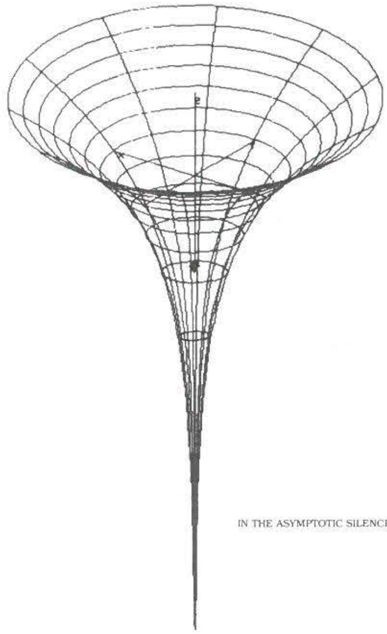
ANDREJS TREIBERGS



IN THE ASYMPTOTIC SILENCE

MIROSLAV HOLUB

The Corporal Who Killed Archimedes

in one bold stroke
he massacred the circle, the
tangent, the point of
intersection at infinity

on pain of
quartering he banned
numbers
from three on up

in Syracuse he now
heads a college of
philosophers squats

on his halberd
and for another thousand
years writes

one two
one two
one two
one two

translated by Jet Wimp

Metaphor and Analogy in Mathematics*

DAVID PIMM

The claim that mathematics is, *inter alia,* a language is often made, but serious investigation of this assertion has been limited. The major focus of research in the area of language and mathematics has been on readability of mathematical texts or difficulty of verbal arithmetic problems [Nesher, 1976; Pimm, to appear; Rothery, 1980]. The sense of approach I wish to explore is that of subjecting certain figures of speech, in particular analogy and metaphor, to close scrutiny. These figures make natural language so powerful and I want to suggest that we can identify comparable processes at work within mathematics itself. The initial inclination is to view analogy, and metaphor in particular, as uncertain methods of working and therefore unsuited to mathematics. "Analogy, however, is not proof, but illustration." I hope to show by means of examples culled from arithmetic, algebra and the calculus that this is far from the case and in fact these processes are as central to the expression of *mathematical* meaning as they are to the expression of meaning in natural language.

Before turning to mathematical examples, however, I wish to discuss a particular view of analogy and how metaphor relates to it. Metaphor is a particularly troublesome phenomenon for philosophers, linguists and literary critics alike, and no common terminology is agreed on by all writers [Black, 1962; Ortony, 1978]. I by no means wish to claim that the main construction identified in this paper characterises *all* cases of metaphor in mathematics; merely that it permits the identification of one class of metaphoric statements which I shall call *structural* metaphors. Some splendid work has been done on the use of extra-mathematical metaphors and the dynamic metaphors implicit in the language of the calculus [Kaput, 1979].

The view of metaphor I wish to employ is one based on analogy (characterised and criticised by Black [op cit.] as the "comparison" view of metaphor). Let us return to the Greek term *analogia.* Szabó [1978] informs us that, "It is less well known that this same word (viz, *analogia)* was not

*This is an expanded version of a talk given at the fourth annual conference of the International Group for the Psychology of Mathematics Education held in Berkeley in August 1980. I should like to thank Fran Rosamond and Rolph Schwarzenberger for commenting on the original version.

originally a grammatical or linguistic term, but a mathematical one... The Greek grammarians of Hellenistic times undoubtedly borrowed their term *analogia* from the language of mathematics. So in the last analysis, we are indebted to Greek mathematics for our word 'analogy'''. The mathematical meaning of *analogia* is that of proportion, currently interpreted as equality of ratios, although David Fowler has recently shown [1979, 1980] how ill-defined the notion of ratio is in Euclid's *Elements* and he has proposed that the celebrated Eudoxan Book V is developing a theory of proportion independently of the notion of ratio.

In a talk, Chaim Perelman characterised the resulting extension of the meaning of analogy as follows, ''analogy is analogical to a proportion'', providing a reverse instance of what I am trying to do in this paper.[1] Thus an analogy links a relationship A:B to another C:D. For example, old age is to one's life as the season of winter is to a year. While proportion is a symmetrical mathematical relation, the use of analogy customarily presumes a unidirectionality, assuming we know more about one relationship than the other. Hence by constructing this link we can thereby illuminate or evaluate the first relationship better. (Two-way flow is also possible: for example, the development of computers and knowledge of the brain: both currently feed off analogies and metaphors from the other.) Educational uses of analogy appear in the images implicit in the terminology used: the mind as a muscle (mathematical exercises), or as a garden (Kindergarten, mental growth) or as a *tabula rasa*. [2]

A metaphor is seen as a condensed analogy and from the analogy A:B::C:D we speak of ''the C of B'' or ''A is a C''. Thus, ''He is in the winter of his life''. If we are only presented with a metaphor, in order to understand it we must identify the latent analogy on which it is based. Thus our interpretation of it is dependent on this reconstruction of it, as often more than one is possible. Consider the statement ''Jane is a lion'' (''A is a C''). It could mean she is brave, that the has a tremendous head of hair (''a mane'') or that she roars a lot. What is to be carried over and what is not needs to be brought out very clearly.

Having outlined some examples in everyday language, let me now turn to some mathematical ones. Below is a table summarising my suggestion for links between certain figures of speech and mathematical notions.

simile (the comparison of one thing with another	⟷	*relation* [equivalence]	
analogy (a resemblance of relations	⟷	*morphism* [isomorphism]	transfer via the morphism
metaphor (change based on an analogy)	⟷	*embedding* [equality]	
(The parenthetic terms provide a description)		[The bracketed terms denote the *strongest* possible instance]	

Thus if I wish to indicate examples of structural metaphors in mathematics, the above makes it clear that I am after statements of equality based on identification by means of a structure-preserving map.

The question, not surprisingly, is one of meaning. As mathematician René Thom has so clearly stated, ''The real problem which confronts mathematics is *not* that of rigour, but the problem of the development of ''meaning'', of the ''existence'' of mathematical objects.'' And again, ''Any mathematician endowed with a modicum of intellectual honesty will recognise then that in each of his proofs he is capable of giving a *meaning* to the symbols he uses'' [Thom, 1972]. Why is the existence of structural metaphors a problem for mathematical education? *The central issue is that a morphism may preserve structure, but it does not preserve meaning.*

My first example comes from the domain of arithmetic. Consider the statement $2 - 3 = -1$. It involves an unexamined metaphor which leads us to presume the identity of certain elements and operations, but there are necessary conflicts as the relationship is not one of identity. The problem of taking a metaphor literally is augmented by its existence being concealed by the symbolism. It is customary from primary mathematics onwards to identify the whole numbers (**N**) with a subset of the integers (**Z**) using the homomorphism $i:\mathbf{N}\rightarrow\mathbf{Z}$ taking n to $+N$, under which N is identified with $i(\mathbf{N})$. Consider in this light Bill Higginson's anagrammatically-phrased insight that ''re-*nameing* is re-*meaning*''. The homomorphism also identifies the operations, and here is a second problem. Discussions of metaphor usually hinge on the distinction between literal and metaphorical meaning. Let us look at the archetypal situations with the natural numbers and operations on them. What are the elements, what are the operations and whence comes their meaning? In brief, for me, natural numbers arise as possible answers to the question ''how many?'', addition from conflating two sets of distinct elements, subtraction from ''take away'', multiplication from repeated addition and division, a model for equal sharing.

What are the equivalent well-springs of meaning for the integers and operations upon them? Under the above identification, the positive integers become the whole numbers by omission. (And what an omission! For me, one of the most important distinctions is between signed and unsigned quantities and the two distinct types of numbers used to measure them.) Too many distinctions have been elided. Because of the structural metaphor, we first require an extended concept of number itself, one including the possibility of directedness (-1 makes no sense as an answer to ''how many?''). More importantly, is it a straightforward identification of two already extant systems, or do the operations on **Z** somehow emerge from those on **N**?

Consider the case of multiplication of elements of **Z**. I can find no real world situation to which we can refer. One force of the identification of $+2$ with 2 is to permit the literal meaning of x in **N** to be used metaphorically in **Z**. Thus $+2 \times +3$ signifies $+3 + +3$. Having associated + with movement to the right — directed — on a number line, we obtain $+6$. $+2 \times -3$ is acceptable also without too much stretching (assuming addition works on the whole of

Z). $-3 \times +2$ we can approach via commutativity, though justification for this is flimsy due to the absence of any situational meaning guiding our operations. The reasons become formal and reflect our desire for logical systems. (Not that this is inappropriate *per se*, but just consider the age at which such operations are customarily taught.) What can we do with -2×-3? Our metaphor cannot help us here and this is, I believe, one of the causes of the problem of multiplying negative numbers.

Why is this a problem? If the metaphoric quality of certain conceptual extensions in mathematics is not made clear to children, then specific meanings and observations (whether intuitive or consciously formulated) about the original setting will be carried over to the new setting where they are often inappropriate. The identification which is at the basis of the metaphor guarantees only that certain structural properties are preserved in the extended system, not the meaning. Consider the observation that subtraction produces something less than either of the two starting numbers. Then think about $4 - (-2) = 6$. Or, for example, that multiplying two whole numbers together results in a number larger than each of the original pair. This is a correct concomitant of viewing multiplication as repeated addition. This no longer holds with fractions. Why do we call $2/3 \times 4/5$ multiplication anyway? What is the root meaning of this operation? The extension of concepts in this fashion can result in the destruction of meaning if no distinction is made between the literal and metaphorical use, because the confusion arising from seeing such (often unanalysed) truths "fail" contaminates not only the extended system but also the one from which it grew. The old concept resists expansion precisely because it is the literal, and hence the most basic, meaning.

Turning to algebra, one finds an abundant metaphoric use of what I want to call the fundamental chain of numbers. This is schematised below.

$$\mathbf{N} \subset \mathbf{Z} \subset \mathbf{Q} \subset \mathbf{R} \subset \mathbf{C}$$

Despite the many complications of construction, these are viewed as set-theoretic inclusions (each one employing a structural metaphor). One (probably not deliberate) mathematical strategy has been to ascend this chain whenever a use has been found for numbers at some point. Consider the case with exponents. a^n, for n a whole number follows the literal meaning for multiplication. Can we make sense of a^{-1}, what about $a^{1/2}$, what about $a^{\sqrt{2}}$, what about a^i (the same choice is available for a)? What about matrices? A^2, $A^{1/2}$,... What about the differential operator $D = {}^{d}/_{dx}$? D^2 is O.K. D^{-1} becomes integration. $D^{1/2}$? A fractional operator calculus has recently been developed.

Another example arises in the calculus. Consider a function of two real variables $z = f(x, y)$. It is possible to assign a vector grad$(f) = (\partial f/\partial x, \partial f/\partial y)$ to every point on the graph (a surface in $\mathbf{R}^3$). Intuitively the direction of the vector grad(f) gives the line of steepest slope. It is a vector in the tangent plane attached to the surface at a point. Yet due to an identification (all 2-dimensional vector spaces are isomorphic) by means of coordinates, this vector becomes situated at the origin, lying in the x-y plane. Co-ordinates allow the identification of too many things which are best kept apart, particular in the case of $\mathbf{R}^3$. Thus an ordered pair determines a point in a plane, a vector, the plane normal to it and so on. A final interesting point can be gained from vector spaces. A distinction is made between the isomorphism: $V \rightarrow V^*$ (the dual) and: $V \rightarrow V^{**}$ (the double dual), the latter being natural in the category-theoretic sense. This shows that not every analogy gives rise to a metaphor.

The focal point of structural metaphor is the use of the symbol for equality, just as it is in natural language where the stress is on the copula (Jane *is* a lion). There seem to be at least two major, different uses of the symbol "=" in mathematics. The first is a fairly low-level, naming use, for example $f(x) = x^2$, where one side has meaning and the other is to be used as a label. Definitions can be of this sort; for instance, consider the statement (claim?) that $a^{-2} = 1/a^2$. (Recall the move from $\mathbf{N}$ to $\mathbf{Z}$ as a source of exponents as another instance of metaphor). Thus whenever we see a^{-2} for example, we transfer the meaning of the other side, giving force to Polya's injunction, "Go back to the definitions" [Polya, 1948].

The second use is more worthy of the same theorem and arises from some kind of identification of meanings and this is where both sides of the equality have meaning in their own right. For instance in the case of $\int_a^b f'(x)dx = f(b) - f(a)$, the two meanings are being equated under certain circumstances (although this, to a Greek, would be abhorrent, as how could an area equal a length?) Is $2 + 3 = 5$ to be classified in the second sense? How about $2 - 3 = -1$? In the former it is a dynamic equality where the left-hand side has been transformed by some equality and the result identified with the right-hand side. Thus $2 + 9$ is not an acceptable answer for me to the question "What is $4 + 7$"?, because it still contains a verb. In addition to examples with elementary operations (for example, leaving a fraction as 7/4 has an incomplete air about it — and what sense can be made of $7 \div 4 = 7/4$?), L. Copes [1980] has pointed out that $\sqrt{\ }$ and $\int dx$ also function in a similar way for many students. Eliminating verbs makes the answer static and hence, permanent, while their continued presence is indicative of a tension yet to be resolved. Consider in this light Wittgenstein's observation that mathematics always identifies processes with results. This provides a partial explanation for the difficulty students often have on seeing the *definition* $\log(x) = \int_1^x 1/t\, dt$. It is a naming equality, yet they already have a meaning for $\log(x)$.

Examples of metaphor in mathematics often go under the name extension of meaning and are often accompanied by great distress.[3] One example is the introduction of negative numbers, another that of complex numbers. Calling these objects *numbers* is a metaphoric extension broadening the notion. The extension of exponent notation firstly to negative numbers, then to rationals then to reals and even complexes provides another example ($e^{i\pi} = -1$ is perhaps the most famous metaphoric claim in mathematics). Another instance can be seen in Klein's *Erlanger Programm* [1872] where he says, "To make use of a modern form of expression, which to be sure is *ordinarily* used only with reference to a particular group, the group of all linear transformations, the problem might be stated as follows: given a manifoldness and a group of transformations of the same; develop the theory of invariants relating to that group". The novelty gets lost with time and with it the metaphoric content of the original insight of useful extension, it becoming a commonplace remark, the literal meaning (cf. "Shakespeare is full of clichés", a dead metaphor being a cliché). Such a situation comes from the use of the same word or symbol for different objects, sometimes referred to as "abus de langage". An example occurs in group theory with the terminology "the *order* of an element" and "the *order* of a group". This produces a forced identification since one possible connection (that the order of an element is the order of the cyclic subgroup it generates) is often omitted and rather than making a positive identification, students fail to discriminate between the two different uses.

With regard to symbols, let us look at the sine function. One possible development involves defining the sine of an angle A as the ratio of two directed line segments. By extending the domain to the reals (with sleight of hand between angle and its numerical measure), we can then show that $\sin(x) = x - x^3/3! + x^5/5! - \ldots$ for all x in **R**. One might observe that the right-hand series converges for all z in C and therefore defines a complex-valued function. What do we call it? Sin(z)! What has $\sin(2 + 3i)$ got to do with angles? The geometric meaning is no longer appropriate.[4] Notice how in this progression we have used both uses of "=" which I have discussed.

There is no trace in the symbolism to indicate that a metaphor is in use. A metaphor skims over a lot and a polished notation permits this riding on the surface. If students are not used to mathematics making sense, then having tried and failed to make literal sense of a statement, they are likely to give up.

Metaphors deny distinctions between things: problems often arise from taking structural metaphors too literally. Because unexamined metaphors lead us to assume the identity of unidentical things, conflicts can arise which can only be resolved by understanding the metaphor (which requires its recognition as such), which means reconstructing the analogy on which it is based. Teachers will often cease to use terms metaphorically, or be conscious of the distinction when their concept is an expanded one, but this will not mirror the situation in most of their student's minds. Kaput [op cit] claims "metaphoring is a central process by which we create and transfer meaning". Are the above metaphors I have identified *necessary?* This is one point which Black raises against the comparison view for a complete account of metaphor. Problems of interpreting e.g. $-2 -2$ or $(-3) - (-1)$ arise, yet disentangling them can lead to an overabundance of notation quite unsuited to the age range at which such manipulations are customarily first taught. As points for further thought let me ask the question, can metaphors be eliminated and, if so, should they be eliminated? Do the notions of literal and metaphorical meaning make sense in mathematics?

I certainly wish to stress that I am not against metaphor. A fruitful metaphor may often suggest theorems (and notation is often suggestive in this regard).[5] We need to be aware of metaphors because unexamined ones lead us to assume the identity of elements and processes which will conflict with our past experience. Such is the power of language that almost any collocation of terms can be assigned some meaning. ("A healthy *book* — did I hear you correctly?") (Consider e^A for some square matrix A.) Our concern as mathematics teachers is surely with the creation and loss of meaning. The unexplained extension of concepts can too often result in the *destruction* rather than the expansion of meaning.

Notes

[1] Part of this paragraph is based on a talk entitled "Analogy and Metaphor in Science, Poetry and Philosophy" given by Chaim Perelman at the University of Wisconsin at Madison on November 8th, 1978. When asked, Dr. Perelman expressed the view that there was no metaphor in mathematics.

[2] Educational metaphors, which is the form in which such analogies often appear, particularly those dealing with mental processes, would make a fascinating and, I feel, enlightening intellectual and etymological study.

[3] For example, the first encounters with negative numbers and zero. "How can anything be less than zero?"

[4] How about the definition $\sin(x + iy) = y/\sqrt{(x^2 + y^2)}$? Consider the geometry of this definition.

[5] Consider infinitesimals and the history of their use from the point of view of attempts to eradicate metaphor, to change the notation and remove the image. [Pimm, 1978]

Bibliography

Unreferenced quotations are from the Oxford English Dictionary entries under *metaphor* or *analogy*.

Black, M. *Models and metaphors*. Ithaca, NY: Cornell University Press, 1962

Copes, L. Personal communication, 1980

Fowler, D. Ratio in early Greek mathematics. *Bulletin of the American Mathematical Society* 1,6 (Nov. 1979)

Fowler, D. Book II of Euclid's *Elements* and a pre-Eudoxan theory of ratio. *Archive for History of the Exact Sciences* (to appear)

Kaput, J. "Mathematics and epistemology". In: Lochhead, J. and Clement, J. (eds) *Cognitive process instruction*. Philadelphia: Franklin Institute Press, 1979

Klein, F. "Das Erlanger Programm" (1872) trans. M. W. Haskell, *Bulletin of the American Mathematical Society* 3, 215-249 (July 1893)

Nesher, P. "Three determinants of difficulty in verbal arithmetic problems", *Educational Studies in Mathematics* 7, 369-388 (1976)

Ortony, A. *Metaphor and thought*. Cambridge: Cambridge University Press, 1978

Pimm, D. Language symbols and meaning in mathematics. M.Sc. Thesis, Cornell University, Ithaca, 1978

Pimm, D. Review of research on linguistic factors related to addition and subtraction (to appear)

Rothery, A. (ed) "Children reading mathematics", a working paper of the Language and Reading in Mathematics Group, U.K.

Szabó, A. *The beginnings of Greek mathematics*. Dordrecht, Holland: Reidel, 1978

Thom, R. "Modern mathematics, does it exist?". In: Howson, A. G. (ed) *Developments in mathematical education*. Cambridge: Cambridge University Press, 1972