TALKING ABOUT ORDER OF OPERATIONS

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I want to share my own experience of witnessing the power of meaningful learning and the manifestation of higher-order thinking among elementary school children, which are not commonly shown. Order of operations became the discussion issue for these grade 2 (8- and 9-year-old students) even though it was not the planned outcome of my instructional plan. A typical teaching method for the order of operations is to provide students with a set of rules to memorize and apply. Certain mnemonic devices are frequently referred to in textbooks to reinforce the rules, such as "Please Excuse My Dear Aunt Sally," which stands for Parentheses, Exponents, Multiplication, Division, Addition, and Subtraction

However, these mnemonic devices alone do not address the mathematical reasoning behind the performance, and frequently lead students to misconceptions. These early misconceptions often continue to impede the learning of more complex arithmetic computations and algebraic calculations in the upper grade levels. My experience with a group of primary grade students, who have completed three years of an implementation of Davydov's curriculum [1] at an American private school, shows how they analyze the mathematical structure inherent in performing mathematical operations instead of memorizing steps. Davydov's curriculum is laid out as a series of questions. An average of two to five questions was discussed in each daily class session. In the class discussion, the children present their solution strategies to questions. If someone disagrees with the solution presented and was able to articulate why, the presenter provided further justification and explanation of their rational. This process continues until all the children agree on the result. At times, there were occasions when all the children agreed with an incorrect explanation. There were also instances where the group could not reach an agreement.

This discussion started in a spontaneous and unexpected way.

Day 1: After checking the previous day's homework, I presented J's solution to the whole class for discussion without mentioning J's name:

$T - 4 - 3 = 75$ [This information was given.]

$T - 7 =$

$T - 8 =$

In her homework, J found the answer 75 for "T - 7" based on the given information. However, she put a question mark for "T - 8," meaning that she could not get the answer because the given information was insufficient. When I presented J's solution, all children, including J, said that they could find the answer. Several children presented their solutions:

A: 74 [She did not elaborate her strategy.]

B: 74 - take away 7 from T is the same as take away 4 and 3 from T because 4 and 3 is 7. Take away 8 from T is 1 less than take away 7 from T.

C: 74 [Student C calculated the value of T]:

$T - 4 - 3 = 75$

$T = 75 + 4 + 3$

$T = 82$

$\therefore T - 4 - 4 = 82 - 4 - 4 = 74$

All the children agreed with these different strategies. The confusion seemed to be resolved. Suddenly, however, A said, "I disagreed with myself" and proposed a completely new solution. A's solution was:

$$\underbrace{T -}_{T} \underbrace{4 - 4}_{0} = T$$

A: It is an easier way to solve this question because "4 - 4" is zero and something minus zero is something.

Immediately, all the children disagreed with A. A was uncomfortable with this situation and the children were distracted from the original topic of discussion. I redirected the discussion by proposing new tasks, using paper strips to show the difference (see Figure 1)

The children were inattentive to this activity. The emotional tension between A and the other children was preventing productive discussion. At this moment, B said, "We cannot take away a part from the other part." However, this comment could not be elaborated upon due to noise and distraction. I also reminded A of a previous session's activity, in which three children had the same number of chips in envelopes and I asked them to give me the same number of chips in different ways. At that time, the final result was:

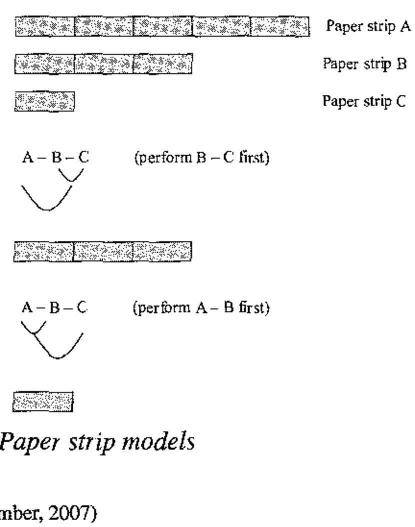

Figure 1: Paper strip models

$x - 2 - 5 = x - 7 = x - 1 - 3 - 3$

I added A's method to the above formula:

$x - 2 - 5 = x - 7 = x - 1 - 3 - 3 = x - 1$

I asked A why "$x - 7 = x - 1$" is true A did not explain it and simply gave up this method. Next, I gave 14 counters to A and asked, "Can you explain '14 - 4 - 4' using counters?" A said, "4 - 4 = 0. So, the answer is 14." [A picked up 4 counters and put them down on the table again.] The other children disagreed with A again by saying "14 - 4 - 4 = 6." This mathematics session ended in an unresolved, chaotic state.

Day 2: We revisited the previous day's discussion. I wrote down the formula that A produced: 14 - 4 - 4 = 14.

I: Have you thought about your method after class?

A: I did, but my answer is the same as yesterday.

At this point, I reminded them of B's comment from the previous day's discussion, to which the children had not paid attention, "We cannot take away a part from a part." I asked A to mark the parts and the whole in her formula. A's first answer was:

14 – 4 – 4 = 14
^   ^

(note: '^' is a symbol for the whole and '_' is the symbol for the part)

A seemed to look at the numbers first before marking them. In other words, A knew that the first 14 was the whole by looking at the structure of the formula. For the second 14, however, the whole sign was marked because the number was the same as the first 14. I gave A another formula having letters and asked the same question:

P - A - B = C

This time, A marked the part and whole signs correctly and also drew a schematic for this formula (Figure 2).

P – A – B = C
^
(whole) (part) (part) (part)

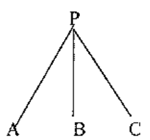

Figure 2: Analyzing the part-whole relationship for a formula having letters

Then, everyone agreed with A. I said, "I want to use A's model." I emphasized that this model was made by A, "If I follow A's model, I can make the parts and the whole in this way for 14 - 4 - 4 = 14" (see Figure 3).

A was not happy with this result but agreed that the solution fit with the model. It was certain that A took away one part from the other part. Meanwhile, the children suggested that we should put x for the unknown answer and drew the schematic again (Figure 4).

I asked them how we could find one part. J said that we should take away the sum of the two parts from the whole (14 - 8). All the children, including A, agreed with J. A withdrew his explanation for "T - 4 - 4" and realized that we should not calculate '4 - 4' first.

14 – 4 – 4 = 14
^
(whole) (part) (part) (part)

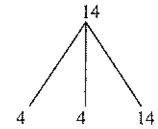

Figure 3: Analyzing the part-whole relationship for a formula having numbers.

14 – 4 – 4 = x
^
(whole) (part) (part) (part)

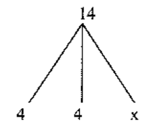

Figure 4: A revised schematic for '14 – 4 – 4'.

Although some of the children had made similar mistakes in their work several times, overall, this type of error was apparently eliminated after this long discussion. In addition, the impact of this discussion was identified when children performed mixed operations in the following year. For example, 10 - 2 × 3 could have been explained by the rule "Dear My Aunt Sally." However, this group of children analyzed the structure of the question based on the part-whole relationship: "10 is the whole and one part is 2 × 3, and the missing part can be found by taking away a part from the whole."

Notes

[1] I taught a Russian elementary mathematics curriculum, developed by Davydov and his colleagues in the former Soviet Union (Davydov, Gorbov, Mikulina and Saveleva, 1999, 2000; and with Tabachnikova, 2001). The most distinctive difference between Davydov's curriculum and other conventional mathematics curricula lies in

> the students' process of tracing the conditions and laws for the origin of the concepts. (Davydov, 1990, p. 348)

In other words, Davydov's curriculum emphasizes theoretical rather than empirical learning.

You can read more about the children and about the curriculum in my unpublished doctoral dissertation (2002), *An analysis of difficulties encountered in teaching Davydov's mathematics curriculum to students in a U.S. setting and measures found to be effective in addressing them*. However, a few background ideas might help if you are unfamiliar with these ideas:

- the concept of quantity is the basis for developing the systems of relationships
- from the beginning, working with literal expressions to examine particular relationships among objects and to abstract their properties, children acquire a general concept of an underlying quantitative relationship and the ability to analyze the properties of mathematical relationships – algebraic expressions are introduced prior to numbers

[The rest of the notes and references can be found on page 37 (ed.)]