SING MATHEMATICS TOGETHER: THOUGHTS ON THE FUTURE OF A SCHOOL SUBJECT

ISTVÁN LÉNÁRT [1]

Can mathematics survive as a school subject for *all*? Is mathematics a subject for *general education*? Has mathematics anything to say for anybody else but a mathematician or an immediate user of mathematics by profession? Or, will mathematics become a subject for the selected few who are to lead society by this mighty tool of human thought?

My wife and I were at a concert recently in Budapest. Before the interval the announcer asked the audience to sing together. Four hundred people, from small children to grandparents, happily sang the Hymn of the European Union, Hungarian folksongs and other pieces. Looking around whilst singing, I was suddenly struck by a thought, *Why can't this happen in mathematics lessons? Why can't we happily work and think and solve and play and laugh together in mathematics lessons? Why have I always had the feeling of being in a pack of [mildly put] rivals among my mathematics peers, from elementary to the highest school?*

Opinions

Ask twenty people at random, "What do you think of mathematics ?" (see Figure 1). Two out of the twenty will say, "I love it"; three, "I like it"; ten, "It's wonderful, but I am at a total loss with it"; three, "I hate it" and the remaining two will leave you without a word. This is surely a clear sign of big trouble? In any business you would take it as a bad omen if you proved that seventy-five percent of your customers were neutral to or hostile towards your product!

It is thought that people do not like to think. Give them food and sex and fairy-tales and nothing else. Look at the TV polls! The worse the program, the more the viewers.

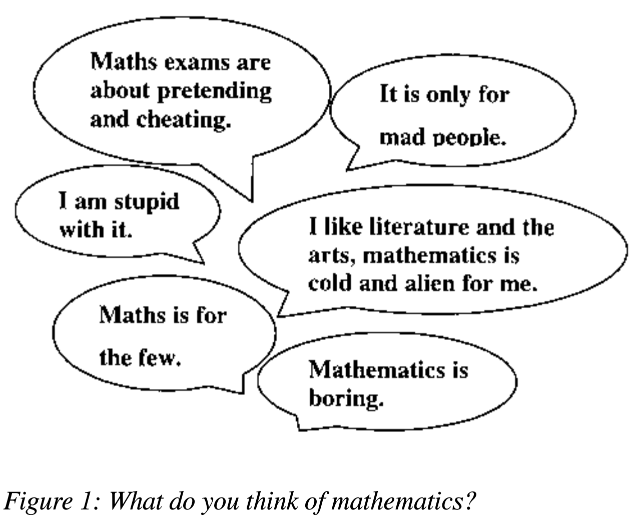


Figure 1: What do you think of mathematics?

People do like to think

Although many people feel bad when faced with mathematics, from experiences in my life I know that they like to think.

- An old lady on the tram works hard on a crossword puzzle.
- Two homeless men play chess on the pavement.
- Members of a family try to solve a crime story.

Why do they rack their brains? Nobody compels them to do so! Children sit in a boring mathematics lesson and secretly play strategy games or draw funny geometric figures, circles and diamonds on grid paper. This is sheer madness: they use their mathematics, their logic and their imaginations in secrecy – during a lesson for mathematics, logic and imagination!

Bertolt Brecht said, "Thinking is the greatest joy of the human race". "Greatest" might be a poetic exaggeration but the essence of this quotation is true for me.

I come of a middle-class family. As a child, I learned middle-class language, and was prepared to spend my life among middle-class friends and ideas. The first impetus to change came from literature, which I took only too seriously; but the final push came from mathematics. I began to think about new systems in arithmetic and geometry, and could not confine myself to the limits of science. If you try to accept non-Euclidean geometry, then you will find it very hard to miscall simple Truth by the word simplicity in everyday life!

In this way, mathematics turned me from my predetermined career, and forced me into difficult, sometimes cruel situations. But, at fifty-six, I am still fond of mathematics; I have retained an amount of self-esteem; and, above all, I still like people around me. That is the clue to everything.

Each idea has its time to come into the world

For many years, I lived among worries – about my theories, my patents, my place under the sun. I remember a remark made some twenty years ago: "This guy has discovered the sphere that had already been discovered by the caveman". It took me decades to understand that those who have already worked on this topic are not my enemies, but my best friends; and our different courses of life can well serve the same purpose.

Three of us whom I know about: David Henderson, a university professor at Cornell, USA; Jan van den Brink, a researcher at the Freudenthal Institute, The Netherlands; and I, a Hungarian mathematics educator teaching at ELTE University, Budapest. Three persons, independently of each

For the Learning of Mathematics **24**, 2 (July, 2004)
FLM Publishing Association, Kingston, Ontario, Canada

other, have ideas that are so similar in many respects it is as if we were consulting each other every day. We look at spherical geometry as a treasure island compared with the traditional plane. We discovered (Gauss had already discovered these ideas two hundred years ago, and this discovery made him suppress all his thoughts on non-Euclidean geometries) that accepting different systems in geometry can lead to changes in thinking about other areas of life, from geography to art, literature to social sciences, and, most importantly, in everyday human communication.

David Henderson (1995) writes in a paper, "I learn mathematics from people who are different from me"; Jan van den Brink works on a geometric-geographic project that connects Islamic culture and science with Western traditions; and I have always looked at my comparative geometry as a tool for changing the world. My educational texts will remain worthy of trust as long as geometry means more for me than mere science [2].

People do like to think: an exhibition

In 2002, a big exhibition called *Dreamers of dreams – world famous Hungarians* was held in my hometown Budapest. There was a lot of text to read, a lot of models to think about and it was not at all an easy ride. Six hundred thousand people came to the exhibition in one year from a city of two million people. I designed the János Bolyai room that displayed planar, spherical and hyperbolic geometry – hyperbolic geometry! – for all who came there was no entrance examination and no age limit – toddlers to centenarians. It was no big deal that some five thousand people attended my geometry talks during the year – the big deal was that they did not go away. They stayed until the end of the talk and were happy and grateful afterwards.

Extracts from the booklet produced for the exhibition [3]

At the beginning of the nineteenth century the German Carl Friedrich Gauss, the Hungarian János Bolyai, and the Russian Nikolai Ivanovitch Lobachevsky, independently of each other, conceived the idea of a reasonable geometry different from that of Euclid's.

We call this new geometry *hyperbolic geometry*. To bridge the gap between Euclidean plane geometry and hyperbolic geometry, we choose an intermediate system, the geometry of the spherical surface. The sphere is familiar to all of us in the shape of a Ping-Pong ball, a watermelon, marbles, soap bubbles, full moon or the earthglobe.

Concepts of geometry are easier to understand when studied in different systems, compared and contrasted with one another.

The simplest line in the plane is the straight line. All the straight lines in the plane that pass through a point are called a pencil of intersecting lines. All the straight lines in the plane that are parallel to each other are called a pencil of parallel lines. On the sphere, the simplest line is the great circle, the cutting line of a perfectly spherical watermelon when it is halved (see Figure 2). All the great circles that pass through a point of the spherical surface are called a spherical pencil of great circles.

Figure 2: On the sphere, the simplest line is the great circle.

Our model of hyperbolic geometry is the transparent hemisphere. The points of the equator are not included in our geometry: they represent a forbidden zone! Which are the simplest lines? The vertical hemicircles that can be visualized as the cutting lines of a halved onion when it is sliced on a breadboard (see figures 6, 7 and 8).

In the plane, an angle of a triangle is always sixty degrees, whether the triangle is big or small (see Figure 3). On the sphere the angles keep growing as the triangles themselves grow (see Figure 4). On the hemisphere, the angles keep diminishing as the triangles themselves grow (see Figure 5). What is the angle sum of the triangles?

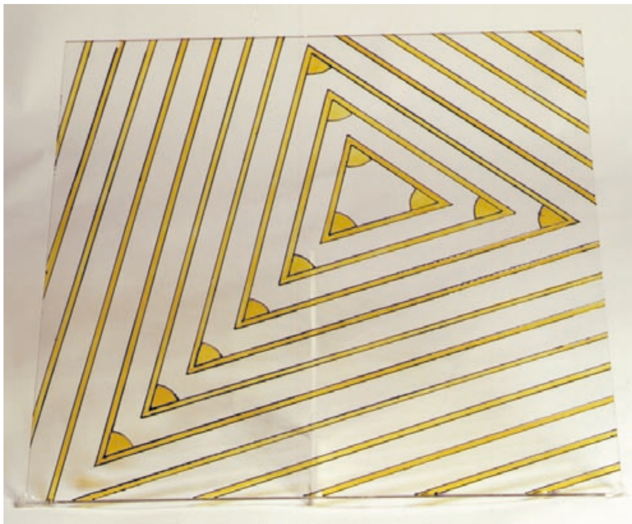
What informed my planning?

In what follows I will try to give the core of the method that I use as a concrete model of my approach to education, and list the main objectives. I will illustrate these ideas using the Plane-Sphere project that has grown into the Plane-Sphere-Hemisphere project in the last few years.

Routes to mathematics: Euclid said, "There is no royal route to geometry". I agree, and also add something, *there is no slavish route, either*. In fear and trembling no one can reach to geometry or mathematics or any science. There is but one route to science: *the route of equal ranks.*

If I read a book by Newton or Gauss or Bolyai or any other scientist, I read it as an equal partner of the author. This is not because I think I am as great as the author, but because the author expects me to be an equal partner. I am not a slave, not a mere crammer of their texts, but a friend to discuss matters of common interest. If you read this article or my book (Lénárt, 1996) and say to yourself, "How smart this guy is", I have failed in my intention. If you read my work and say, "How smart I am", then I have hit my target.

One-system mathematics and infallible mathematics: Two kinds of mathematics are dead and unteachable:



Figures 3, 4 and 5: Sum of the angles in a triangle in planar, spherical and hyperbolic geometries.

One-system mathematics means that you teach only one fixed system, one way of thinking in a given branch of mathematics, such as arithmetic, algebra, number theory or geometry. This method contrasts sharply with everyday experience in a modern society. If you are free to choose among different options in your job, government, private life or goods in the department store, then you are likely to expect freedom of choice in mathematics as well.

Mathematical theories are made by human minds and no human theory can be onmipotent and all-answering. I accept a concept if, and only if, the counter- examples mark out its validity limits. It is just the contrast, the comparison of different concepts that gives me the clue.

Infallible mathematics means that the teacher aspires to be the infallible mouth of the great mathematicians of the past. This role is inhuman and unacceptable. Those great people have brought forth their treasures out of many mistakes and failures. And for us teachers – we are very far from being infallible in our educational, professional or private life. Why pretend infallibility, if our task is to urge students to go beyond past knowledge? How can we expect them to be engaged by a supercilious science full of self-conceit?

Plane-Sphere-Hemisphere project: From elementary level upwards, I teach plane geometry and spherical geometry together, continuously compared and contrasted with each other. I always start from manipulation with the help of hands-on construction tools, supported and complemented by software materials whenever it seems necessary. Later on, I expand to the hemispherical model of hyperbolic geometry, in order to show the third possibility beyond traditional planar and spherical geometries (see Figures 2-5 showing the Lénárt sphere).

Children like working with the sphere. They are eager to scribble and draw on the plane and sphere. They play fortune-telling, go-around-the-world and draw-the-teacher-with-glasses-on-his-nose (Lénárt, 1991). There is a spherical ruler that they initially put on their heads with the torus for the sphere to sit securely on around their wrists. And suddenly they draw a triangle on the sphere and say: “It has two right angles!”

Yes! It is the counterexample that explains the theory. If any triangle has its sum of angles adding up to 180˚, then it seems senseless to talk about this ’obvious’ result. However, if our students meet a shape which can be treated as a triangle, but its sum of angles clearly differs from 180˚, then the whole question becomes interesting and thought-provoking. A monologue of one system changes into a dialogue between two or three different worlds of geometry.

Geometries: At the end of the nineteenth century, my great-grandpa spent his apprentice years wandering in Hungary, Austria, Germany and Italy. When he came home, he kept saying to his family: “I wandered all over the world”. His whole world could very neatly be depicted on one page of the school atlas, and understood with the help of Euclidean plane geometry. However, our life in the twenty-first century is closely connected in many ways with the entire globe of the Earth. We are earthlings, and we need spherical geometry to describe our planet.

Hyperbolic geometry has played a central role in the creation of the world concept of modern physics. It is becoming more and more important in other sciences as well. On the sphere the basic ideas of hyperbolic geometry can be shown. Very importantly, students avoid developing an inferiority complex towards a subject that is heard of by many, but understood by few.

Axiomatization: Various branches of mathematics and a number of natural, social and economical sciences are built on different systems of axioms. The *New Mathematics* educational movement aimed at teaching the axiomatic way of thinking. It never really took off. I believe that axiomatization cannot be taught – even at university level – by simply listing our axioms and theorems. Axiomatization, as Freudenthal clearly saw in the early seventies, can only be grasped by individual action. Students should work on their own, set up local axioms for themselves, and improve them step by step as their experience allows them to do.

In this regard, the central task for us teachers is to raise interest for this process. It is exactly this interest that can be reached via comparative geometry. Two different systems offer a lot of opportunities to compare differences and likenesses, and to set up, check and change local or global axioms.

Motivation: As you see from the illustrations, I propose that students draw manually and construct on plastic surfaces. In the third millennium, is it not ridiculous to scribble on a sphere with a marker, instead of playing on the keyboard of a computer? *No, it is not!* You can display any line on the screen with the help of your keyboard, but only when your hand is already familiar with that line. You must feel the line drawn by your hand when you push the key on the board! This is true for the plane, and much more so for the sphere. Space perception and understanding must always start from direct experience. Manipulation is not in opposition to computer technology. On the contrary, they are best friends, complementary to each other.

Beyond mathematics

If you teach your students to accept different approaches, different standpoints in mathematics, then, instinctively or deliberately, you teach them the same attitude in other areas of life as well. They learn to accept another person with different traditional, cultural or social background, from a different age or sex group. They learn things that are more important than any mathematics, namely, creative debate, human communication, empathy and understanding.

Quite frankly, what has your philosophy done to your own life?

Notes

[1] In Hungary, István's name would be written Lénárt István. I first met him whilst visiting Budapest with a group of student mathematics teachers from the UK as part of an exchange. I have been struck over the years by how different groups of my students have engaged with geometrical ideas through being taught by István. During the session in April 2003 I acted as scribe, writing down what was said and what happened. After the session I

Figures 6, 7 and 8: Pencils of lines (drawn in red) in hyperbolic geometry intersecting on the surface of the open hemisphere, on the equator (no common point in the geometry) and somewhere outside the equator and again, outside the geometry and corresponding circles (continuous blue lines that are perpendicular to every line in the pencil).

gave the notes to István and asked him to write about what he thought informed the planning of the session (see also Lénárt, 2004) (ed.).
[2] The Lénárt Sphere is produced by Key Curriculum Press http://www.keypress.com/manipulatives.

References

Henderson, D. (1995) *Experiencing geometry on the plane and sphere*, Englewood Cliffs, NJ, Prentice Hall.
Lénárt, I. (1991) 'The drawing ball set', *Mathematics Teaching* **134**, 30-33.
Lénárt, I. (1996) *"Non-Euclidean adventures on the Lénárt sphere'*, Key Curriculum Press.
Lénárt, I. (2004) 'The plane-sphere project', *Mathematics Teaching* **187**, 22-26.

Figure 1: "One of the defining characteristics of a 'culture' or a 'tribe'" (see the beginning of the article by Guy Claxton, opposite).