MODELLING WITH GRAPHICAL REPRESENTATIONS

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"[O]ne essential answer (of course not the only one) to the question as to *why* all human beings ought to learn mathematics, is that it provides a means for understanding the world around us, for coping with everyday problems, or for preparing for future professions" (Blum, 2002, p. 151). Thus, it is not surprising that "applications and modelling" is a very important theme in mathematics education. Evidence for the extent of the interest in this area includes, for example, the recent ICMI study and the biennial conference held by the International Community of Teachers of Mathematical Modelling and Applications (ICTMA). As well, comprehensive curricular approaches have been developed and investigated in which situations (from the real world or from within mathematics) are the departure points for learning mathematics. The most salient initiative with this spirit is the world famous and successful Dutch project called "Realistic Mathematics Education", initiated by the late Professor Hans Freudenthal.

What is modelling? "The essence of modelling for me is a movement between worlds: from the world of the 'problem' … to another familiar world, such as the world of symbols" (Mason, 2003, p. 42). Many other descriptions of modelling can be found. For example,

> The starting point is normally a certain *situation* in the real world. Simplifying it, structuring it and making it more precise – according to the problem solver's knowledge and interests – leads to the formulation of a *problem* and to a *real model* of the situation. … If appropriate, real data are collected in order to provide more information on the situation at one's disposal … the objects, data, relations, and conditions involved in it are translated into mathematics, resulting in a *mathematical model* of the original situation… mathematical methods come into play, and are used to derive *mathematical results.* These have to be re-translated into the real world that is *interpreted* in relation to the original situation. (Blum, 2002, pp. 151–152)

Some aspects of mathematical modelling are usually taught at school at all levels within any curriculum – including, for example, word problems in elementary school, distance velocity problems in middle school, and max-min problems in high school calculus (or pre-calculus) courses. Although there are several exceptions, modelling in school mathematics traditionally consists of translating a problem from words into "mathematical language" (usually symbols), operating with the symbols, and obtaining a solution which is then re-interpreted within the problem situation. Much can be said about the differences between this kind of "mathematical modelling" and the practice of modelling by professional mathematicians or engineers facing a genuine problem from the real world.

Modelling with graphs

Rather than discussing how to bridge between the two apparently dissimilar practices of schooling and the professional world, we suggest that much can be done within school mathematics in order to enrich the present practice of mathematical modelling in order to learn meaningful mathematical content and useful tools in a meaningful way, even when "real" professional practices are not yet fully incorporated.

The modelling that is the subject of this paper refers to translations within mathematics. In other words, we focus on the activity of studying one mathematical phenomenon by using mathematical means other than those of the phenomenon itself. We start with a geometrical situation – namely the exploration of the variation of areas of figures within a dynamic geometry environment. The dynamics allow for a visualization of the area changes, which in turn enables one to become familiar with the variables and even to hypothesize about expected outcomes. Moreover, in the dynamic environment, it is possible to see the changes of the area as functions of a variable (*e.g.*, a side) in a Cartesian graph built *ad hoc,* in which the graph and the geometrical phenomena change simultaneously and in real time, as requested by the user. In general, geometrical situations that involve change are modelled by algebraic symbols – that is by the algebraic expression of the corresponding function. In the modelling presented below, the algebraic representation of the phenomenon is postponed in favor of the study of two visual, yet very different, representations. In comparing, contrasting and translating between the changing geometrical phenomenon and its Cartesian graph, students may notice and learn about aspects of the situation that go unnoticed when only the situation is analyzed. At the same time aspects of graphing a phenomenon become apparent. It is through the non-algebraic mediation between a phenomenon and its Cartesian representation that the characteristics of the phenomenon can be studied, avoiding the potentially distracting need of symbolic manipulations. When we finally bring in the algebraic representation, it "comes alive" (Noss & Hoyles, 1996, p. 245), both because the representation reflects information found in the previous graphical explorations and because it may contribute to explanations and insights.

In the following, these characteristics are exemplified discussed in detail, including the way we use them with

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students and some of their reactions. The first is a well-known problem, followed by others less known, which carry some surprises.

> **Problem 1**
> Given a rectangle of fixed perimeter 12 cm, investigate the variation of its *area* as a *function* of its *side*.

A traditional solution path consists of establishing a "mathematical model" – or, in other words, translating the conditions of the problem into a symbolic relationship, $A = x(6 - x)$, and investigating it. According to what was described above, we propose to investigate this problem in a different way, by making use of a computerized environment that enables us to model the situation graphically in a dynamic way. In Figure 1, there is a display of a typical screen of such an environment in which the rectangle of fixed perimeter (12 cm) was constructed, a measurement of its side AB and of its area ABCD was performed, and a Cartesian graph was defined by inputting the dependent variable (in this case, the area of the rectangle) into the y-axis and the independent variable (its side) into the x-axis. Note that the broken line connecting two of the frames in the computer display indicates the input of measured values of a variable into the corresponding axis of the graph.

Figure 2 shows four screens. On the upper left side of each screen, there are snapshots of the different rectangles as they change dynamically while one of its vertices is dragged. The lower left side of each snapshot shows the corresponding measurements (of the area and one side, which change accordingly). Simultaneously, on the right side of each screen, we see the Cartesian graph (of the variation of the area as a function of its side) as it is being traced.

This environment enables one to study the situation through its graphical model by, for example, considering the following issues:

- When does the maximum occur in the graph and what does it represent?
- What are the graph minima, and what do they represent?
- Explain the symmetry of the resulting graph in terms of the situation.
- When is the rate of increase of the area slower?

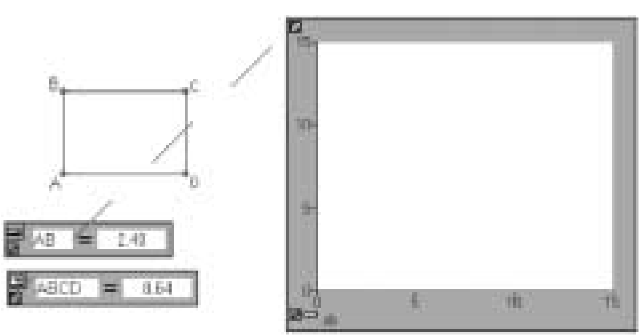

Figure 1: Initial screen displaying the rectangle and the Cartesian graph.

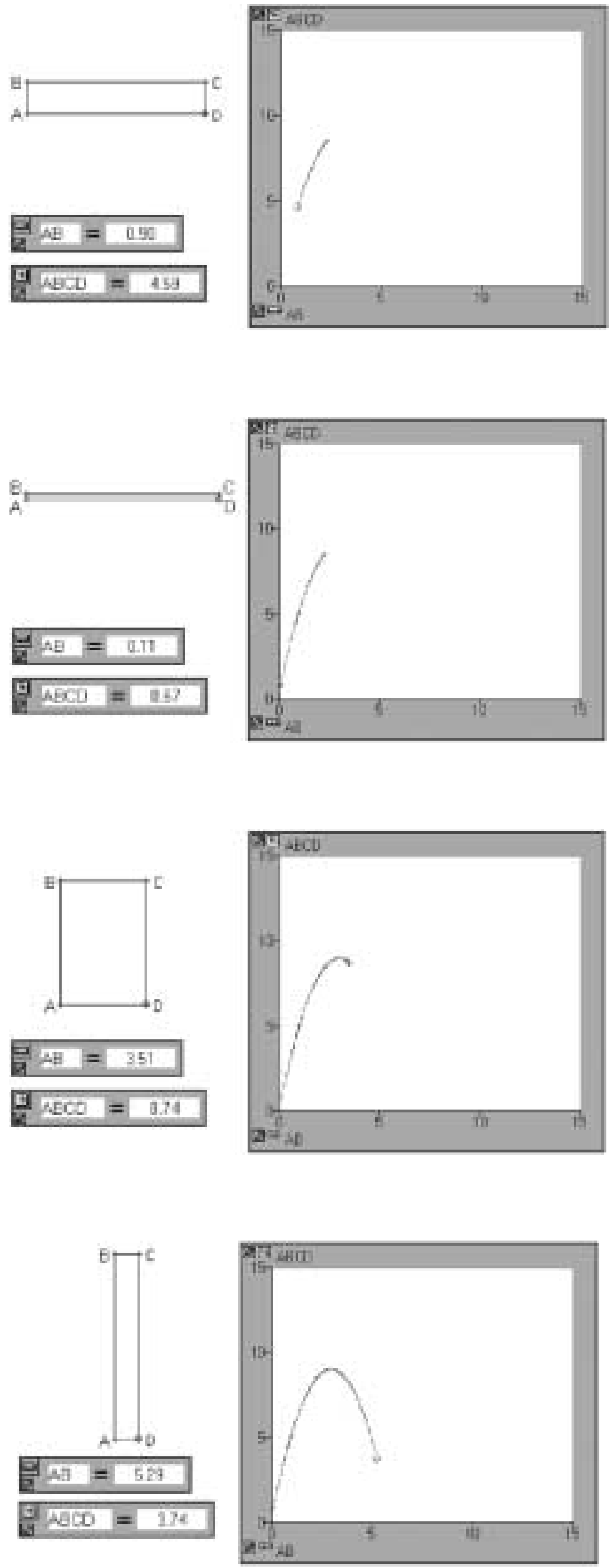

Figure 2: Snapshots of the rectangle changes and the Cartesian graph

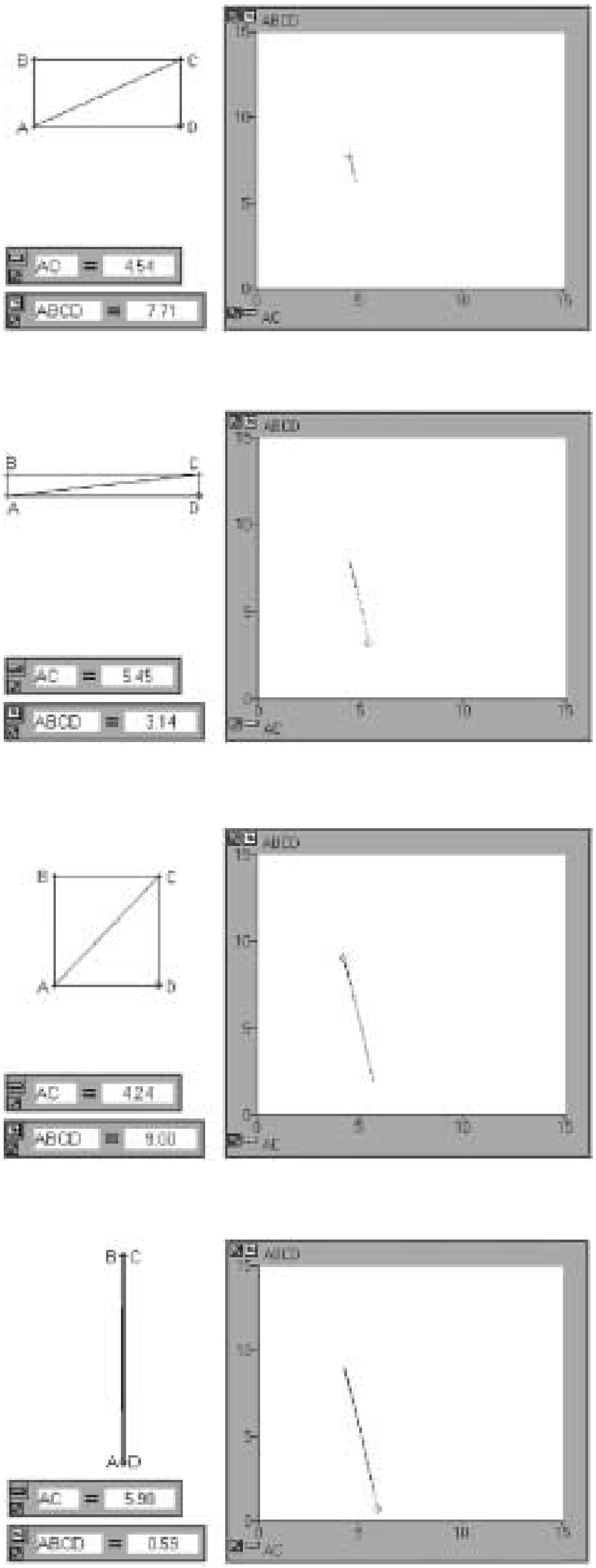

Figure 3: Snapshots of the rectangle changes and the corresponding Cartesian graph

Problem 2
Given a rectangle of fixed perimeter 12 cm, investigate the variation of its *area* as a *function* of its *diagonal*.

The variation of the area of the rectangle when seen through the graph enables us to follow visually its increase, decrease and rates of change; it highlights the fact that the situation is symmetrical with respect to the "mid-point" of the domain (all the possible values for the length of the side); and it shows that its maximum is attained when the rectangle becomes a square (*i.e.*, the only figure that does not occur twice when one varies the side over its full domain). In the following, we illustrate further how the graphical model can become an insightful tool for discussing similar issues.
This problem is less common in traditional school textbooks. As previously, we postpone the modelling of this problem with algebraic symbols, and proceed to study first the graphical model. We propose to our students to make a conjecture about the shape of the graph, before proceeding. Figure 3 shows snapshots of the graphing as it is being created.

This time the graph is a bit surprising for many. The previous example may have set up the expectation of a symmetrical graph (parabola), however, here as we change the rectangle dynamically by dragging one of its vertices, the Cartesian graph traces as "running" back and forth along what appears to be a segment of a straight line! (See last snapshot of Fig. 3.) Is it so? In order to make sense of these surprises in our trial classrooms, we first investigate the following:

- What are the domain and the co-domain of this function?
- Explain the situations represented by the extreme points of the segment.
- In the previous problem the graph is symmetrical. Here it is not. Explain why.
- In Figure 4, the graphs from the previous problem and this one are juxtaposed. Explain:
  - a) why the two graphs have the same height,
  - b) the meaning of the proximity of the two graphs, especially when the independent variable increases. Do the graphs intersect? If so, where?

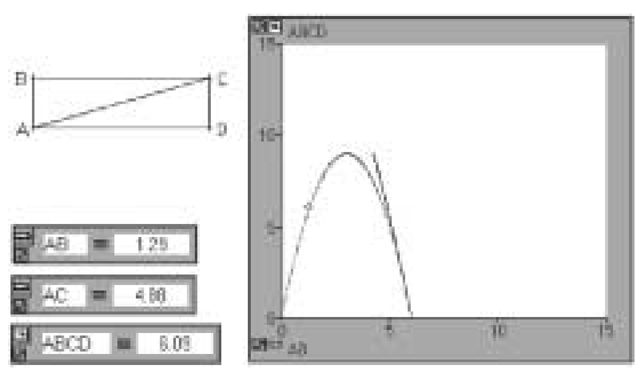

Figure 4: The two graphs juxtaposed

Replying to these questions requires a fruitful interplay between the situation and its graphical model. For example, when looking at the graph for this problem, many students and teachers notice for the first time that the domain (*i.e.*, all possible values taken by the independent variable) is very limited – between a bit more than 4 and bit less than 6. In order to make sense of this finding, we re-turn our attention to the situation itself and observe with more care the variation of the length of the diagonal as we change the rectangle. We notice that:

a) the diagonal becomes longest when the rectangle "flattens" and it "degenerates" into a segment – namely when the diagonal and the side collapse and both are of length 6; and

b) the shortest diagonal is obtained when the rectangle becomes a square, in which case the diagonal is $3\sqrt{2} \approx 4.24$. Note that the smallest diagonal yields the maximum area.

Another feature of the graphical model, which redirects attention to the situation and leads us to re-inspect it, is that the point tracing the graph runs up and down on it as the rectangle changes. This is in contrast to the previous problem in which the graph was a symmetrical curve. In order to explain this discrepancy between these global graph features, one needs to look again at the situation as it changes dynamically. One realises that the diagonal decreases from one extreme case in which it is longest (when there is no rectangle) to its shortest length (in the case of the square), and then it starts increasing when we "push" the square further. In other words, the independent variable (the length of the diagonal) travels back and forth along its domain for one full cycle of the rectangle changes. This is in contrast to the previous problem, in which the side increases from its smallest value (0) to its largest value (6) and on its way "covers" all the rectangles, including those who have the same area (for different values of the domain), thus creating a symmetrical curve. In this case, it was the unexpected shape of the Cartesian graph that re-directed the attention to a more careful observation of the changes in the rectangles.

The juxtaposition of the two graphs highlights features that should also be explained. Their equal height is due to the fact that in both cases the dependent value is the area – although this fact is not always immediately obvious to students. Its maximum value remains the same, although the dependent variable is different and therefore this maximum is attained for different values of the domain. The graphs approaching each other reflect the fact that, for increasing values of the domain (for the independent variables in both cases), when the side and the diagonal approach the same value of 6, the area tends to 0.

The educational moral we propose to draw so far is that the dynamical graphical model highlights aspects of the situation that were not as salient had we investigated it alone, or even by modelling it symbolically. In other words, by virtue of the observation and analysis of the graphical model, we notice and explore many features of the situation that become apparent and explicit precisely because they are highlighted by the model.

Let us turn now to a result that puzzled us: is the graph indeed a segment of a straight line? It is useful to stimulate intuitive answers first. For example, some students – and also some teachers – say that it is unlikely that an area as a function of a line segment would be linear. The linearity must thus be an illusion due to scaling. However, a change of scales to discern visually the type of variation does not help much, In this case, the appearance of linearity persists, further fueling our doubts. Certainly, the question cannot be rigorously resolved by graphical means only, and a discussion of the limitations of this model and the need for a different kind of model is now in place. Only the symbolical model will ultimately help us to settle the matter. Thus, in our investigations, we do not rule out the fundamental role of a symbolic model; we merely propose to postpone it and to invoke it when we need it and when it can provide us with precise answers to well formulated questions that cannot be found elsewhere.

In this case, certain symbolic skill and symbol sense (Arcavi, 1994, 2005) may be needed to create a symbolic model. We look for how to express the area as a function of the diagonal, which, for students, may be less straightforward than it seems. The area of a rectangle is $A = xy$, its half perimeter is $x + y$ (which, in our case, is fixed at 6), and its diagonal is $D = \sqrt{x^2 + y^2}$. A good candidate to combine all these expressions in one formula would be $(x + y)^2 = x^2 + 2xy + y^2$. In our particular case, $36 = x^2 + y^2 + 2A$ yields $A = (36 - D^2)/2$. This symbolic expression for the function dispels our initial thought that the graph may be linear. The "segment" displayed in the graph must be one of the branches of the parabola, of which only some part of it (*i.e.*, the part drawn on the screen) is meaningful as a model for the present situation. In order to appreciate how close this branch of the parabola is to a straight line, we ask students to find the equation of the linear function joining the two extremes – namely (4.24,9) and (6,0). Once this equation is obtained, we propose to graph the function of the difference between the linear function and the function of the area. The graph of the difference, in the relevant domain, highlights not only how close these two functions are, but also their maximum/minimum differences and when they occur.

The symbolic model can also provide an opportunity to double check if and how all the information we have gathered is represented by the symbols. By inspecting the formula, we can indeed see that when the diagonal is 6, the area of the rectangle is 0. However, the symbolic model does not prevent us from substituting any value for D in the interval $0 \leq D \leq 6$. It is from our acquaintance with the situation – and from the graphical model attached to it that we explored – that we know that the minimum value for D is $3\sqrt{2} \approx 4.24$. If we substitute this value in the area function, we indeed obtain the area of the square of sides 3, which is the maximum possible area (occurring for the minimum value of the diagonal).

At this point, we propose the following:

Problem 3
Given a rectangle of fixed perimeter 12 cm, investigate the variation of its *diagonal* as a *function* of its *side.*

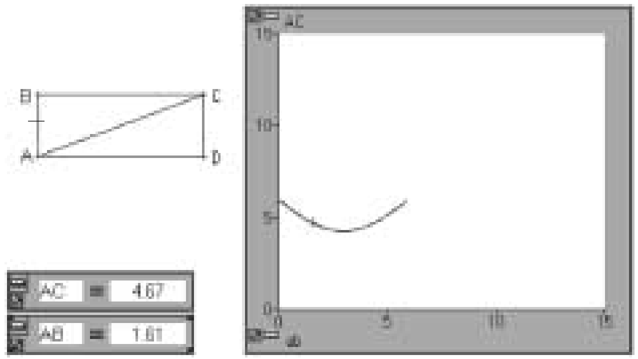

Figure 5: The diagonal as a function of the side

As with the previous problems, it is very productive to ask students to predict the shape of the expected graph. Making predictions can serve several purposes. First, it engages students with a more careful observation of the situation in order to form an image of the expected result. It thus supports explicit representations of knowledge and understandings related to the problem. Second, prediction creates an affective commitment with the problem and can stimulate classroom discussions. Third, the expectation for a certain result, especially when the actual result is different from the predicted, serves as a strong anchor against which to compare and contrast the expectation with the actual result. Such reflection can become a rich source of learning (Arcavi & Hadas, 2000).

From the previous problems, we know that as the side increases from 0 onwards, the diagonal decreases up to a minimum and then starts increasing again, symmetrically. We can also establish some values: when the side is 0, the diagonal is 6, it decreases until it reaches the minimum value $3\sqrt{2} \approx 4.24$, and then it starts increasing until it becomes again 6. So most would predict a symmetrical shape. Given the previous experience with problem 1 and the widespread tendency towards prototypicality (Schwarz & Hershkowitz, 1999), it is very tempting to suggest "parabola". Figure 5 shows the drawn graph.

As expected, the graph is symmetrical, with a shape that resembles that of a parabola. However, as we learned from experience, the rigorous resolution cannot be done only on the basis of graphs. Indeed, the algebraic model shows that $D = \sqrt{(6 - x)^2 + x^2}$, which is not an expression for a parabola. (Actually, it is one branch of a hyperbola.) Besides, the symbolic form of the function serves as a tool to double check the

Problem 4

Given a fixed rectangle ABCD, investigate the variation of the *area* of ABEF as a *function* of AF.

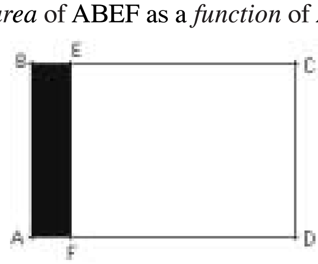

Figure 6: Area of ABEF

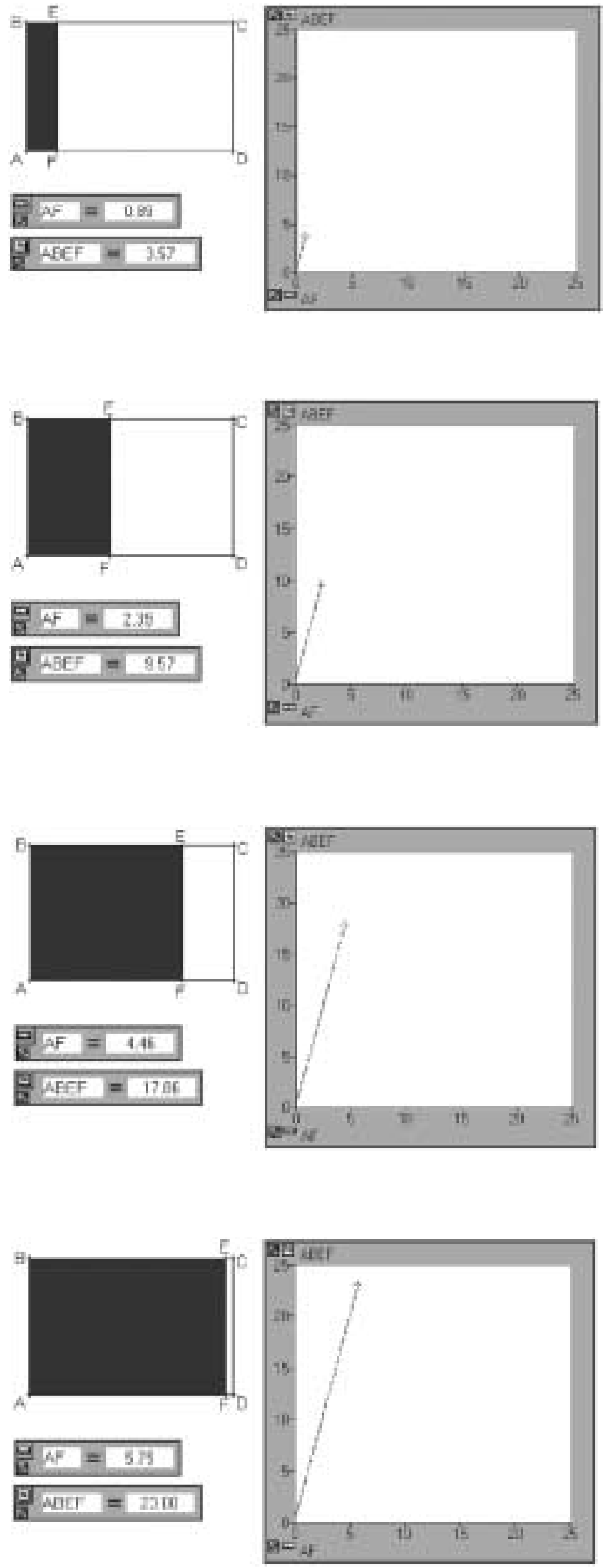

Figure 7: The changing area of a rectangle as a function of one variable side

information gathered from the situation (*e.g.*, extreme values for x and corresponding values for D, *etc.*).

In the previous problems we explored the variation of the area of a rectangle of fixed perimeter as both a function of its side and a function of its diagonal. To further illustrate the exploration of graphical models, let us now turn to the variation of the area of a fixed rectangle in a way that renders that variation visually perceptible. This concretization may well serve as an intuitive visual introduction to the notion of area under a curve (*i.e.*, definite integral).

Using graphical modelling as our main working tool, we first try to envision the graph of the area of ABEF as AF increases or decreases. The task posed at this stage is to make a conjecture about the shape of the graph. Some students conjecture that the graph is a horizontal line – due to erroneous visual transfer from features of the situation to features of the graph representing it. (This phenomenon was first described in detail by Bell & Janvier, 1982.) Figure 7 shows snapshots of the graph as it being created while the side increases dynamically and the area is colored as it increases.

The graphical model suggests that the change is linear. Is it?

To elucidate this question, one can proceed in different ways even before resorting to a full-fledged symbolic model. [1] For example, one may go back to the situation in order to find out whether the characterization of linearity applies there – namely, whether the rate of change is constant. In our case, for the same increases in the length of AF, would we always obtain the same increase in the area of ABEF? Since the area depends only on AF (and also on the width of the rectangle, but the width is constant), the answer is positive – although this type of reasoning is not always immediate for students. A less qualitative way to express this reasoning is to model symbolically the function for the area as ABEF = $a \times$ AF – or, in other words, $y = ax$, which describes a linear relationship. In order to make sense of the parameter a in terms of the situation, we ask students how to change the original rectangle ABCD in such a way that the linear graph (obtained when dynamically changing the area of ABEF) will be steeper? In order to answer this question, some students would experiment and obtain a desired graph by trial and error. Others would do an *a priori* reflection. Either *a priori* or *a posteriori,* the reflection will connect among concepts making use of both graphical and symbolic features, for example, as follows: a steeper graph should have a larger slope, namely a greater value of a in the formula $y = ax$. Since a represents the width of the rectangle, a larger width will yield a steeper graph.

The following problem seems to be a natural follow up inspired by a "What if …?" question. In this case, we propose, what if the rectangle is a triangle?

Problem 5
Given a fixed triangle ABC, investigate the variation of the *area* of ADE as a *function* of AE.

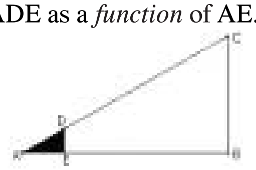

Figure 8: Area of ADE

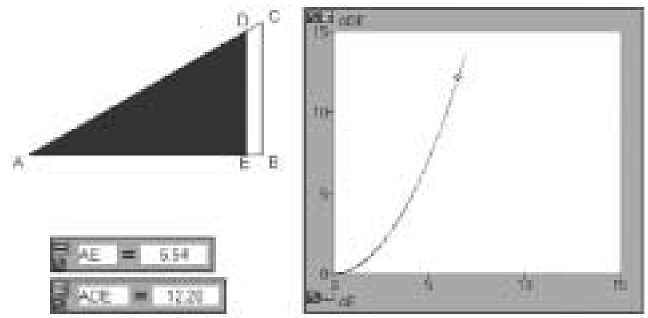

Figure 9: Area of ADE as a function of AE

Again, we propose to pause and make a conjecture about the shape of the graph. Here also, the graphical model may have in store a surprise for many (see Fig. 9).

The graph resembles a branch of a parabola. But, is it? Why? A qualitative analysis shows that this case is different from the previous because, as the length of AE increases, the same increases in the length of AE produce different increases in the area of ADE. Thus the rate of change is not constant. Moreover, the graph highlights that the rate of change increases. Can we make sense of this in terms of the situation? The question is aimed at the visual realisation that as we increase AE by equal amounts, the corresponding changes in area increase. Also, we help students to realise that the area of the changing triangle now depends on two variable quantities AE and DE. We notice that the ratio DE/AE is constant – that is, these two variables are proportionally related. Thus, the area of the triangle, A = (AE × DE)/2 is a quadratic expression of AE. Now, symbols support what we have found from the graph and the situation in a more formal way.

Let us turn the triangle around and proceed to the next problem.

Problem 6
Given a fixed triangle ABC, investigate the variation of the *area* of ACDE as a *function* of AE.

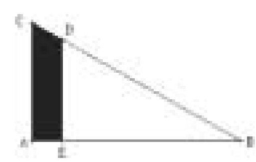

Figure 10: Area of ACDE

What would be the shape of the graph of the area of this trapezoid? Some students claim that the area decreases – a confusion that likely stems from attributing the visually salient decrease of the rate of change to the area, which surely increases. Once we agree that the area increases as AE increases, we ask whether the graph is linear. As previously, when we increase AE, the same change in AE produces different changes in the area of ACDE. Thus the rate of change cannot be constant and the function is not linear. Is the graph for this situation similar to the graph of the previous problem? In a sense, it is. That graph corresponds to an increasing function – and it probably is, again, a branch of a parabola. However, it must reflect the decrease of the rate of change (see Fig. 11).

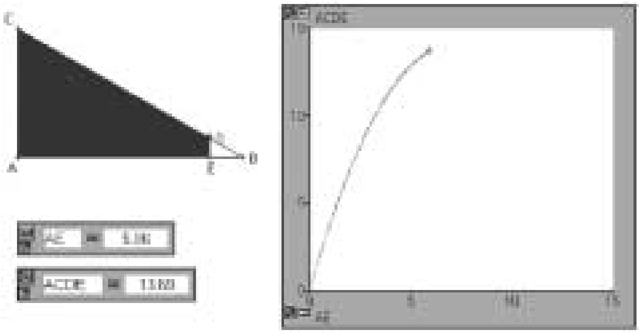

Figure 11: Area of ACDE as a function of AE

Our final problem combines the last problems, as follows.

Problem 7
Explore problems 5 and 6 by simultaneously varying the areas of the triangle and the trapezoid as functions of the same side length.

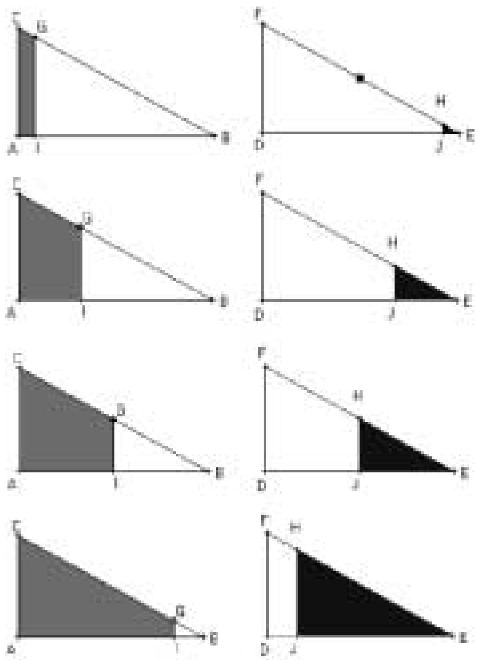

Figure 12: Areas of the trapezoid and the little triangle varying simultaneously

Note that the variation is linked in such a way that the segments AI of the trapezoid and JE of the triangle are always the same. Figure 13 shows the two graphs juxtaposed.

The juxtaposition of the graphs suggests some questions. For example, what is the meaning of the intersection points of the graphs? How do you interpret that one of the graphs is fully above the other? The intentions here are to notice interesting features of the graphs and to look back into the situation for what these features represent. We notice, for example, that the area of the trapezoid CGIA is always larger than the area of the triangle HJE, and these areas are equal only when we cover all or none of the area of the larger triangles (ABC, EDF).

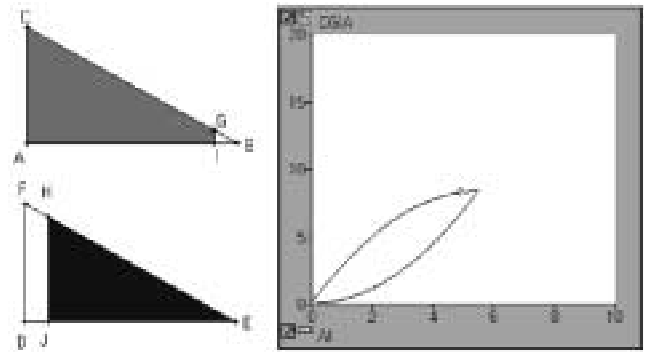

Figure 13: Juxtaposition of the two graphs

The next and final issue we propose to students is to make conjectures on the shape of the graph of the two combined areas (CGIA + HEJ) as a function of the side AI (or JE). This question may seem strange at first, and when we explore it in the environment, it produces a surprise (see Fig. 14).

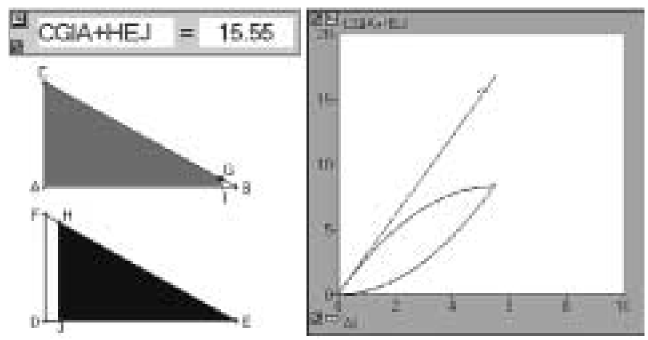

Figure 14: Graph of the two combined areas

The graph of the sum of the areas seems to produce a linear graph! How could it be? Why? The new seemingly linear graph must be the addition, point by point, of the two non-linear graphs (branches of parabolas), is that possible? With these questions in mind we first go back to the situation and then resort to symbols in order to make sense of this puzzling result.

By playing with the figures one starts to visualize what the sum really represents. Figure 15 shows the inversion of one of the two large triangles and its positioning above the other.

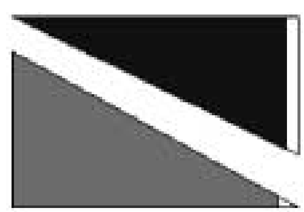

Figure 15: Relocation of the two fixed triangles

The combined areas give a rectangle with one fixed side and one variable side. In other words, we are back to problem 4! Indeed the graph must be linear. An informal symbolic reasoning will assure us that this is perfectly possible: the lower graph is a branch of an increasing parabola, the other branch of a parabola is decreasing. Thus, since the quadratic terms have the same parameter with opposite signs, they cancel out and yield a linear function.

Final remarks

Modelling geometrical situations with dynamic Cartesian graphs is a promising avenue for new ways of learning mathematical concepts and tools as exemplified above. The following are the main reasons and the possible morals drawn from the work with the above examples.

Functions. The concept of function is a central idea in mathematics and it has a prominent place in most middle and high schools throughout the world. Function is an important mathematical idea in itself, but no less important is its role in developing "students' facility with using patterns and functions to represent, model, and analyze a variety of phenomena and relationships in mathematics problems or in the real world" (NCTM, 2000, p. 227). Also, "teachers may find it helpful to compare and contrast situations that are modeled by functions from various classes" (*ibid.*, p. 297). Function is the core idea of all the activities that model the geometrical situations described above. Moreover, the concept, by virtue of being implemented within a computerized environment, "recovers its dynamism, as a genuine model for change and variation since its graphical representation is being created in real time describing the phenomenon as it occurs" (Arcavi & Hadas, 2000, p. 42)

Graphical models. The most widespread representation of functions is the symbolic (algebraic). Traditionally, graphs are invoked later as just an illustrative tool, usually sketched on the bases of features (such as extrema, increase, decrease, etc.) derived from applying analytic techniques to symbolic expressions. Some characteristics of the situation may remain opaque when we model a situation in this way, and for many students the activities are technical and mostly devoid of meaning. In the problems above, the graphical model is produced, explored and interpreted *first*. "Both the situation and its graph are looked at dynamically, all information gathered is intimately related and expressed in terms of the situation, and is put at the service of better understanding it" (Arcavi & Hadas, 2000, p. 41). Moreover, whereas a main goal for modelling is to understand better the situation we model, sometimes the situation contributes to better understanding of the modelling tool (*e.g.*, certain features of the graphs, such as "addition of graphs" in the final problem above).

Symbolical models. In the above examples, symbols are introduced when we need them in order to transcend limitations of the graph or when we want to re-inspect in a different way information already gathered. In this way, rather than engaging in mostly technical manipulation of symbols, the "algebraic expression comes alive" (Noss & Hoyles, 1996, p. 245). It is preceded by meaning making activities that drive the way we look at symbols, use them, and interpret them. This trajectory of working with graphs first and symbols later is not meant to imply that the models should be treated independently. Quite the contrary, once all the models are introduced, the interplay among them is very fruitful – as illustrated in most of the examples.

Surprises. Surprises can be a good source for learning. The contrast between an outcome that is expected and one that is surprising can prompt students to pose meaningful and productive questions, followed by attempts to resolve them. However, it is up to mathematics educators to set the scene for these surprises to occur. A first challenge is to design activities in which hands-on (and "minds-on") experimentation leads to puzzling and unexpected results. A second challenge is to nudge students to explicate their predictions in order to make visible the sources of their reasoning and conceptualizations and also as a means to make them more committed to the task. Explicit predictions, or conjectures, make the surprise (if/when it happens) more meaningful and provides rich opportunities to revise our knowledge, its sources, and the tools we use. A third challenge arises around classroom discussions through which one might take maximum advantage of the discrepancies between the predictions and the actual results.

Experimental arena. The computerized environment we use constitutes a rich experimental arena. Students receive the feedback as a direct consequence of their actions and not as a judgmental statement from their teachers. They can use the environment as a springboard for conversation and discussions with their peers and teachers. They are free to experiment, to produce questions of their own, and to embark in further explorations.

Intuitive introduction to formal topics. All the characteristics above combined offer us new learning paths and opportunities, mostly by helping to build informal intuitive knowledge infrastructures as sound foundations for the more advanced mathematics. It is our belief that such foundations would prevent many of the student failures in higher mathematics – which are mostly due to the lack of sense making we hereby propose to stimulate and support.

Coda

We started with a shortened quote from Mason (2003) about modelling. We conclude with the quote in full – because, as we understand it, it faithfully reflects the spirit of the present work:

> The essence of modelling for me, is a movement between worlds: from the world of the 'problem', through the world of imagery in which an 'essence' is sought in the abstract, the pure, the ideal, the simplified, to another familiar world, such as the world of symbols, of scaled down material objects, or of premade simulations. Finally there is a movement back through imagery again to the original problematic situation, and this cycling may be repeated several times at various levels of detail before some sort of conclusion is reached and recorded. (p. 42)

Acknowledgement

The problems hereby described are taken from the booklet *"On Geometric Variation and Graphs"* co-authored by Dr. Nurit Hadas, and myself. The booklet (in Hebrew), published by the Department of Science Teaching at the Weizmann Institute of Science, contains a collection of similar problems. Dr. Hadas's research is described in Hadas & Arcavi (2001) and in Hadas, Hershkowitz, & Schwarz (2002). I am very grateful to Dr. Hadas for her advice in the preparation of this manuscript.

Note

[1] The students for whom these problems are intended have no formal

knowledge of calculus. It is thus unlikely that they would connect this problem with the notion of integral as the area under a curve. Hence we do not consider this as a possible answer.

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The Oxford Problem Café

[Editor's note: Leo Rogers has kindly agreed to select interesting mathematical questions to fill some of the empty spaces that appear in FLM. His brief account of the origins of these problems and some of the criteria for their selection follows.]

A group of Mathematics Teachers and their friends have been meeting in the Oxford area on the first Thursday of the month to solve problems that have the broad and rather vague criteria that they have 'something to do with teaching and learning'. The intention is that the solutions should be accessible using, at most, what might be called 'High School' mathematics. Of course, it's more than just that.

Not many of these problems are original, and although we have kept some records, not all of them are attributable. So, on behalf of the group, we offer our apologies to the originators, and thank them for giving us much pleasure, interest, and sometimes trouble! We thought you would like to share some of the frustrations with us.

A Powerful Birthday

I celebrated my 72nd birthday recently, and a friend sent me this greeting:

$72^5 = 19^5 + 43^5 + 46^5 + 47^5 + 67^5$

I can show it is correct, but how did he discover this?