics and even in their criteria for the validity of a proof, they did share an emphasis on the importance of formal proof, and it is this emphasis, rather than the differences among the schools, that has so greatly influenced the mathematics curriculum.

The central assertion of logicism is that mathematics is part of logic. Accordingly, the aim of the logicists was to produce the corpus of mathematics without introducing concepts indefinable in logical terms of theorems which cannot be proved from the primitive sentences of a logical calculus using its tightly-defined rules of proof. Thus formal proof played a central role in the logicist agenda

This was true of the formalist effort as well In fact, the thesis of the formalist school was precisely that mathematics is a science of formal systems: that it deals with the manipulation of strings of symbols to which no meaning need be assigned. In the formalist view, the validity of any mathematical proposition rests upon the ability to demonstrate its truth through rigorous proof within an appropriate formal system.

The intuitionists, too, assigned importance to formal proof Their differences from the logicist and formalist schools centred upon the types of proof which should be admitted as valid, with the intuitionists taking a mor restrictive view Intuitionism is the belief that mathemat ics and mathematical language are two separate entities mathematics being essentially a languageless activity o the mind. Mathematical activity then consists of "intro spective constructions", rather than axioms and theo rems. But for the intuitionist the assertion of a mathemat ical proposition was equivalent to the assertion that ther is a construction of a finite nature which produces th proposition—and such a construction had to obey rule of rigour

More recent views of mathematics

In the last two decades several mathematicians an mathematics educators have challenged the tenet that th most significant aspect of mathematics is reasoning b deduction, culminating in formal proofs. In their view there is much more to mathematics than formal systems This view recognizes the realities of mathematical prac tice. Mathematicians admit that their proofs can hav different degrees of formal validity—and still gain th same degree of acceptance. Mathematicians agree, fur thermore, that when a proof is valid by virtue of its forn only, without regard to its content, it is likely to add ver little to an understanding of its subject and ironically ma not even be very convincing.

In these more recent views, a proof is an argumen needed to validate a statement, an argument that ma assume several different forms as long as it is convincing Proof has been described as a "debating forum" [Davis 1986], as having "a certain openness and flexibility" [Tymoczko, 1986], and as possibly depending for it validity on "correct or reasonable social practice" [Kitch er, 1984]

In examining how proof in mathematics takes a socia process into account and hence goes beyond the concep of formal proof often reflected in the mathematics curric ulum. I will discuss ideas advanced by Lakatos [1976].

Lakatos has expressed the point of view that mathematics is fallible by its very nature His account o mathematics is thus at odds with both logicism and formalism, undoubtedly influenced by their failures Though mathematics is not an empirical science, Lakatos shows that its methods are very similar to those of the empirica sciences; he refers to mathematics as quasi-empirical. Mathematics, in fact, grows through an incessant "improvement of guesses by speculation and criticism, by the logic of proof and refutation" [Lakatos, 1976] Thus no proof is final, and indeed it is the essentially social process of negotiation of meaning, rather than the application of formal criteria from the outset, which leads to the improvement of a proof and its growing acceptance

For the Learning of Mathematics 9, 1 (February, 1989)
FLM Publishing Association, Montreal, Quebec, Canada

According to Kitcher [1984], to understand the development of mathematical knowledge one must focus on the development of mathematical practice: mathematical knowledge owes its growth to rational modifications to this practice. Mathematical practice has five components: (1) a language, (2) a set of accepted statements, (3) a set of accepted questions, (4) a set of accepted reasonings, and (5) a set of metamathematical views. The latter component includes standards for proof and definition, as well as claims about the scope and structure of mathematics [p. 163]. Thus in his view it is not only the corpus of mathematical results which develops, but also the very ways in which mathematics is done.

Further, Kitcher does not accept the apriorist view that mathematical knowledge is based on proof. He attacks the conception that "proposes to characterize the types that count as proofs in *structural* terms" [p. 36]. It is furthermore historically incorrect to assume that change in mathematics has consisted only of the discovery of earlier mistakes and their replacement by new, correct demonstrations. In his view mathematical knowledge is always sensitive to peer challenges and is sustained, in part, by community approval of assumptions and techniques.

Citing examples from the work of Euler, Cauchy, Weierstrass and Newton, Kitcher concludes that mathematical proof is not always necessary to mathematical knowledge, and that it may not even be rational to attempt to accumulate a series of certainties; a demand for rigour may even be a hindrance to the growth of mathematics, because it impedes problem solving. Indeed, within the set of accepted reasonings (mentioned above as a component of mathematical practice), the most interesting are those which occupy an intermediate position in this set: the unrigorous ones.

Tymoczko is a philosopher of mathematics who thinks that what mathematicians actually do has a bearing on philosophical questions about mathematical knowledge. According to him, the concept of the "ideal" mathematician, the totally rational agent who needs only follow formal deductive procedures to generate eternal and infallible knowledge, is not one which is helpful in the philosophy of mathematics [Tymoczko, 1986].

Tymoczko's account of mathematical knowledge centres upon the community. It views not only mathematics teaching, but also the concept of proof and the practice of proving theorems, as *public* activities. Regarding the concept of proof, he agrees with Lakatos that "proof ideas" are subject to criticism and even invite it. In his view

> Mathematical proofs . . . generally have a certain openness and flexibility. They can be paraphrased, restated and filled out in various ways, and to this extent they transcend any particular formal system. We might say that an informal proof determines an open-ended class, or family, to use Wittgenstein's term, of more specific proofs [p. 49]

Tymoczko goes on to say that informal proofs are often convincing and can lead to new discoveries. They are codified in terms of simple proof ideas and become the property of a network of mathematicians.

Davis [1986] states that a proof can play several different roles. It can serve as a validation, it can lead to new discoveries, it can be a focus for debate, and it can help eliminate errors. According to him, as to Tymoczko, the traditional philosophies of logicism, formalism and intuitionism are "private theories" that describe an ideal mathematics. But mathematics, being a social activity, requires a public theory.

In the real world of mathematicians, Davis believes, a proof is never complete and furthermore cannot be completed. Routine calculations will invariably be omitted. There will always be an appeal to intuition, to pictures. There will be some metamathematical objections, but such a part-proof will nevertheless be convincing because it is addressed to people who share a mathematical subculture in which an incomplete argument is understood, appreciated and seen as adequate. A typical college lecture in advanced mathematics will include formulations such as "it is easy to show", "you can verify that", "by an elementary computation which I leave to you", and so forth. It is considered perfectly proper to transmit mathematics in this elliptical way.

Davis is quite explicit in his view of formal proof:

> There is a view of proof or a view of mathematics which I disagree with and which I think is a myth, which says that mathematics is potentially, totally formalizable and, therefore, one can say, in advance, what a proof is, how it should work, etc. [p. 336]

Factors in acceptance

Clearly the acceptance of a theorem by practising mathematicians is a social process which is more a function of understanding and significance than of rigorous proof. Indeed, the presence of any proof, rigorous or otherwise, is only one of several determining elements in acceptance. This process is by no means capricious: the community judges by certain criteria, as I will discuss. But the significance of a theorem for mathematics as a whole, and an understanding of its underlying concepts, play a much greater role in creating this acceptance than does the existence of a rigorous proof.

The development of mathematics and the comments of practising mathematicians suggest that most mathematicians accept a new theorem where some combination of the following factors is present:

1. They understand the theorem, the concepts embodied in it, its logical antecedents, and its implications; and there is nothing to suggest it is not true;
2. The theorem is significant enough to have implications in one or more branches of mathematics (and is thus important and useful enough to warrant detailed study and analysis);
3. The theorem is consistent with the body of accepted mathematical results;
4. The author has an unimpeachable reputation as an expert in the subject matter of the theorem;
5. There is a convincing mathematical argument

for it (rigorous or otherwise), of a type they have encountered before.

If there is a rank order of criteria for admissibility, then these five criteria all rank higher than rigorous proof.

Perhaps the situation is best discussed in terms borrowed from Maslow's theory of social motivation [Maslow, 1970] Understanding, significance, compatibility, reputation, and convincing argument are "positive motivators" to acceptance: it is these factors which focus the attention of practising mathematicians on a new theorem and move them to its active acceptance, lifting it above the great body of equally valid but less attractive theorems which confront them in the mathematical literature.

On the other hand, the structural validity of the mathematical argument for a new theorem, that is, the actual or potential validity of its form as distinct from its content, is merely a "hygiene factor," a factor recognized as essential but taken for granted. There is presumption that any convincing proof appearing in a reputable journal is in fact valid in terms of its form, or could be made so without violence to its content. The publication of a rigorous proof would provide no additional positive motivation for active acceptance, and in fact such a proof would not be examined at all in the absence of the motivating factors enumerated above

The social process

The following will briefly examine the social process of acceptance and the central role played in that process by the factors of understanding, significance, compatibility, reputation, and convincing argument suggested above.

The Russian logician Yu. Manin is among those who have stressed that the acceptance of a proof depends much more on a social process than on some ideal objective criterion:

> A proof becomes a proof after the social act of "accepting it as a proof." This is true of mathematics as it is of physics, linguistics, and biology [Manin, 1977, p. 48]

Manin then goes on to explain that a new proof needs to be accepted and approved by other mathematicians —who often decide to refine and improve it. The scrutiny to which mathematicians subject a proof, he points out, is aimed more at weighing the plausibility of the results than at verifying the deductive process. It is only when they are skeptical of a result that mathematicians will put any great effort into discovering counter-examples Manin cites this as the reason why the truth of a theorem in the eyes of the mathematical community becomes established indirectly, that is, not because the proof has been verified as error-free, but because the results are compatible with other accepted results and the arguments used in the proof are similar to ones used in other proofs.

Of the estimated 200,000 theorems published yearly [Ulam, 1976], only a very few are actively accepted by the mathematical community. It is the theorems judged significant that have their proofs scrutinized, corrected, and refined, while the proofs of other theorems go unexamined. Clearly an alleged proof of Fermat's last theorem or of the four-color theorem, when submitted by reputable mathematicians, would attract meticulous review, while the proof of a theorem of no apparent consequence is likely to be ignored, no matter how original or sophisticated the proof might be in its own right.

Indeed, as Davis [1972] notes, most proofs in research papers are never checked. Many of them are rife with errors, in fact. This is borne out by the many mistakes found in those published proofs which have been checked, and is also supported by the contention of a former editor of *Mathematical Reviews* that as many as half the proofs published are false, though the theorems they purport to prove are true. When an error is detected in the proof of a significant theorem, it is often the proof that is changed, of course, while the theorem itself stands unquestioned

The role of proof in the process of acceptance is similar to its role in discovery. Mathematical ideas are discovered through an act of creation in which formal logic is not directly involved They are not derived or deduced, but developed by a process in which their significance for the existing body of mathematics and their potential for future yield are recognized by informal intuition While a proof is considered a prerequisite for the publication of a theorem, it need be neither rigorous nor complete Indeed the surveyability of a proof, the holistic conveyance of its ideas in a way that makes them intelligible and convincing, is of much more importance than its formal adequacy [Hanna, 1983]. Since fully adequate, step-by-step proof i in most cases impracticable, and since surveyability is los when proofs become too long, proofs are conventionall elliptical and brief.

The conclusion therefore is that an orientation toward extreme formalism in proof is not reflective of curren mathematical practice or current philosophies of mathematics. There are, as has been shown above, good reason for this. As Tymoczko has put it, "Mathematicians, eve ideal mathematicians, are able to do mathematics and t know mathematics only by participating in a mathema ical community."

The impact of formal proof on the curriculum

Despite the secondary nature of proof, it is easy to se how misunderstandings about the nature of mathematic arise. Mathematical results published for a mathematic audience are invariably presented in the form of theoren and proofs They retain this form, reflecting as it does tl nature of mathematics as a highly structured body knowledge held together by the concept of logical prec dence, even though the proofs are not judged by criteria completeness or rigour To a person only partially train in mathematics, to someone who is neither fully equipp to assess significance nor able to make the intuitive jud ments necessary in successfully surveying a proof, it mig easily appear that the manner of presentation—with possible implication that full rigor is the ideal form— the core of mathematical practice Thus competence mathematics might readily be misperceived as synon mous with the ability to create the form, a rigorous pro

It is only one step further, then, to assume that learni mathematics must involve training in the ability to create this form. To teach a beginning student is assumed to involved teaching the formalities of proof Paradoxically, such an emphasis omits the crucial element. When a mathematician reads a proof, it is not the deductive scheme that commands most attention It is, in fact, the mathematical ideas, whose relationships are illuminated by the proof in a new way, which appeal for understanding, and it is the intuitive bridging of the gaps in logic that forms the essential component of that understanding. When a mathematician evaluates an idea, it is significance that is sought, the purpose of the idea and its implications, not the formal adequacy of the logic in which it is couched

It would therefore appear that what needs to be conveyed to students is the importance of careful reasoning and of building arguments that can be scrutinized and revised While these skills may involve a degree of formalization, the emphasis must be clearly placed on the clarity of the ideas.

Teaching

That reasoning is a pedagogical issue at all bespeaks a conviction that the learning of mathematics is a dynamic rather than static process, in which students progress towards deeper levels of insight and skill. Thus a classroom activity that includes formal or informal reasoning can be judged to be of value only to the degree that is promotes greater understanding.

The starting point for understanding is the naive mathematical idea rooted in everyday experience. To provide a basis for further progress, this naive idea must be developed and made explicit. This requires a degree of formalism A language must be created: symbols defined, rules of manipulation specified, the scope of mathematical operations delineated Greater precision must be taught, so that the essential can be separated from the nonessential and greater generality achieved.

But this has its price Distanced from the original intuitive context, the student may lose sight of reality and become a symbol pusher. Experienced mathematicians have learned to handle this danger by acquiring the ability to make mental shifts in moving among levels of generality and formalism, and by building on specific examples, drawing only upon those characteristics pertinent to the more general situation under study They are able to exploit symbolism and algorithms to work automatically and efficiently, while retaining the ability to intervene in their own work to monitor its accuracy and effectiveness.

What are the issues to be kept in mind in teaching mathematics, then, and in particular in developing the power of reasoning?

1. Formalism should not be seen as a side issue, but as an important tool for clarification, validation and understanding. When a need for justification is felt, and when this need can be met with an appropriate degree of rigour, learning will be greatly enhanced.
2. It is not enough to provide mathematical experiences. It is the reflection on one's experiences which leads to growth. As long as students see mathematics as a black box for the instantaneous production of "answers", they will not develop the patience necessary to cope with the many and erratic paths their minds will take in trying to grasp what mathematics is about One goal of pedagogy should be to help pupils maintain the level of concentration required to negotiate a line of reasoning
3. Ironically for a discipline touted as precise, the student of mathematics has to develop a tolerance for ambiguity Pedantry can be the enemy of insight. Sometimes an explanation is better given pictorially, loosely, by example or by analogy Sometimes distinctions are better left blurred (e.g., the various roles of the minus sign and the use of "$f(x)$" as both the function and the value of the function at x). Sometimes the role of a symbol in the discussion should be allowed to vary (e.g., the parameter which is sometimes held constant, sometimes allowed to vary)
4. At the same time, when there is a danger that genuine confusion might develop, the student must learn to become conscious of looseness and to apply the necessary amount of rigour. It is this judgmental aspect of reasoning, so essential in mathematics education, that must be communicated to students.

Acknowledgment

Preparation of this article was supported in part by a grant from the Deutscher Akademischer Austauschdienst

The ideas in the last section, and their formulation, owe much to discussions with E J Barbeau of the University of Toronto

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